

The class of congruence meet semidistributive varieties is not strong Maltsev

Seminar Algebra–Geometrie–Kombinatorik

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The interpretability lattice and Maltsev conditions

Definition

Let $\mathcal{V}_1 = \text{Mod}(\Sigma_1)$ and $\mathcal{V}_2 = \text{Mod}(\Sigma_2)$ be varieties over the signatures τ_1, τ_2 , respectively. An *interpretation* of $\mathcal{V}_1$ in $\mathcal{V}_2$ is a mapping $I : \tau_1 \rightarrow \text{Terms}(\tau_2)$ which preserves the satisfaction of Σ_1 identities, i.e.

$$f(x_1, \dots, x_n) \approx g(x_1, \dots, x_n) \in \Sigma_1 \implies \mathcal{V}_2 \models (If)(x_1, \dots, x_n) \approx (Ig)(x_1, \dots, x_n).$$

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This induces a preorder $\preceq$ on the class of varieties and we write $\mathcal{V}_1 \preceq \mathcal{V}_2$ to indicate that $\mathcal{V}_1$ interprets in $\mathcal{V}_2$. If both $\mathcal{V}_1 \preceq \mathcal{V}_2$ and $\mathcal{V}_2 \preceq \mathcal{V}_1$, we say $\mathcal{V}_1$ and $\mathcal{V}_2$ are *equi-interpretable*.

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- The variety $\mathcal{V}$ in the signature with a single ternary operation $m(xyz)$ axiomatized by the Maltsev identities $m(yxx) \approx m(xxy) \approx y$ interprets in the variety of groups, by mapping $m(xyz)$ to the term $xy^{-1}z$.

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- A class of varieties $\mathcal{C}$ is said to be characterized by a *strong Maltsev condition* if there exists a finitely presented (finite signature and finitely based) variety $\mathcal{V}$ such that $\mathcal{C} = \{\mathcal{W} : \mathcal{V} \preceq \mathcal{W}\}$.

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- A class of varieties $\mathcal{C}$ is said to be characterized by a *Maltsev condition* if there exists a countable sequence of finitely presented varieties $\mathcal{V}_1 \succeq \mathcal{V}_2 \succeq \cdots \succeq \mathcal{V}_i \succeq \cdots$ such that $\mathcal{C} = \bigcup_{1 \leq i} \{\mathcal{W} : \mathcal{V}_i \preceq \mathcal{W}\}$.

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A variety $\mathcal{V}$ of algebras is *congruence meet semidistributive* if each congruence lattice of its members satisfies the implication

$$\gamma \wedge \alpha = \gamma \wedge \beta \implies \gamma \wedge (\alpha \vee \beta) = \gamma \wedge \alpha.$$

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TABLE 1. The six conditions

Type Omitting Class	Equivalent Property	Strong for l.f. varieties?	Strong in general?
$\mathcal{M}_{\{1\}}$	satisfies a nontrivial idempotent Maltsev condition	YES (Siggers)	YES (Olšák)
$\mathcal{M}_{\{1,5\}}$	satisfies a nontrivial congruence identity (see [17])	NO (KKVW)	NO
$\mathcal{M}_{\{1,4,5\}}$	congruence n -permutable, for some $n > 1$	NO (KKVW)	NO
$\mathcal{M}_{\{1,2\}}$	congruence meet semidistributive	YES (KKVW)	??
$\mathcal{M}_{\{1,2,5\}}$	congruence join semidistributive (see [17])	NO (KKVW)	NO
$\mathcal{M}_{\{1,2,4,5\}}$	congruence n -permutable for some n and congruence join semidistributive	NO (KKVW)	NO

Table taken from ‘Characterizations of several Maltsev conditions’ by Kozik, Krokhin, Valeriote, Willard.

