

# Introduction to the QCSP

Dmitriy Zhuk

Charles University

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# Quantified Equality Constraints

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**Lemma [Zhuk, Martin, 2021]**

$\text{QCSP}(\mathbb{N}; x = y \rightarrow y = z)$  is PSpace-hard.

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### Lemma [Zhuk, Martin, 2021]

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### Theorem [Zhuk, Martin, Bodirsky, Chen, 2021]

Suppose relations  $R_1, \dots, R_s$  are definable by some Boolean combination of atoms of the form  $(x = y)$ . Then  $\text{QCSP}(\mathbb{N}; R_1, \dots, R_s)$  is either tractable, NP-complete, or PSpace-complete.

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**Main Question**

What is the complexity of QCSP( $\Gamma$ ) for different  $\Gamma$ ?

| $\Sigma$ | dual- $\Sigma$ | Classification | Complexity Classes |
|----------|----------------|----------------|--------------------|
|          |                |                |                    |
|          |                |                |                    |
|          |                |                |                    |
|          |                |                |                    |
|          |                |                |                    |
|          |                |                |                    |

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|--------------------------------|----------------|----------------|--------------------|
| $\{\exists, \forall, \wedge\}$ |                |                |                    |
|                                |                |                |                    |
|                                |                |                |                    |
|                                |                |                |                    |
|                                |                |                |                    |
|                                |                |                |                    |
|                                |                |                |                    |

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|--------------------------------|------------------------------|----------------|--------------------|
| $\{\exists, \forall, \wedge\}$ | $\{\exists, \forall, \vee\}$ |                |                    |
|                                |                              |                |                    |
|                                |                              |                |                    |
|                                |                              |                |                    |
|                                |                              |                |                    |
|                                |                              |                |                    |

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|--------------------------------|------------------------------|----------------|--------------------|
| $\{\exists, \forall, \wedge\}$ | $\{\exists, \forall, \vee\}$ | ???????????    | ???????????        |
|                                |                              |                |                    |
|                                |                              |                |                    |
|                                |                              |                |                    |
|                                |                              |                |                    |
|                                |                              |                |                    |

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|--------------------------------|------------------------------|----------------|--------------------|
| $\{\exists, \forall, \wedge\}$ | $\{\exists, \forall, \vee\}$ | ???????????    | ???????????        |
| $\{\exists, \vee\}$            | $\{\forall, \wedge\}$        | Trivial        | L                  |
|                                |                              |                |                    |
|                                |                              |                |                    |
|                                |                              |                |                    |
|                                |                              |                |                    |

**Given** a sentence  $\exists y_1 \dots \exists y_t (R_1(\dots) \vee \dots \vee R_s(\dots))$ ,  
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| $\{\exists, \forall, \wedge\}$ | $\{\exists, \forall, \vee\}$ | ???????????    | ???????????        |
| $\{\exists, \vee\}$            | $\{\forall, \wedge\}$        | Trivial        | L                  |
| $\{\exists, \wedge\}$          | $\{\forall, \vee\}$          | CSP Dichotomy  | P, NP-complete     |
|                                |                              |                |                    |
|                                |                              |                |                    |
|                                |                              |                |                    |

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| $\{\exists, \wedge, \vee\}$    | $\{\forall, \wedge, \vee\}$  | Trivial iff<br>the core has<br>one element | L<br>NP-complete   |
|                                |                              |                                            |                    |
|                                |                              |                                            |                    |

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| $\{\exists, \forall, \wedge, \vee\}$ |                              | Positive equality<br>free tetrachotomy     | P, NP-complete<br>co-NP-complete<br>PSPACE-complete |
|                                      |                              |                                            |                                                     |

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| $\{\exists, \wedge\}$                      | $\{\forall, \vee\}$          | CSP Dichotomy                              | P, NP-complete                                      |
| $\{\exists, \wedge, \vee\}$                | $\{\forall, \wedge, \vee\}$  | Trivial iff<br>the core has<br>one element | L<br>NP-complete                                    |
| $\{\exists, \forall, \wedge, \vee\}$       |                              | Positive equality<br>free tetrachotomy     | P, NP-complete<br>co-NP-complete<br>PSPACE-complete |
| $\{\exists, \forall, \wedge, \vee, \neg\}$ |                              | Trivial iff<br>$\Gamma$ is trivial         | L<br>PSPACE-complete                                |

**Given** a sentence

$$\exists y_1 \forall x_1 \dots \exists y_t \forall x_t ((\neg R_1(\dots) \vee R_2(\dots)) \wedge \neg R_3(\dots)),$$

where  $R_1, \dots, R_3 \in \Gamma$ .

