Hierarchies galore: Toward a Uniform Algorithm and Uniform Reduction

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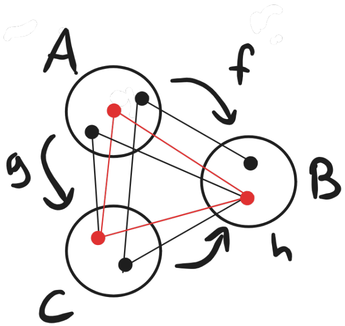
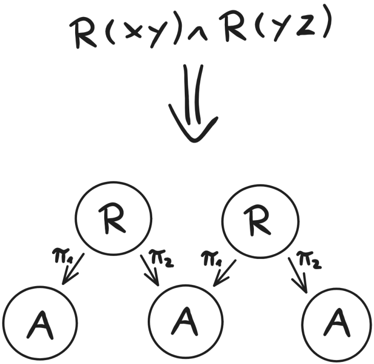
Label Cover

Input: some sets and maps



Task: pick one element out of each set, compatibly.

CSP to LC: Let $(A, R \subseteq A^2)$ be a CSP template

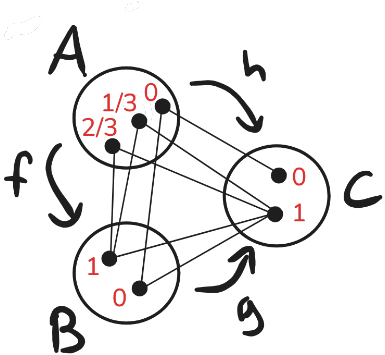


LC to LP: elements of the sets are variables $a \in [0,1]$

$$1 = \sum_{a \in A} a$$

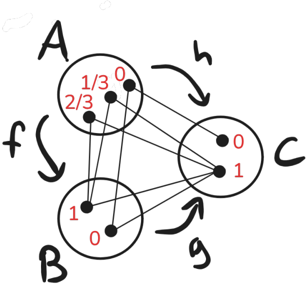
$$b = \sum_{f(a)=b} a$$

replace $[0,1]$ with $\mathbb{Z}$, $\mathbb{Z}_p$, vectors, ...



Sherali-Adams

LP to LP:



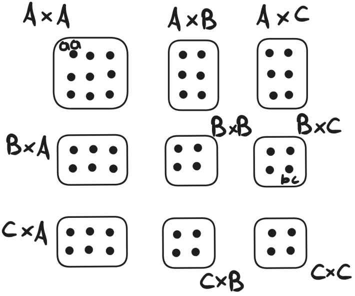
$$1 = \sum_{a \in A} a, \quad b = \sum_{f(a)=b} a$$