The connection to the commutator and 2-congruences

Definition

Let $\mathbb{A}$ be an algebra and let θ_1, θ_2 be congruences of $\mathbb{A}$. We define the *algebra of (θ_1, θ_2) -matrices* as follows.

$$M(\theta_1, \theta_2) = \text{Sg}_{A^{2^2}} \left(\left\{ \begin{array}{cc} x & \text{---} & y \\ | & & | \\ x & \text{---} & y \end{array} : (x, y) \in \theta_1 \right\} \cup \left\{ \begin{array}{cc} y & \text{---} & y \\ | & & | \\ x & \text{---} & x \end{array} : (x, y) \in \theta_2 \right\} \right).$$

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We then say that θ_1 *term condition centralizes* θ_2 if no matrix of $M(\theta_1, \theta_2)$ has one column which determines a pair of equal elements, while the opposite column determines a pair of unequal elements.

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Informally, the above theorem is stating that the class of congruence meet semidistributive varieties is exactly the class of varieties which have no nontrivial abelian congruences, and the latter condition (2) holds for a variety $\mathcal{V}$ if and only if there are no abelian principle congruences, which is true if and only if the ‘free’ principle congruence for $\mathcal{V}$ is neutral.

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for congruences θ_1 and θ_2 of an algebra $\mathbb{A}$. We say θ_1 *hypercentralizes* θ_2 if no matrix of $\Delta(\theta_1, \theta_2)$ has one column which determines a pair of equal elements, while the opposite column determines a pair of unequal elements. We denote by $[\theta_1, \theta_2]_H$ the corresponding commutator.

Let $S \leq A^{2^2}$. We define

- $H(S) := \left\{ \begin{array}{cc} b & \text{---} & d \\ | & & | \\ a & \text{---} & c \end{array} : \exists e, f \left(\begin{array}{cc} b & \text{---} & f & , & f & \text{---} & d \\ | & & | & & | & & | \\ a & \text{---} & e & & e & \text{---} & c \end{array} \in S \right) \right\} \text{ and}$
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It is not hard to see that

$$\Delta(\theta, \theta) = \bigcup_{n \geq 0} (V \circ H)^n(M(\theta, \theta)).$$

In particular, $M(\theta, \theta) \subseteq \Delta(\theta, \theta)$, so it follows that $[\theta_1, \theta_2]_{TC} \leq [\theta_1, \theta_2]_H$.

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Theorem

Let $\mathbb{A}$ be an algebra and let θ be a congruence of $\mathbb{A}$. The following are equivalent.

1. $(x, y) \in [\theta, \theta]_H$, and

2.
$$\begin{array}{ccc} x & \text{---} & y \\ | & & | \\ x & \text{---} & x \end{array} \in \Delta(\theta, \theta)$$

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Theorem (Follows from a result of Kearnes and Szendrei)

Let $\mathbb{A}$ be a Taylor algebra and let α be a congruence of $\mathbb{A}$. Then

$$[\alpha, \alpha]_{TC} = [\alpha, \alpha]_H.$$

We define for a variety $\mathcal{V}$ and the (x, y) -*elementary matrices*:

$$E_{\mathcal{V}}(x, y) := \text{Sg}_{\mathbb{F}_{\mathcal{V}}(x, y)^{2^2}} \left(\left\{ \begin{array}{cc|cc} x & \text{---} & x & y & \text{---} & y \\ | & & | & | & & | \\ x & \text{---} & x & y & \text{---} & y \end{array} , \begin{array}{cc|cc} y & \text{---} & y & x & \text{---} & x \\ | & & | & | & & | \\ y & \text{---} & y & x & \text{---} & x \end{array} , \begin{array}{cc|cc} y & \text{---} & y & x & \text{---} & x \\ | & & | & | & & | \\ x & \text{---} & x & y & \text{---} & y \end{array} , \begin{array}{cc|cc} x & \text{---} & x & y & \text{---} & y \\ | & & | & | & & | \\ y & \text{---} & y & x & \text{---} & x \end{array} , \begin{array}{cc|cc} x & \text{---} & y & y & \text{---} & x \\ | & & | & | & & | \\ x & \text{---} & y & y & \text{---} & x \end{array} , \begin{array}{cc|cc} y & \text{---} & x & y & \text{---} & x \\ | & & | & | & & | \\ y & \text{---} & x & y & \text{---} & x \end{array} \right\} \right).$$

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Theorem

Let $\mathcal{V}$ be a variety. The following are equivalent.

