A CSP approach to Graph Sandwich Problems

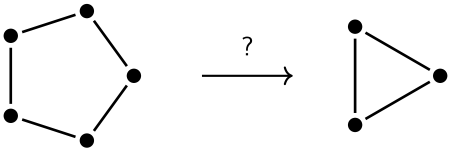
Manuel Bodirsky and **Santiago Guzmán-Pro**

Institute of Algebra
TU Dresden

SODA 2026, Vancouver, Canada

ERC Synergy Grant POCOCOP (GA 101071674)

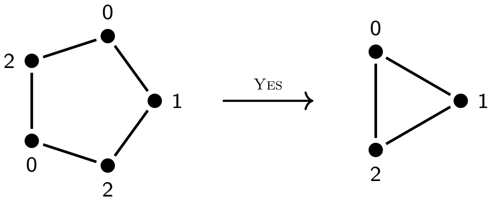
CSPs = Homomorphism Problems



Input: G
(Always finite)

Template: H
(Might be infinite)

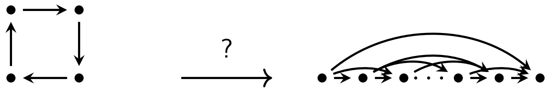
CSPs = Homomorphism Problems



Input: G
(Always finite)

Template: H
(Might be infinite)

CSPs = Homomorphism Problems



Input: G
(Always finite)

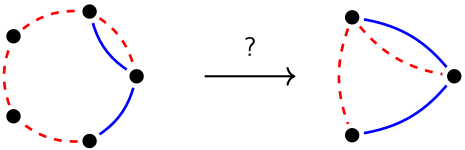
Template: $(\mathbb{Q}, <)$
(Might be infinite)

CSPs = Homomorphism Problems

Acyclic digraphs $\xrightarrow{\text{Yes}}$ 

Input: G Template: $(\mathbb{Q}, <)$

CSPs = Homomorphism Problems

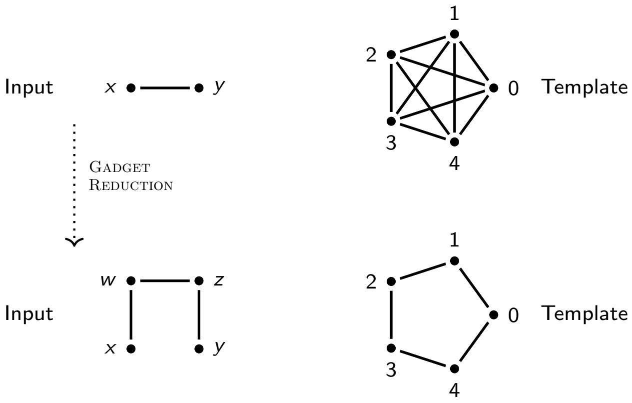


Input: G
(Always finite)

Template: H
(Might be infinite)

