

A categorical perspective on Constraint Satisfaction: The Wonderland of Adjunctions

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Lisbon – LICS 2026



Funded by the European Union (ERC, POCOCOP, 101071674). Views and opinions expressed are however those of the author only and do not necessarily reflect those of the European Union or the European Research Council Executive Agency. Neither the European Union nor the granting authority can be held responsible for them.

Constraint Satisfaction Problems

Input: - variables: $x_1 \dots x_n$
- constraints: $C_1(x_i \dots x_j), C_2(x_k \dots x_\ell), \dots$

Task: assign values to variables, such that *all* constraints are satisfied

Example: 2-SAT

variables: Boolean

constraints: binary logic gates

$x_i \vee x_j, \neg(x_k \wedge x_\ell), \dots$

Example: 3-coloring

variables take values $\{0, 1, 2\}$

constraints: $x \neq y$

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CSP template:

(finite) set of values

list of possible constraints

Complexity?

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More examples:

graph homomorphism
linear equations mod p
 st -reachability
HornSAT
...

CSP template:

(finite) set of values
list of possible constraints

Complexity?

Categorical encoding

Template: finite category S and $A: S \rightarrow \text{set}$. $\text{CSP}(A)$:

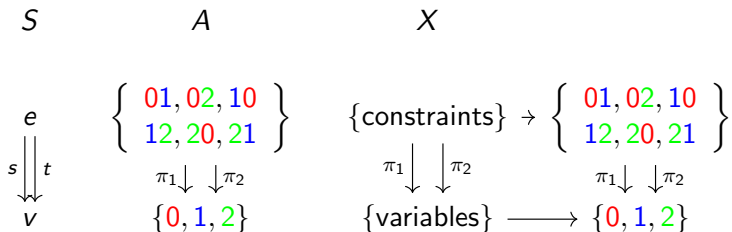
Input: functor $X: S \rightarrow \text{set}$

Task: find a natural transformation $X \rightarrow A$

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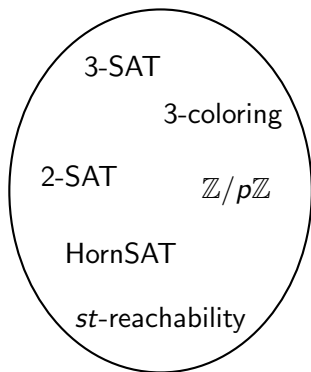
Reductions

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hard

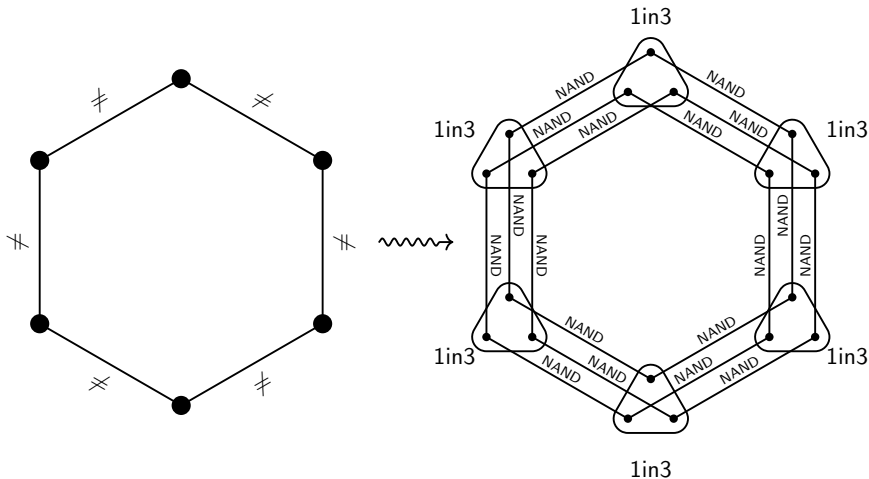


easy

" $\text{CSP}(A)$ is harder than $\text{CSP}(B)$ "

Reductions and Adjunctions

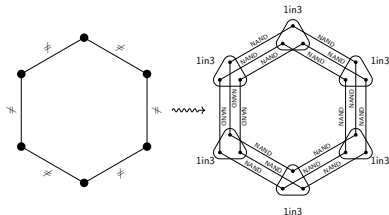
3-coloring to 3-SAT:



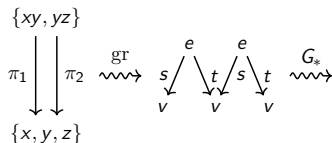
replace variables x with ... replace constraints $x \neq y$ with ...