1. $\mathcal{V}$ is congruence meet semidistributive,

2. $\begin{array}{cc} x & \text{---} & x \\ | & & | \\ x & \text{---} & y \end{array} \in \Delta(\gamma, \gamma)$, where γ is the congruence of the two generated free algebra $\mathbb{F}_{\mathcal{V}}(x, y)$ in $\mathcal{V}$ generated by the pair (x, y) , and

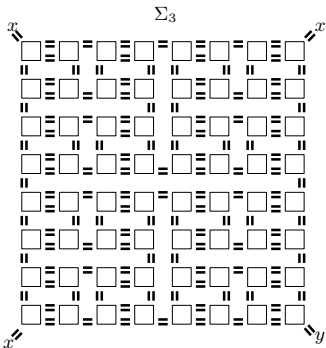
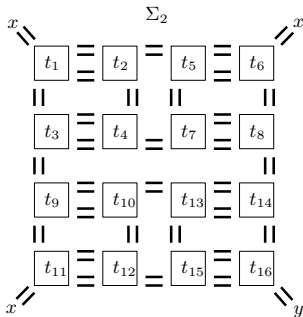
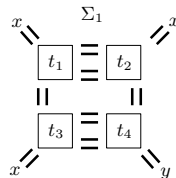
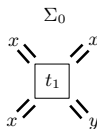
3. $\begin{array}{cc} x & \text{---} & x \\ | & & | \\ x & \text{---} & y \end{array} \in \text{Cg}_2(E_{\mathcal{V}}(x, y)) = \bigcup_{n \geq 0} (V \circ H)^n(E_{\mathcal{V}}(x, y)).$

$$\boxed{t} \equiv \boxed{s}$$

stands for

$$\begin{array}{ccc} t(xxyyyy) \text{---} t(xyxyyx) & = & s(xxyyyy) \text{---} s(xyxyyx) \\ | & & | \\ t(xyxyxy) \text{---} t(xyyyxx) & = & s(xyxyxy) \text{---} s(xyyyxx) \end{array}$$

The equational conditions $\Sigma_1, \Sigma_2, \dots, \Sigma_i, \dots$ which determine a Maltsev condition for congruence meet semidistributivity:



- We denote by $\mathcal{V}_n$ the variety of algebras in the signature $\{t_1, \dots, t_{4^n}\}$ which satisfy the package of identities Σ_n .

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- To show that there is no strong Maltsev condition that characterizes congruence meet semidistributivity, it suffices to produce for each $n \geq 0$ some congruence meet semidistributive variety $\mathcal{W}$ that does not interpret $\mathcal{V}_n$.

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$$\begin{array}{ccc} xyxx - xxyx & & s(xyxx) - s(xxyx) \\ | \quad s \quad | & \text{stands for} & | \quad | \\ yyxx - yxyx & & s(yyxx) - s(yxyx) \end{array}$$

$$\begin{array}{ccccccc} x & & & & & & x \\ // & & & & & & // \\ xxxx - xyxx = xyxx - xxyx = xxyx - xyxx = & \dots & = xxyx - xyxx = xyxx - xyyy & & & & \\ | \quad s_0 \quad | & & | \quad s_{2l} \quad | & & | \quad s_{2l+1} \quad | & & \\ yxxx - yyxx = yyxx - yxyx = yxyx - yyxx = & \dots & = yxyx - yyxx = yyxx - yyyy & & & & \\ x & & & & & & y \\ // & & & & & & // \end{array}$$

The condition Λ_l

Term analysis

Recall that

$$E_{\mathcal{W}_l}(x, y) := \text{Sg}_{(\mathbb{F}_l)^{2^2}} \left(\left\{ \begin{array}{ccc|ccc} x & \text{---} & x & y & \text{---} & y & y & \text{---} & y & x & \text{---} & x & x & \text{---} & y & y & \text{---} & x \\ \hline & & & & & & & & & & & & & & & & & \\ x & \text{---} & x & y & \text{---} & y & x & \text{---} & x & y & \text{---} & y & x & \text{---} & y & y & \text{---} & x \end{array} \right\} \right).$$

We want to show that for any n , there exists l so that it is impossible to glue together such squares to obtain a diagram witnessing the condition Σ_n .