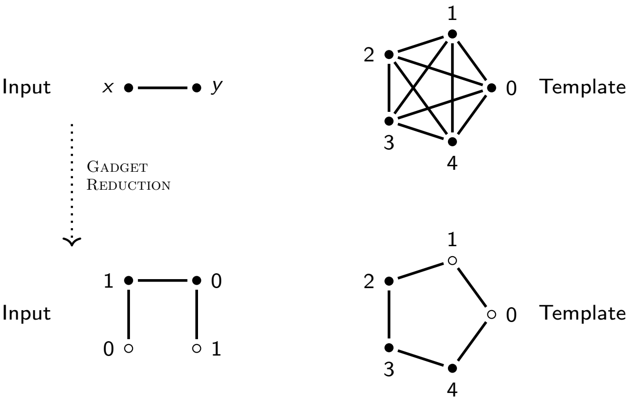
CSPs = Homomorphism Problems

Primitive positive constructions



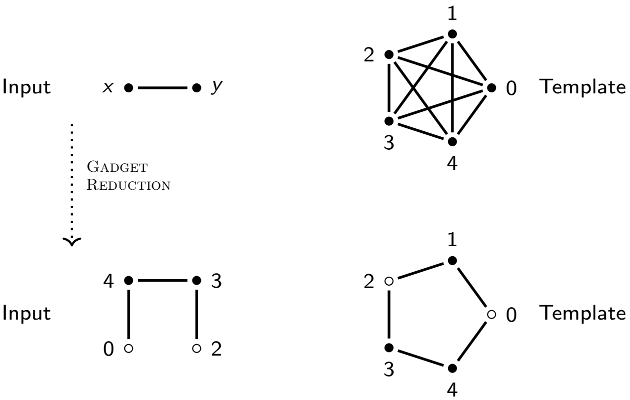
CSPs = Homomorphism Problems

Primitive positive constructions



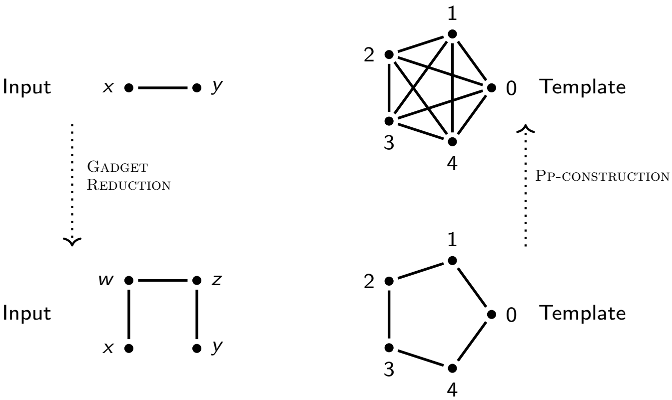
CSPs = Homomorphism Problems

Primitive positive constructions



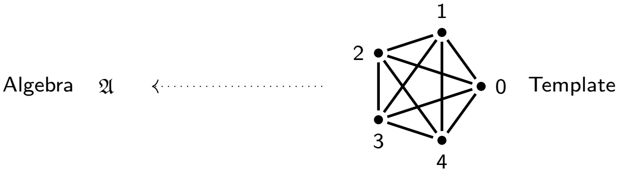
CSPs = Homomorphism Problems

Primitive positive constructions



CSPs = Homomorphism Problems

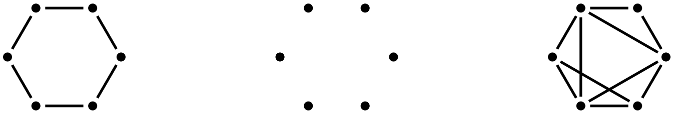
Polymorphisms



- If $\mathfrak{A}$ satisfies Π, then CSP(T) is solved by algorithm M
- If $\mathfrak{A}$ satisfies Σ, then T pp-constructs K_3

Graph Sandwich Problems

Template: $\mathcal{C}$



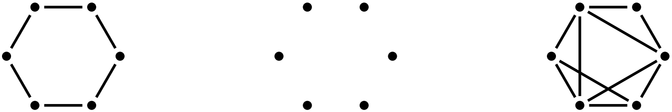
(V, E_1) Input

$(V, E) \in \mathcal{C}$ with $E_1 \subseteq E \subseteq E_2$?

(V, E_2) Input

Graph Sandwich Problems

Template: Split graphs



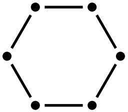
(V, E_1) Input

(V, E) split graph with $E_1 \subseteq E \subseteq E_2$?

(V, E_2) Input

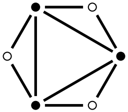
Graph Sandwich Problems

Template: Split graphs



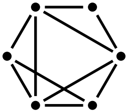
(V, E_1)

Input



(V, E) split graph with $E_1 \subseteq E \subseteq E_2$?

YES



(V, E_2)

Input

Graph Sandwich Problems

What has been done?