**Decide** whether it holds.

| $\Sigma$                                   | dual- $\Sigma$               | Classification                             | Complexity Classes                                  |
|--------------------------------------------|------------------------------|--------------------------------------------|-----------------------------------------------------|
| $\{\exists, \forall, \wedge\}$             | $\{\exists, \forall, \vee\}$ | ??????????                                 | ??????????                                          |
| $\{\exists, \vee\}$                        | $\{\forall, \wedge\}$        | Trivial                                    | L                                                   |
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### Quantified Constraint Satisfaction Problem:

**Given** a sentence  $\exists y_1 \forall x_1 \dots \exists y_t \forall x_t (R_1(\dots) \wedge \dots \wedge R_s(\dots))$ ,  
where  $R_1, \dots, R_s \in \Gamma$ .

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$A$  is a finite set,

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**Decide** whether the formula is satisfiable.

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$f$  **preserves**  $\Gamma$  (equivalently  $f \in \text{Pol}(\Gamma)$ ) if  $f$  preserves every  $R \in \Gamma$ .

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### Theorem [Bulatov, Zhuk, 2017]

- ▶ CSP( $\Gamma$ ) is solvable in polynomial time (tractable) if there exists a weak near-unanimity operation preserving  $\Gamma$ ,
- ▶ CSP( $\Gamma$ ) is NP-complete otherwise.

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Examples:  $x \vee y, x \wedge y, xy \vee xz \vee yz, x + y + z, 0, \min(x, y), \dots$

P

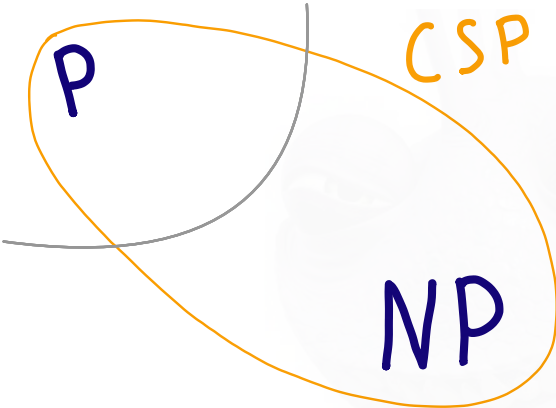
CSP

NP

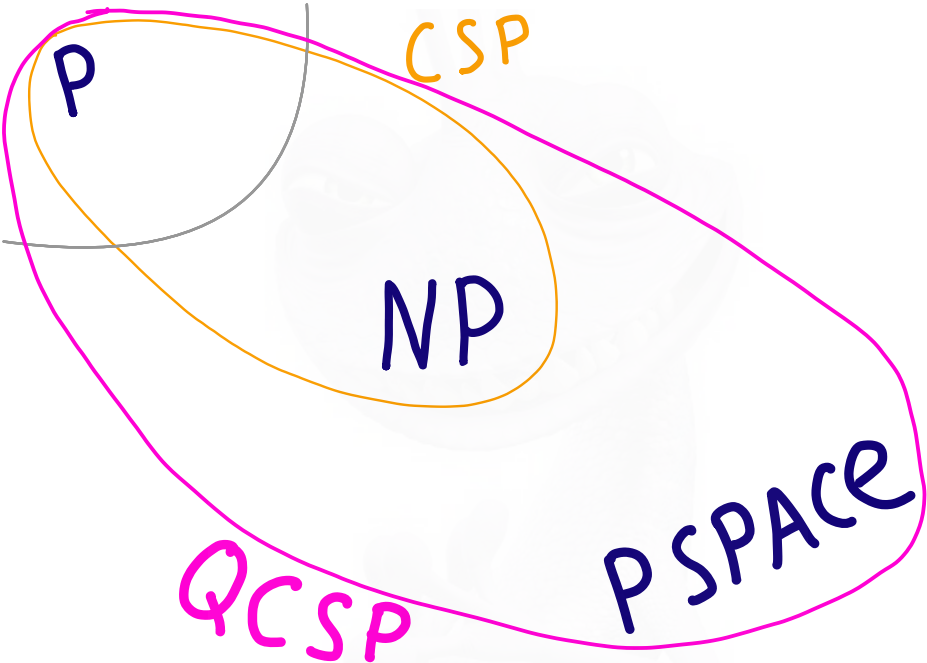
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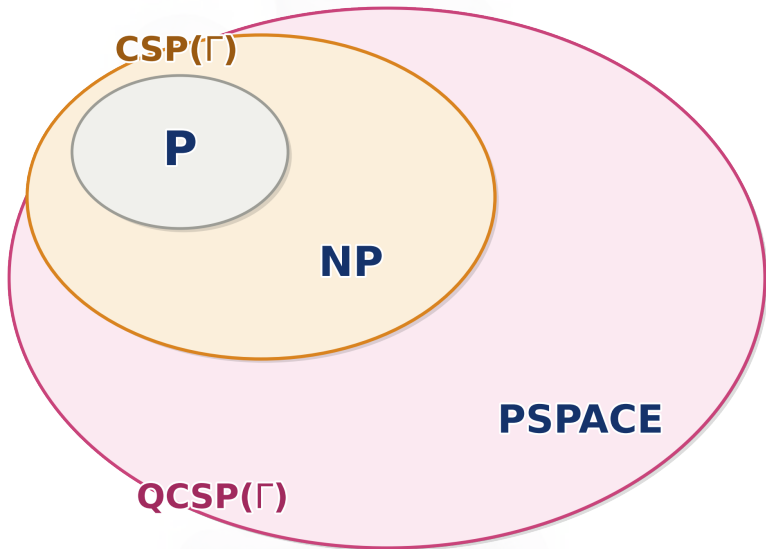
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PSPACE