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We want to show that for any n , there exists l so that it is impossible to glue together such squares to obtain a diagram witnessing the condition Σ_n . Let $\tau_l = \{s_0, \dots, s_{2l+1}\}$ be the signature corresponding to Λ_l . Consider the sets $E_0 \subseteq E_1 \subseteq \dots \subseteq E_k \subseteq \dots$ whose union is $E_{\mathcal{W}_l}(x, y)$ defined by

$$E_0 = \left\{ \begin{array}{ccc|ccc} x & \text{---} & x & y & \text{---} & y & y & \text{---} & y & x & \text{---} & x & x & \text{---} & y & y & \text{---} & x \\ \hline & & & & & & & & & & & & & & & & & \\ x & \text{---} & x & y & \text{---} & y & x & \text{---} & x & y & \text{---} & y & x & \text{---} & y & y & \text{---} & x \end{array} \right\} \text{ and}$$

$$E_{k+1} = \{r^{(\mathbb{F}_l)^{2^2}}(\alpha, \beta, \gamma, \delta) : r \in \tau_l \text{ and } \alpha, \beta, \delta, \gamma \in E_k\} \text{ for } k \geq 0.$$

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Our goal is to show that, for large enough l , there is no k where

$$\begin{array}{ccc} x & \text{---} & x \\ | & & | \\ x & \text{---} & y \end{array} \in (V \circ H)^n(E_k).$$

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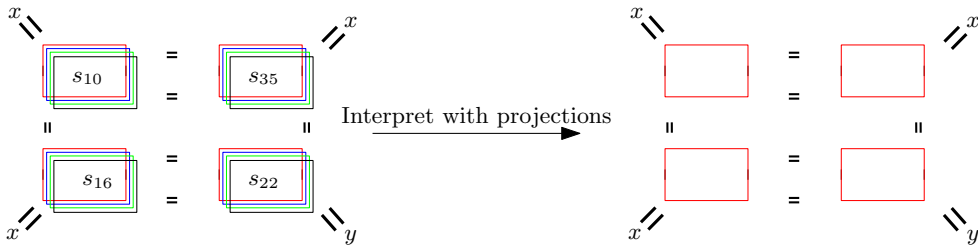
The condition Λ_l

$$\begin{array}{ccccccc}
 x & & & & & & x \\
 \parallel & & & & & & \parallel \\
 xxx - yxx & = & yxx - xxy & = & xxy - yxx & = & \dots = xxy - yxx = yxx - xyy \\
 | \quad s_0 & | & | \quad s_1 & | & | \quad s_2 & | & | \quad s_{2l} & | & | \quad s_{2l+1} \\
 yxx - yyy & = & yyy - yxy & = & yxy - yyy & = & \dots = yxy - yyy = yyy - yyy \\
 x & & & & & & y \\
 \parallel & & & & & & \parallel
 \end{array}$$

The condition $\Lambda_{l,1}$ is modeled by projections.

$$\begin{array}{ccc}
 x & & x \\
 \parallel & & \parallel \\
 xxx - yxx & & xxy - yxx = \dots = xxy - yxx = yxx - xyy \\
 | \quad s_0 & & | \quad s_2 & | & | \quad s_{2l} & | & | \quad s_{2l+1} \\
 yxx - yyy & & yxy - yyy = \dots = yxy - yyy = yyy - yyy \\
 x & & y \\
 \parallel & & \parallel \\
 \pi_4 & & \pi_1 & & \pi_1 & & \pi_1
 \end{array}$$

delete s_1



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- We define a sequence of sets $\{x, y\} = F_l^0 \subseteq F_l^1 \subseteq \dots \subseteq F_l^{k-1} \subseteq F_l^k \dots$, where each F_l^k is the domain of a partial τ_l -algebra $\mathbb{F}_l^k$ with all τ_l operations defined on $(F_l^{k-1})^4$, for every $k \geq 1$.