Reductions and Adjunctions

3-coloring to 3-SAT:

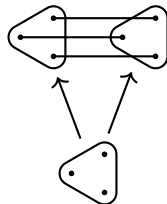
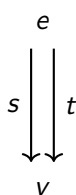


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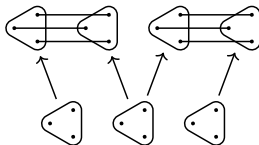
CSP(A) to CSP(B):

Gadget $G : S^{\text{op}} \rightarrow \text{Fun}(\mathcal{T}, \text{set})$

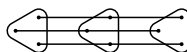


$L_G : \text{Fun}(S, \text{set}) \rightarrow \text{Fun}(\mathcal{T}, \text{set})$

$X \mapsto \text{colim}(G \circ \text{gr}X)$



colim



Reductions and Adjunctions

Definition

Call $L: \mathcal{C} \rightleftarrows \mathcal{D}: R$ adjunction if

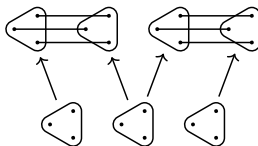
$$\text{hom}(L(X), B) \simeq \text{hom}(X, R(B))$$

- L_G is left adjoint
- every left adjoint

$$L: \text{Fun}(\mathcal{S}, \text{set}) \rightarrow \text{Fun}(\mathcal{T}, \text{set})$$

is of the form L_G

$$\begin{array}{c} \{xy, yz\} \\ \downarrow \pi_1 \quad \downarrow \pi_2 \\ \{x, y, z\} \end{array} \xrightarrow{\text{gr}} \begin{array}{c} e \quad e \\ s \searrow \quad \swarrow t \quad s \searrow \quad \swarrow t \\ v \quad v \quad v \quad v \end{array} \xrightarrow{G_*}$$



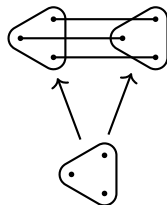
colim



CSP($\mathcal{A}$) to CSP($\mathcal{B}$):

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$$\begin{array}{c} e \\ \downarrow s \quad \downarrow t \\ v \end{array}$$



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Call $L: \mathcal{C} \rightleftarrows \mathcal{D}: R$ *adjunction* if

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is of the form L_G

Lemma

L reduces $\text{CSP}(A)$ to $\text{CSP}(B)$ iff

$$A \overset{\exists}{\rightleftarrows} R(B)$$

Proof: ($\Leftarrow$)

$$X \overset{\exists}{\rightarrow} A$$

$$\text{iff } X \overset{\exists}{\rightarrow} R(B)$$

$$\text{iff } L(X) \overset{\exists}{\rightarrow} B$$

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When does such an L exist?

Kan extensions and the universal gadget

Lemma

Each $\text{CSP}(A)$ is "equivalent" to $\text{CSP}(M)$ for some $M: \text{set} \rightarrow \text{set}$

Need reductions from $\text{CSP}(A)$ to $\text{CSP}(M)$ and backwards:

There are functors $\text{Lan}_A, \text{Ran}_A: \text{Fun}(S, \text{set}) \rightarrow \text{Fun}(\text{set}, \text{set})$ with

$$\text{hom}(\text{Lan}_A(X), N) = \text{hom}(X, N \circ A)$$

$$\text{hom}(N \circ A, Y) = \text{hom}(N, \text{Ran}_A(Y))$$

$\implies$ choose M to be $\text{Ran}_A(A)$

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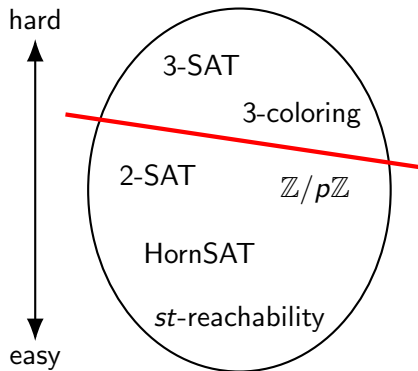
Idea: reduce $\text{CSP}(A)$ to $\text{CSP}(B)$ via $\text{Lan}_A(-) \circ B$

This works iff $\text{Ran}_B(B) \xrightarrow{\exists} \text{Ran}_A(A)$

Theorem [Bulatov, Jeavons, Krokhin 2005]

$\text{CSP}(A)$ reduces to $\text{CSP}(B)$ via some L iff $\text{Ran}_B B \xrightarrow{\exists} \text{Ran}_A A$

In particular, if $\text{Ran}_B B \xrightarrow{\exists} \text{id}_{\text{set}}$ then $\text{CSP}(B)$ is **NP-complete**



Theorem [Bulatov; Zhuk 2017]

If $B \not\xrightarrow{\exists} \text{id}_{\text{set}}$ then $\text{CSP}(B)$ is in **P**

Generalizations

Valued CSP

[BBKVŽ'24]

Promise CSP

[BBKO'21][HJO'26]

Infinite templates

[BOP'17]

Quantum CSP

[CJM'26]