## QCSP Complexity Classes



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**Theorem [Schaefer 1978 + Creignou et al. 2001/ Dalmau 1997.]**

Suppose  $\Gamma$  is a constraint language on  $\{0,1\}$ . Then

- ▶ QCSP( $\Gamma$ ) is in P if  $\Gamma$  is preserved by an idempotent WNU operation,
- ▶ QCSP( $\Gamma$ ) is PSPACE-complete otherwise.



# QCSP Complexity Classes



## QCSP Complexity Classes

- Put  $A' = A \cup \{*\}$ ,  $\Gamma'$  is  $\Gamma$  extended to  $A'$ . Then  $\text{QCSP}(\Gamma')$  is equivalent to  $\text{CSP}(\Gamma)$ .



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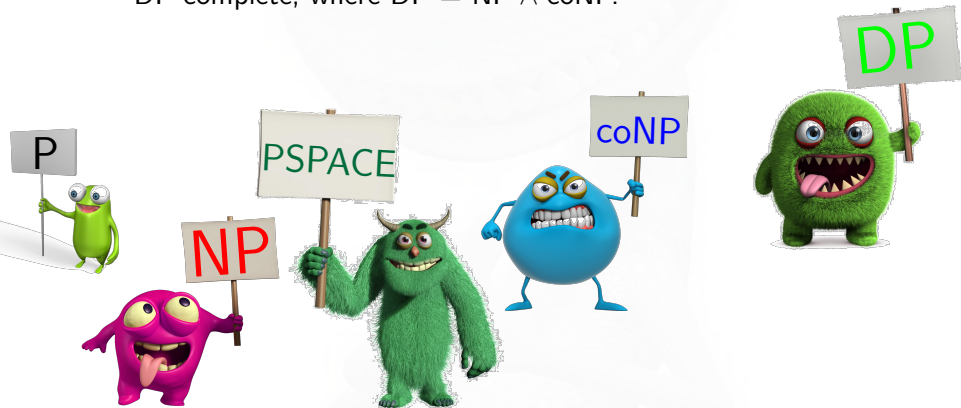
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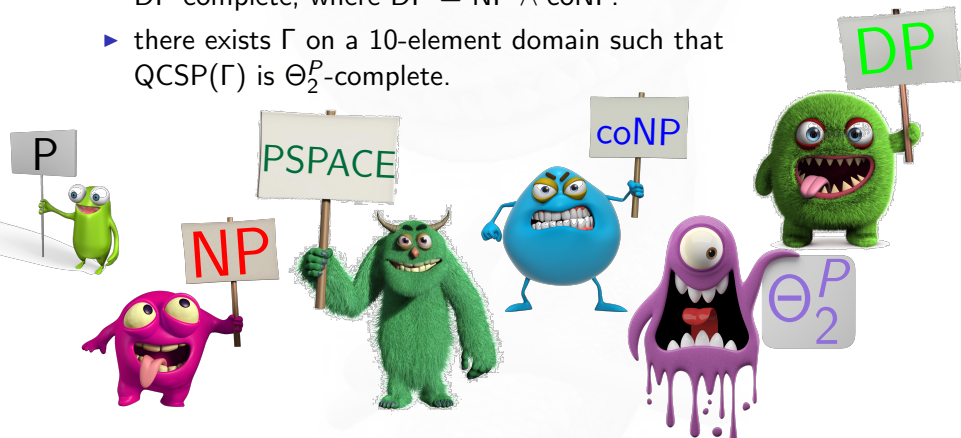
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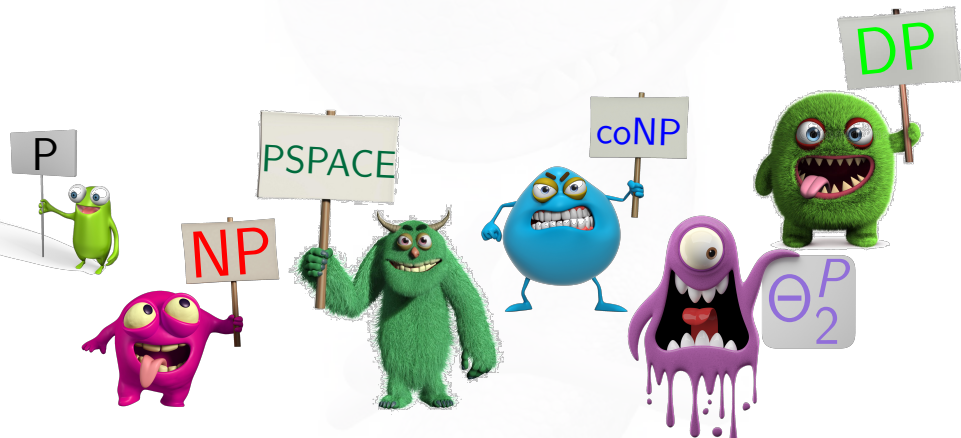