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	$s_0^{\mathbb{F}_l^1}$	$s_1^{\mathbb{F}_l^1}$	$s_2^{\mathbb{F}_l^1}$	$\dots$	$s_{2l}^{\mathbb{F}_l^1}$	$s_{2l+1}^{\mathbb{F}_l^1}$
$xxxx$	x	x	x	$\dots$	x	x
$xxxy$	$s_0(xxxy)$	$s_1(xxxy)$	$s_2(xxxy)$	$\dots$	$s_{2l}(xxxy)$	$s_{2l+1}(xxxy)$
$xyyx$	$s_0(xyyx)$	$s_1(xyyx)$	$\leftarrow \mathbf{s}_1(\mathbf{xyyx})$	$\dots$	$s_{2l}(xyyx)$	$s_{2l+1}(xyyx)$
$xyyy$	$s_0(xyyy)$	$\leftarrow \mathbf{s}_0(\mathbf{xyyy})$	$s_2(xyyy)$	$\dots$	$s_{2l}(xyyy)$	$\leftarrow \mathbf{s}_{2l}(\mathbf{xyyy})$
$xyxx$	$s_0(xyxx)$	$\leftarrow \mathbf{s}_0(\mathbf{xyxx})$	$s_2(xyxx)$	$\dots$	$s_{2l}(xyxx)$	$\leftarrow \mathbf{s}_{2l}(\mathbf{xyxx})$
$xyxy$	$s_0(xyxy)$	$s_1(xyxy)$	$\leftarrow \mathbf{s}_1(\mathbf{xyxy})$	$\dots$	$s_{2l}(xyxy)$	$s_{2l+1}(xyxy)$
$xyyx$	$s_0(xyyx)$	$s_1(xyyx)$	$s_2(xyyx)$	$\dots$	$s_{2l}(xyyx)$	$s_{2l+1}(xyyx)$
$xyyy$	y	$s_1(xyyy)$	$s_2(xyyy)$	$\dots$	$s_{2l}(xyyy)$	x
$yxxx$	x	$s_1(yxxx)$	$s_2(yxxx)$	$\dots$	$s_{2l}(yxxx)$	y
$yxxxy$	$s_0(yxxxy)$	$s_1(yxxxy)$	$s_2(yxxxy)$	$\dots$	$s_{2l}(yxxxy)$	$s_{2l+1}(yxxxy)$
$yxyyx$	$s_0(yxyyx)$	$s_1(yxyyx)$	$\leftarrow \mathbf{s}_1(\mathbf{yxyyx})$	$\dots$	$s_{2l}(yxyyx)$	$s_{2l+1}(yxyyx)$
$yxyyy$	$s_0(yxyyy)$	$\leftarrow \mathbf{s}_0(\mathbf{yxyyy})$	$s_2(yxyyy)$	$\dots$	$s_{2l}(yxyyy)$	$\leftarrow \mathbf{s}_{2l}(\mathbf{yxyyy})$
$yyxx$	$s_0(yyxx)$	$\leftarrow \mathbf{s}_0(\mathbf{yyxx})$	$s_2(yyxx)$	$\dots$	$s_{2l}(yyxx)$	$\leftarrow \mathbf{s}_{2l}(\mathbf{yyxx})$
$yyxy$	$s_0(yyxy)$	$s_1(yyxy)$	$\leftarrow \mathbf{s}_1(\mathbf{yyxy})$	$\dots$	$s_{2l}(yyxy)$	$s_{2l+1}(yyxy)$
$yyyx$	$s_0(yyyx)$	$s_1(yyyx)$	$s_2(yyyx)$	$\dots$	$s_{2l}(yyyx)$	$s_{2l+1}(yyyx)$
$yyyy$	y	y	y	$\dots$	y	y

Input tuples $(a, b, c, d) \in F_l^k$ satisfy $\{a, b, c, d\} \cap (F_l^k \setminus F_l^{k-1}) \neq \emptyset$