Introduced by Golumbic, Kaplan and Shamir (1995). Since then:

- Chordal
- Comparability
- Circle
- Path graphs
- Directed path graphs
- Split
- Permutation
- Trivially perfect
- Cographs
- Circular-arc
- Interval graphs
- Proper circular-arc
- Unit circular-arc
- Proper interval
- Threshold
- (k, l)-graphs
- Clique-helly
- Hereditary clique-helly
- Strongly chordal
- Bipartite chain
- Odd-hole-free
- Even-hole-free
- $3PC(\cdot, \cdot)$-free
- C_n-free
- Complete multipartite
- P_n-free

Alvarado, Cameron, Chaniotis, Chudnovsky, Dantas, Dourado, Faria, de Figueiredo, Golumbic, Kaplan, Klein, Maffray, Petito, Rautenbach, Shamir, da Silva, Spirkl, Sritharan, Teixeira, Vušković (and possibly others)

Graph Sandwich Problems

Dear colleagues,

We are pleased to invite you to the online event:

Event: *30 Years of Graph Sandwich Problems: A Celebration*
Date: March 27, 2025
Time: 2:00 PM (GMT -3), São Paulo, Brazil

Link: https://meet.google.com/sur-pmun-evy

In 1995, the publication of the seminal paper:

"M.C. Golumbic, H. Kaplan, R. Shamir, "Graph Sandwich Problems," Journal of Algorithms 19 (1995) 449-473"

opened a rich and extensive research area that continues to inspire publications worldwide.

Graph Sandwich Problems

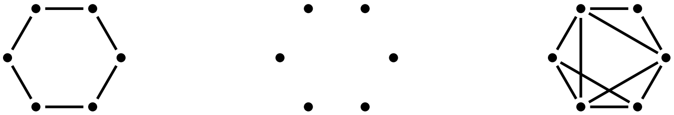
What is open?

Many things, but... "the complexity of the Perfect-Graph-Sandwich-Problem remains one of the most prominent open questions in this area" Cameron, Chianotis, de Figueiredo, Spirkl (2025).

The approach

The approach

Template: Split graphs



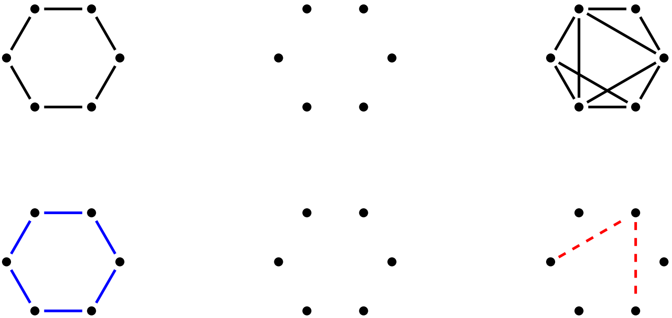
(V, E_1)
Input

(V, E) split graph
with $E_1 \subseteq E \subseteq E_2$?

(V, E_2)
Input

The approach

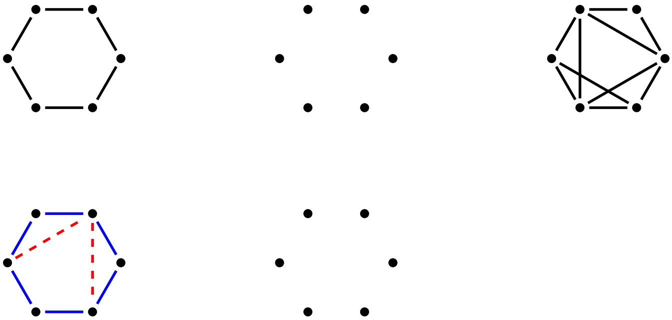
Template: Split graphs



(V, E_1)	(V, E) split graph with	(V, N)
Input	$E_1 \subseteq E$ and $E \cap N = \varnothing$?	Input

The approach

Template: Split graphs



(V, E_1, N)

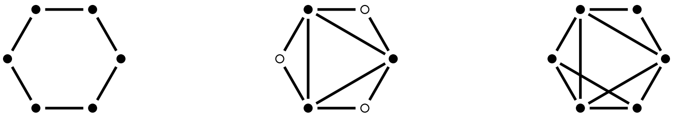
Input

(V, E) split graph with

$E_1 \subseteq E$ and $E \cap N = \varnothing$?