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- ▶ there exists  $\Gamma$  on a 10-element domain such that  $\text{QCSP}(\Gamma)$  is  $\Theta_2^P$ -complete.



# QCSP Complexity Classes



# QCSP Complexity Classes

## Theorem [Zhuk, Martin, 2019]

Suppose  $\Gamma$  is a constraint language on  $\{0, 1, 2\}$  containing  $\{x = a \mid a \in \{0, 1, 2\}\}$ . Then  $\text{QCSP}(\Gamma)$  is

- ▶ in P, or
- ▶ NP-complete, or
- ▶ coNP-complete, or
- ▶ PSPACE-complete.



## QCSP Dichotomy



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Then there exists  $S \subseteq A^n$  with  $|S| \leq |A|^2 \cdot (n \cdot |A|)^{2^{|A|}|A|+1}$

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We can restrict the Universal Player to polynomially many moves and he still wins.

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$\text{QCSP}(\Gamma)$  is not PSpace-hard  $\Rightarrow \text{QCSP}(\Gamma)$  is in  $\Pi_2^P$ .

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### Corollary

QCSP( $\Gamma$ ) is not PSpace-hard  $\Rightarrow$  QCSP( $\Gamma$ ) is in  $\Pi_2^P$ .

**Proof:**

$$\Pi_2^P = \{ \forall X^{|X| < p(|Z|)} \exists Y^{|Y| < q(|Z|)} \mathcal{V}(X, Y, Z) : \mathcal{V} \in \mathcal{P}, p, q - \text{polynomials} \}$$

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Then there exists  $S \subseteq A^n$  with  $|S| \leq |A|^2 \cdot (n \cdot |A|)^{2^{2|A|^{|A|+1}}}$  s. t.

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QCSP( $\Gamma$ ) is not PSpace-hard  $\Rightarrow$  QCSP( $\Gamma$ ) is in  $\Pi_2^P$ .

**Proof:** For each polynomial-size  $S \subseteq A^n$  check the existence of a winning strategy for the Existential Player.

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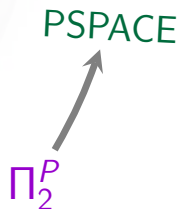
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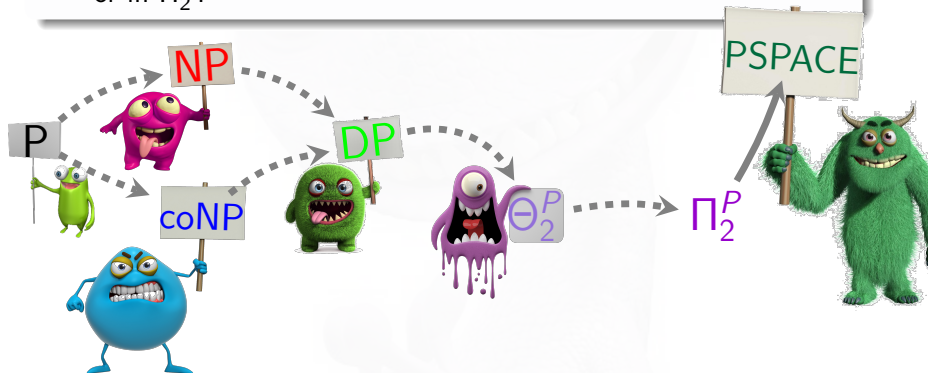