	$\mathbb{F}_l^{k+1}$ s_0	$\mathbb{F}_l^{k+1}$ s_1	$\mathbb{F}_l^{k+1}$ s_2	$\dots$	$\mathbb{F}_l^{k+1}$ s_{2l}	$\mathbb{F}_l^{k+1}$ s_{2l+1}
$pppp$	p	p	p	$\dots$	p	p
$pppq$	$s_0(pppq)$	$s_1(pppq)$	$s_2(pppq)$	$\dots$	$s_{2l}(pppq)$	$s_{2l+1}(pppq)$
$ppqp$	$s_0(ppqp)$	$s_1(ppqp)$	$\leftarrow s_1(\mathbf{ppqp})$	$\dots$	$s_{2l}(ppqp)$	$s_{2l+1}(ppqp)$
$ppqq$	$s_0(ppqq)$	$\leftarrow s_0(\mathbf{ppqq})$	$s_2(ppqq)$	$\dots$	$s_{2l}(ppqq)$	$\leftarrow s_{2l}(\mathbf{ppqq})$
$pqpp$	$s_0(pqpp)$	$\leftarrow s_0(\mathbf{pqpp})$	$s_2(pqpp)$	$\dots$	$s_{2l}(pqpp)$	$\leftarrow s_{2l}(\mathbf{pqpp})$
$pqpq$	$s_0(pqpq)$	$s_1(pqpq)$	$\leftarrow s_1(\mathbf{pqpq})$	$\dots$	$s_{2l}(pqpq)$	$s_{2l+1}(pqpq)$
$pqqp$	$s_0(pqqp)$	$s_1(pqqp)$	$s_2(pqqp)$	$\dots$	$s_{2l}(pqqp)$	$s_{2l+1}(pqqp)$
$pqqq$	q	$s_1(pqqq)$	$s_2(pqqq)$	$\dots$	$s_{2l}(pqqq)$	p

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	$\mathbb{F}_l^{k+1}$ s_0	$\mathbb{F}_l^{k+1}$ s_1	$\mathbb{F}_l^{k+1}$ s_2	$\dots$	$\mathbb{F}_l^{k+1}$ s_{2l}	$\mathbb{F}_l^{k+1}$ s_{2l+1}
$abcd$	$s_0(abcd)$	$s_1(abcd)$	$s_2(abcd)$	$\dots$	$s_{2l}(abcd)$	$s_{2l+1}(abcd)$

Lemma

Let $l \geq 1$, and $m \geq 1$. Consider a set of τ_l -terms of the form

$$T = \{r_j(a_j, b_j, c_j, d_j) : r_j \in \tau_l \text{ and } a_j, b_j, c_j, d_j \in F_l, \text{ for } 1 \leq j \leq m\}$$

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and let

$$Z = \{z : \text{there exists } r_j(a_j, b_j, c_j, d_j) \in T \text{ with } r_j = s_z\}$$

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where $\mathbb{G} = \mathbb{F}_{\mathcal{W}_{l,i}}(\bigcup_{1 \leq j \leq m} \{a_j, b_j, c_j, d_j\})$ is the algebra for $\mathcal{W}_{l,i}$ freely generated by

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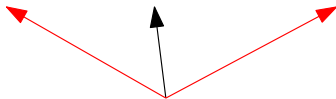
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$ppqp$	$s_0(ppqp)$	$s_1(ppqp)$	$\leftarrow \mathbf{s_1(ppqp)}$	...	$s_{2l}(ppqp)$	$s_{2l+1}(ppqp)$
$ppqq$	$s_0(ppqq)$	$\leftarrow \mathbf{s_0(ppqq)}$	$s_2(ppqq)$	...	$s_{2l}(ppqq)$	$\leftarrow \mathbf{s_{2l}(ppqq)}$
$pqpp$	$s_0(pqpp)$	$\leftarrow \mathbf{s_0(pqpp)}$	$s_2(pqpp)$	...	$s_{2l}(pqpp)$	$\leftarrow \mathbf{s_{2l}(pqpp)}$
$pqpq$	$s_0(pqpq)$	$s_1(pqpq)$	$\leftarrow \mathbf{s_1(pqpq)}$	...	$s_{2l}(pqpq)$	$s_{2l+1}(pqpq)$
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$pqqq$	q	$s_1(pqqq)$	$s_2(pqqq)$	...	$s_{2l}(pqqq)$	p



Intution: if $i \in Z$, where Z is the set of indices of basic τ_l -operation symbols being used, then we *at most* need to use identities involving s_{i-1} , s_i , and s_{i+1} to find the normal form for some $s_i(a, b, c, d)$.

Hence, for fixed n , there is obviously l large enough so that the lemma applies when $|Z| = 4^n$.