new variable $ab \in [0,1]$ for every pair

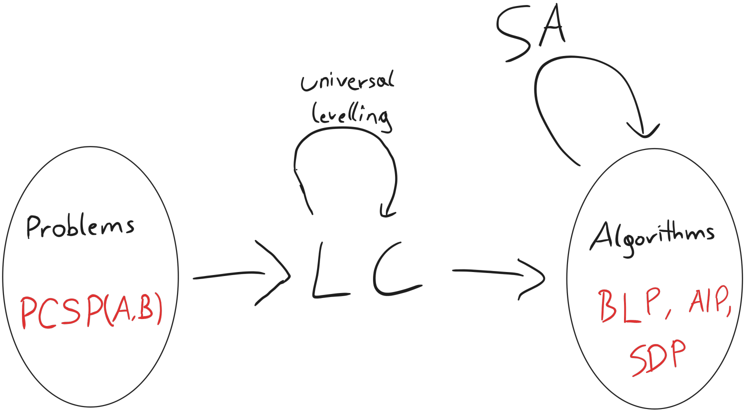
$$ab = ba, \quad aa = a,$$

$$ab = 0 \text{ if } a, b \in \text{same potato}$$

$$c = \sum_{a \in A} ac, bc = \sum_{f(a)=b} ac$$



Philosophy

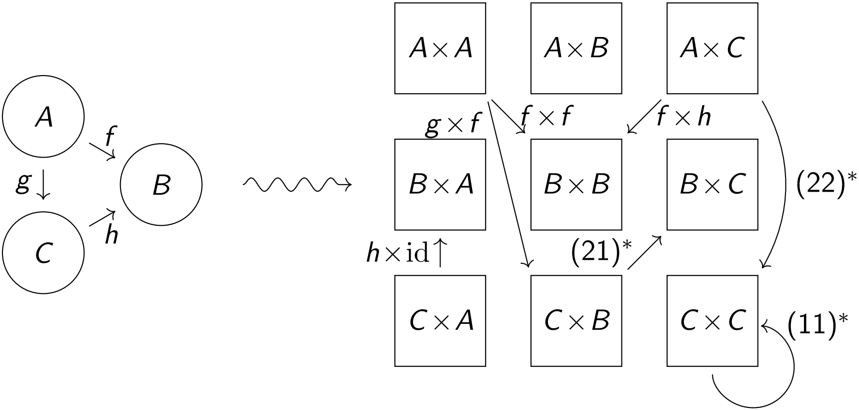


SA
universal levelling
Problems
PCSP(A,B)
LC
Algorithms
BLP, AIP, SDP

Hierarchies for anything

given a LC-instance $\mathcal{D}$, build a new LC-instance:

- a potato $A_1 \times \cdots \times A_k$ for every k-tuple of potatoes $A_1, \ldots A_k$ in $\mathcal{D}$
- add all "obvious" constraints



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given a LC-instance $\mathcal{D}$, build a new LC-instance:

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- add constraints

$$A_1 \times \cdots \times A_k \xrightarrow{f_1 \times \cdots \times f_k} B_1 \times \cdots \times B_k$$

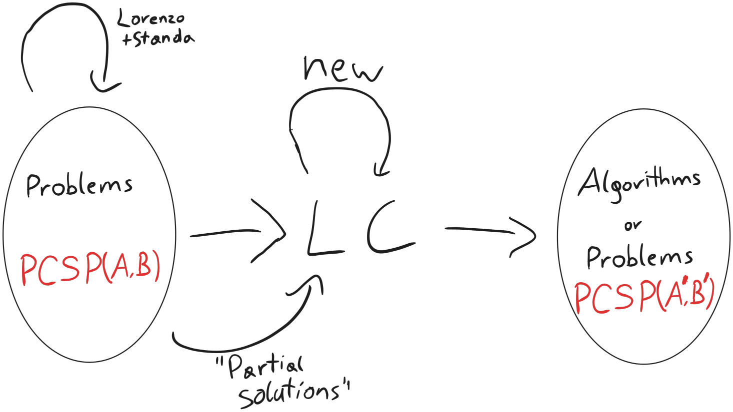
  where each f_i is from $\mathcal{D}$ or identity

- for each $\sigma \colon [k] \to [k]$ add a constraint

$$A_1 \times \cdots \times A_k \xrightarrow{\sigma^*} A_{\sigma(1)} \times \ldots A_{\sigma(k)}$$

$$a_1 \ldots a_k \longmapsto a_{\sigma(1)} \ldots a_{\sigma(k)}$$

More philosophy



Minions

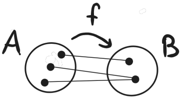
Definition: A minion $\mathcal{M}$ is a functor $\mathrm{set} \to \mathrm{set}$, i.e.

- for every set A, there is a set $\mathcal{M}^{(A)}$
- for every map $f\colon A \to B$, there is a map

$$\mathcal{M}^{(f)}\colon \mathcal{M}^{(A)} \to \mathcal{M}^{(B)}$$

- with axioms $\mathcal{M}^{(\mathrm{id}_A)} = \mathrm{id}_{\mathcal{M}^{(A)}}$ and $\mathcal{M}^{(f)} \circ \mathcal{M}^{(g)} = \mathcal{M}^{(f \circ g)}$

Example: $\mathcal{M}_{\mathrm{BLP}}^{(A)} = \{P : A \to [0,1] \mid \sum_{a \in A} P(a) = 1\}$



$$\mathcal{M}^{(f)}: \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x \\ y+z \end{pmatrix}$$