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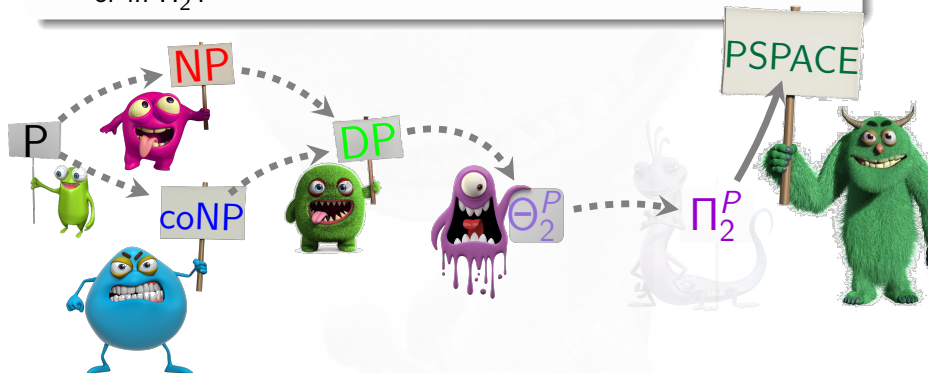


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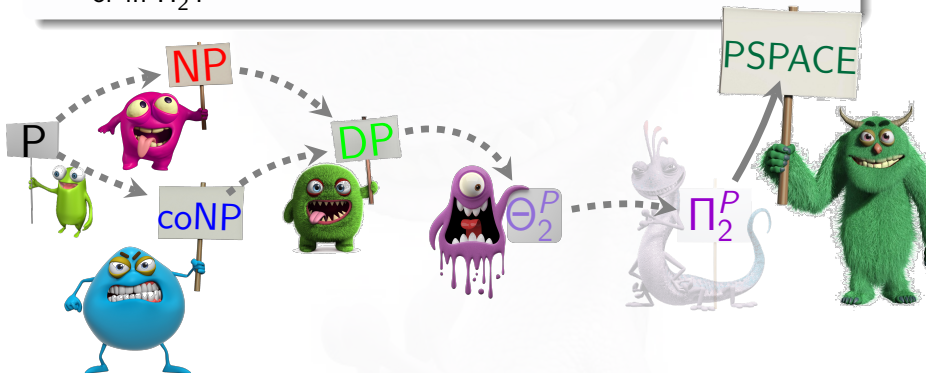


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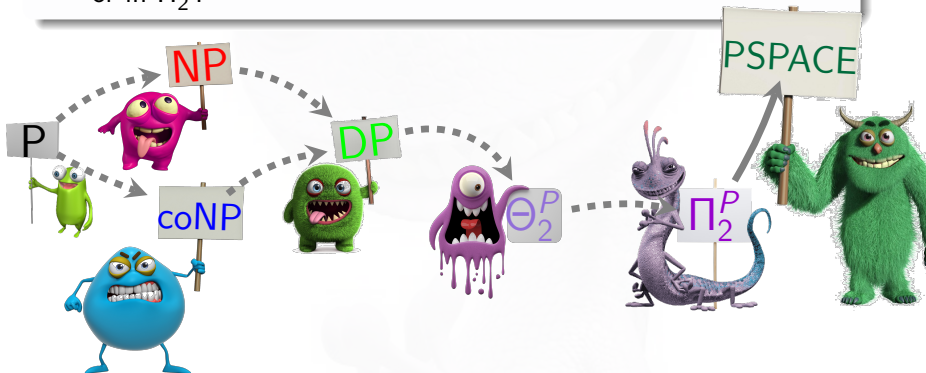


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## Lemma

There exists  $\Gamma$  on a 6-element set such that QCSP( $\Gamma$ ) is  $\Pi_2^P$ -complete.