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$$\begin{array}{ccc} x & \text{---} & x \\ | & & | \\ x & \text{---} & y \end{array} \in \bigcup_{n \geq 0} (V \circ H)^n (E_{\mathcal{V}}(x, y)).$$

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We defined $E_0 = \left\{ \begin{array}{ccc} x & \text{---} & x \\ | & & | \\ x & \text{---} & x \end{array}, \begin{array}{ccc} y & \text{---} & y \\ | & & | \\ y & \text{---} & y \end{array}, \begin{array}{ccc} y & \text{---} & y \\ | & & | \\ x & \text{---} & x \end{array}, \begin{array}{ccc} x & \text{---} & x \\ | & & | \\ y & \text{---} & y \end{array}, \begin{array}{ccc} x & \text{---} & y \\ | & & | \\ x & \text{---} & y \end{array}, \begin{array}{ccc} y & \text{---} & x \\ | & & | \\ y & \text{---} & x \end{array} \right\}$ and

$$E_{k+1} = \{r^{(\mathbb{F}_l)^{2^2}}(\alpha, \beta, \gamma, \delta) : r \in \tau_l \text{ and } \alpha, \beta, \delta, \gamma \in E_k\} \text{ for } k \geq 0.$$

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- Choose k minimal so that $\begin{array}{c} x \text{ --- } x \\ | \quad \quad | \\ x \text{ --- } y \end{array} \in \bigcup_{n \geq 0} (V \circ H)^n (E_k(x, y))$. Obviously, $k \neq 0$.

We apply the lemma and find that $k - 1$ works also, contradiction.

$$\begin{array}{ccccccc}
r_1^{\mathbb{F}_l}(xxxx) & & & & & & r_6^{\mathbb{F}_l}(xxxx) \\
// & & & & & & // \\
\boxed{r_1(\alpha_1, \beta_1, \gamma_1, \delta_1)} & \equiv & \boxed{r_2(\alpha_2, \beta_2, \gamma_2, \delta_2)} & = & \boxed{r_5(\alpha_5, \beta_5, \gamma_5, \delta_5)} & \equiv & \boxed{r_6(\alpha_6, \beta_6, \gamma_6, \delta_6)} \\
|| & & & || & || & & || \\
\boxed{r_3(\alpha_3, \beta_3, \gamma_3, \delta_3)} & \equiv & \boxed{r_4(\alpha_4, \beta_4, \gamma_4, \delta_4)} & = & \boxed{r_7(\alpha_7, \beta_7, \gamma_7, \delta_7)} & \equiv & \boxed{r_8(\alpha_8, \beta_8, \gamma_8, \delta_8)} \\
|| & & & & & & || \\
\boxed{r_9(\alpha_9, \beta_9, \gamma_9, \delta_9)} & \equiv & \boxed{r_{10}(\alpha_{10}, \beta_{10}, \gamma_{10}, \delta_{10})} & = & \boxed{r_{13}(\alpha_{13}, \beta_{13}, \gamma_{13}, \delta_{13})} & \equiv & \boxed{r_{14}(\alpha_{14}, \beta_{14}, \gamma_{14}, \delta_{14})} \\
|| & & & || & || & & || \\
\boxed{r_{11}(\alpha_{11}, \beta_{11}, \gamma_{11}, \delta_{11})} & \equiv & \boxed{r_{12}(\alpha_{12}, \beta_{12}, \gamma_{12}, \delta_{12})} & = & \boxed{r_{15}(\alpha_{15}, \beta_{15}, \gamma_{15}, \delta_{15})} & \equiv & \boxed{r_{16}(\alpha_{16}, \beta_{16}, \gamma_{16}, \delta_{16})} \\
// & & & & & & // \\
r_{11}^{\mathbb{F}_l}(xxxx) & & & & & & r_{16}^{\mathbb{F}_l}(yyyy)
\end{array}$$

$$\boxed{r(\alpha, \beta, \gamma, \delta)} \quad \text{stands for} \quad r^{\mathbb{F}_l^{2^2}} \left(\boxed{\alpha} \quad \boxed{\beta} \quad \boxed{\gamma} \quad \boxed{\delta} \right)$$

Thank you for your attention!