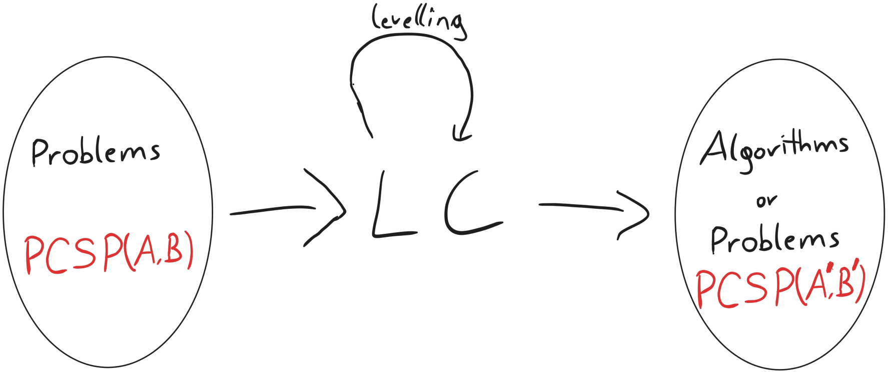
Minion homomorphisms

Definition: a minion homomorphism is a natural transformation $f\colon \mathcal{M} \to \mathcal{N}$, i.e. a tuple of maps $\alpha_A\colon \mathcal{M}^{(A)} \to \mathcal{N}^{(A)}$ s.t.

$$\begin{array}{ccccc} A & \qquad & \mathcal{M}^{(A)} & \xrightarrow{\alpha_A} & \mathcal{N}^{(A)} \\ {\scriptstyle \forall f}\downarrow & & {\scriptstyle \mathcal{M}^{(f)}}\downarrow & \circ & \downarrow{\scriptstyle \mathcal{N}^{(f)}} \\ B & & \mathcal{M}^{(B)} & \xrightarrow[\alpha_B]{} & \mathcal{N}^{(B)} \end{array}$$

Example:

$$\mathcal{M}_{\mathrm{BLP}} \to \mathrm{Pol}(\mathrm{hornSAT})$$
$$P \mapsto \bigwedge_{\substack{a \in A \\ P(a) > 0}} a : \{0,1\}^A \to \{0,1\}$$


Problems
PCSP(A,B)
levelling
LC
Algorithms
or
Problems
PCSP(A',B')

Characterizations

$\mathrm{R} =$ BLP, AIP, $\mathcal{Z}_p \ldots$

[BBKO21]:

$$\mathcal{M}_{\mathrm{R}} \to \mathrm{Pol}(A, B) \iff R \text{ solves } \mathrm{PCSP}(A, B)$$
$$\mathrm{Pol}(A', B') \to \mathrm{Pol}(A, B) \implies \mathrm{PCSP}(A, B) \leq_{\mathrm{P}} \mathrm{PCSP}(A', B')$$

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new: for any minion $\mathcal{M}$, there is a minion $\mathrm{lvl}_k \mathcal{M}$ s.t.

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combine with Victor+Jakub:

$$\mathrm{lvl}_k(\omega\mathrm{Pol}(A', B')) \to \mathrm{Pol}(A, B) \iff \mathrm{PCSP}(A, B) \leq_{\mathrm{Datalog}} \mathrm{PCSP}(A', B')$$

Levels of Z_p

have better minions for levels of $\mathcal{Z}_p$:

$$(\mathrm{lvl}_2\mathcal{Z}_p)^{(A)} \ni P\colon A \to \mathbb{Z}_p^N \text{ s.t.} \begin{cases} \sum_{a\in A} P(a) = \bar{1} \\ P(a) \perp P(b) \text{ for } a \neq b \end{cases}$$

Same for $k > 2$, but stronger orthogonality condition

Consequences:

second level of $\mathcal{Z}_2$ solves D_4

p-th level of $\mathcal{Z}_p$ solves $\mathcal{Z}_{p^2}$

$\mathbb{Z}_8$???