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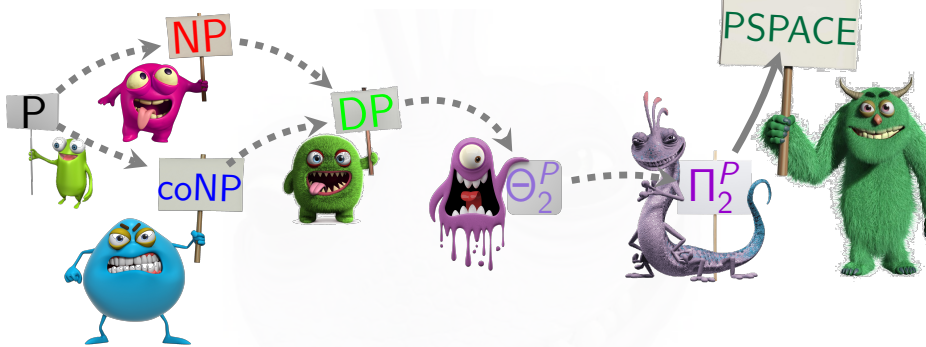
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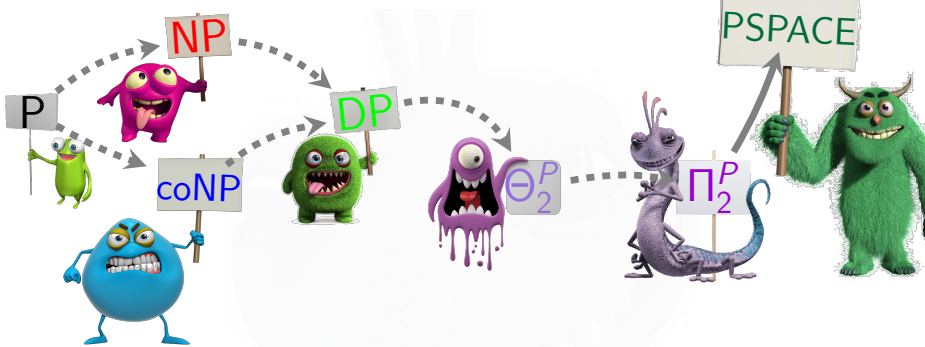


## Lemma

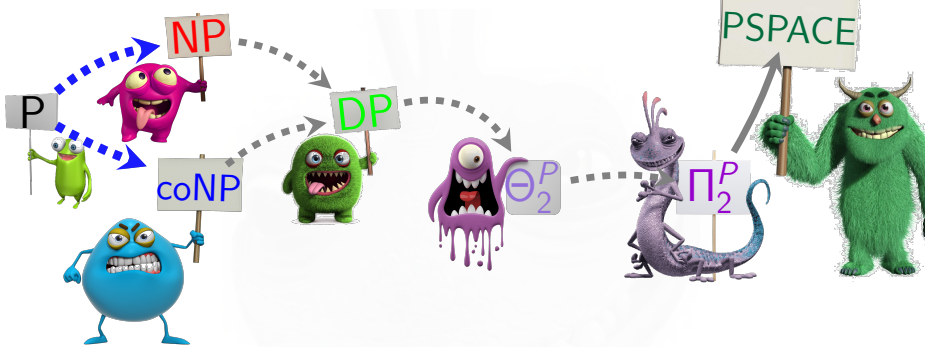
There exists  $\Gamma$  on a 6-element set such that QCSP( $\Gamma$ ) is  $\Pi_2^P$ -complete.

Are there any other complexity classes?



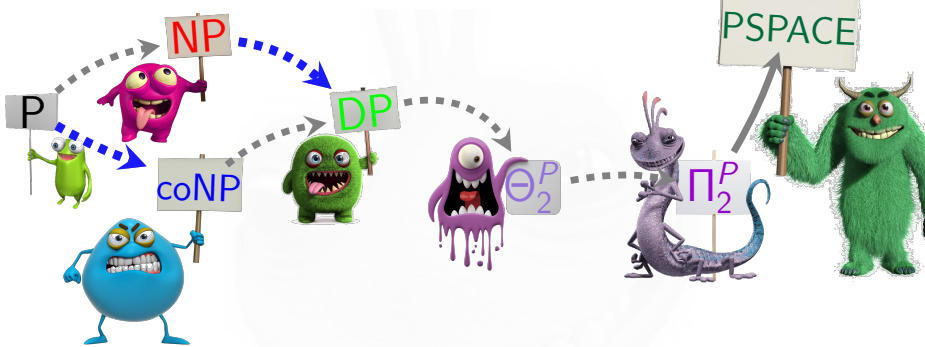


QCSP Hepta-chotomy to prove



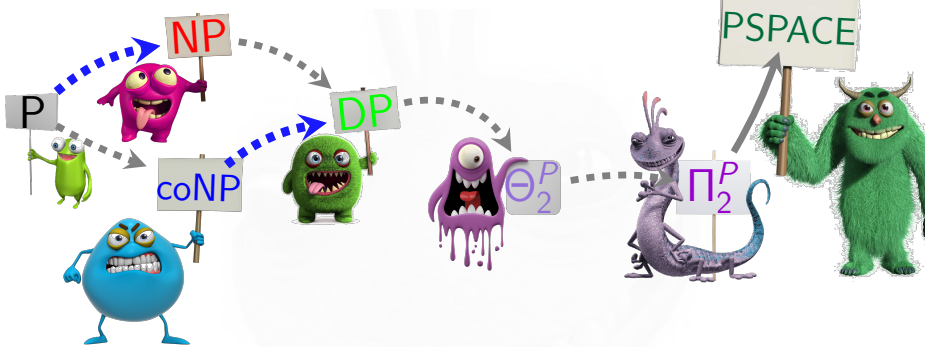
## QCSP Hepta-chotomy to prove

1. **P vs NP-hard** (under Turing reductions).



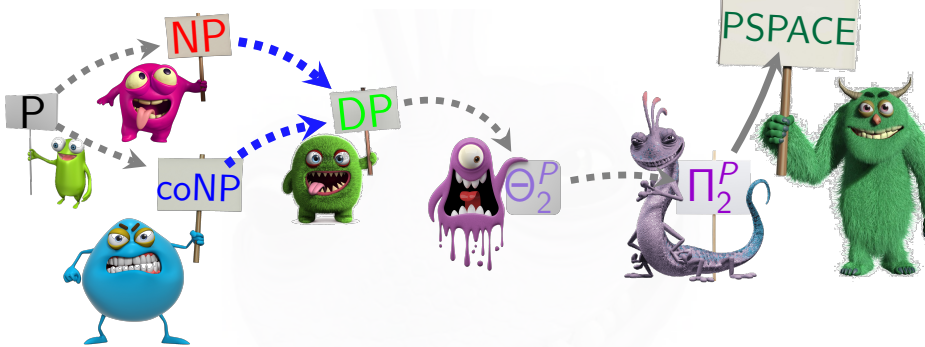
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2. **NP** vs **coNP**-hard



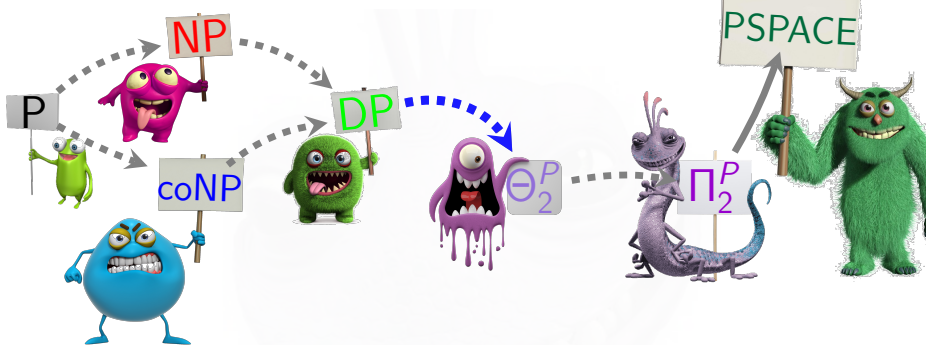
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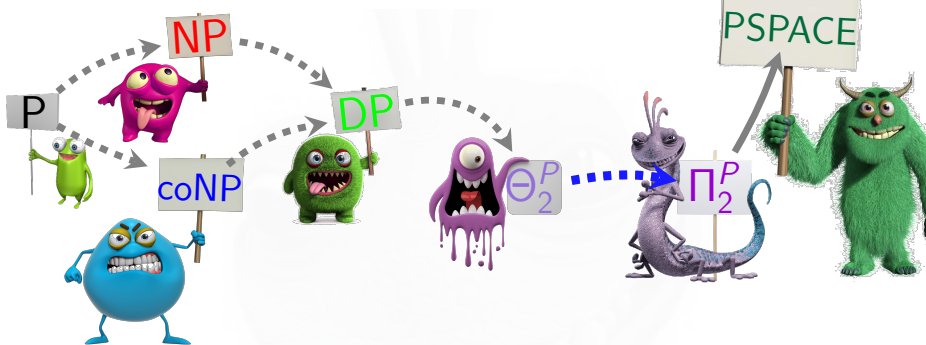
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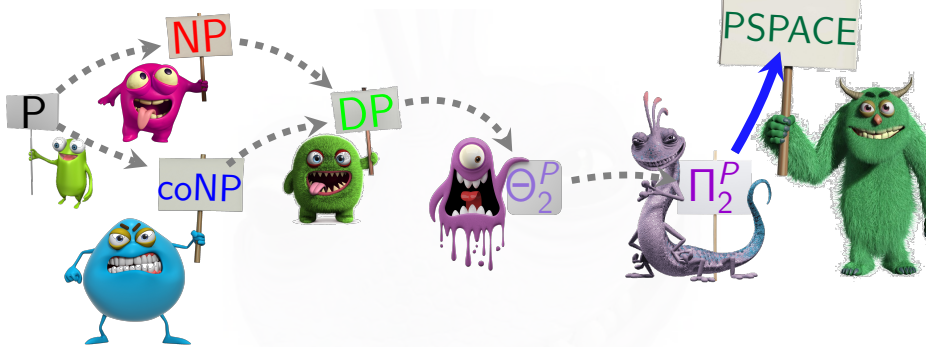
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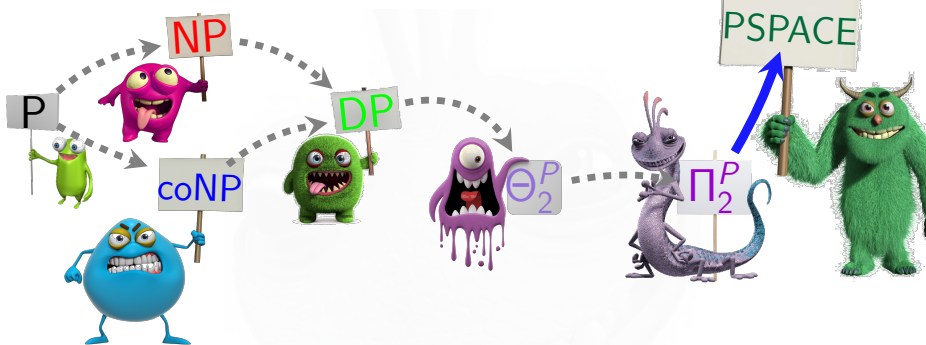
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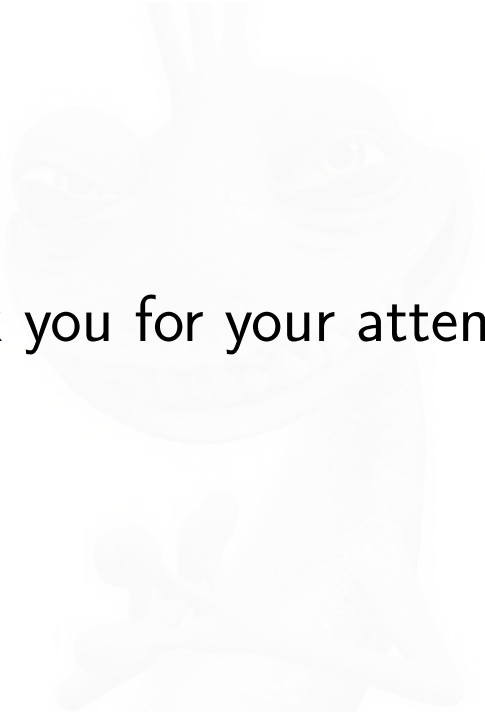
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3. coNP vs NP-hard
4.  $\text{NP} \cup \text{coNP}$  vs DP-hard
5. DP vs  $\Theta_2^P$ -hard
6.  $\Theta_2^P$  vs  $\Pi_2^P$ -hard
7.  $\Pi_2^P$  vs PSpace-hard



### QCSP Hepta-chotomy to prove

1. P vs NP-hard (under Turing reductions).
2. NP vs coNP-hard
3. coNP vs NP-hard
4.  $\text{NP} \cup \text{coNP}$  vs DP-hard
5. DP vs  $\Theta_2^P$ -hard
6.  $\Theta_2^P$  vs  $\Pi_2^P$ -hard
7.  $\Pi_2^P$  vs PSpace-hard (Done!!!)



Thank you for your attention



Thank you for your attention

A faded, light purple cartoon alien with large, expressive eyes and a wide, toothy smile. It has three long, thin antennae on its head and is waving its right hand. The alien's skin has a bumpy, textured appearance. The text "Thank you for your attention" is overlaid in the center of the image.

Thank you for your attention

Thank you for your attention