The approach

Template: Split graphs

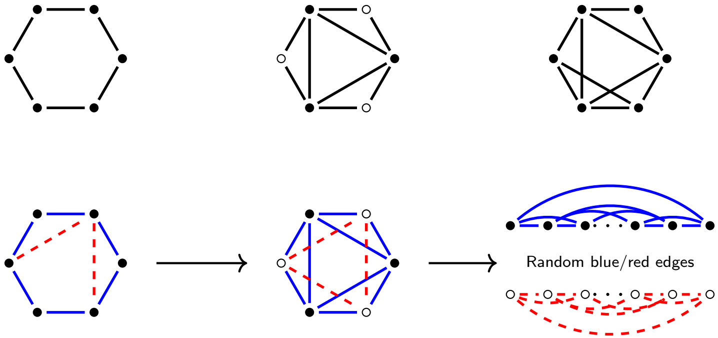


(V, E_1, N)
Input

(V, E) split graph with
$E_1 \subseteq E$ and $E \cap N = \varnothing$?

The approach

Template: Split graphs



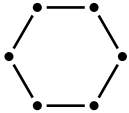
(V, E_1, N)
Input

(V, E) split graph with
$E_1 \subseteq E$ and $E \cap N = \varnothing$?

H

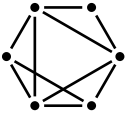
The approach

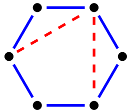
Template: Split graphs



(V, E) split graph

with $E_1 \subseteq E \subseteq E_2$?

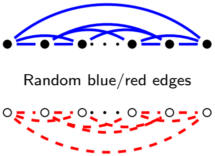




(V, E_1, N)

Input

?



Template: H

The approach

When is the SP for $\mathcal{C}$ a CSP?

- $\mathcal{C}$ is a hereditary class,
- $\mathcal{C}$ has the joint embedding property, and
- $\mathcal{C}$ is preserved under split blow-ups.

The approach

For instance. . .

- Chordal
- Comparability
- Circle
- Path graphs
- Directed path graphs
- Split
- Permutation
- Trivially perfect
- Cographs
- Circular-arc
- Interval graphs
- Proper circular-arc
- Unit circular-arc
- Proper interval
- Threshold
- (k, l)-graphs
- Clique-helly
- Hereditary clique-helly
- Strongly chordal
- Bipartite chain
- Odd-hole-free
- Even-hole-free
- $3PC(\cdot, \cdot)$-free
- C_n-free
- Complete multipartite
- P_n-free

. . . and of course, for perfect graphs

The approach

Moreover ...

- **Complete multipartite graphs**. Algorithm from Dantas, Figueiredo, da Silva, and Teixeira is a *Datalog program*.
- **Split graphs**. Algorithm from Golumbic, Kaplan, and Shamir is a *reduction to the finite*.
- **Threshold graphs**. Tractability also explained by the *algebraic approach* to CSPs.
- **Comparability graphs**. Hardness follows from the classifications of CSPs of reducts of the random poset (Kompatscher and van Pham, 2018).
- **Generalized split graphs**. The P vs. NP-complete classification of the sandwich problem for (p, q)-split graphs (Dantas, Figueiredo, da Silva, and Teixeira) recovered in terms of pp-constructions.
- **Permutation graphs**. Hardness proof of Golumbic, Kaplan and Shamir is a pp-construction of $(\mathbb{Q}, Betw)$.

The approach

Moreover ...

- **Complete multipartite graphs**. Algorithm from Dantas, Figueiredo, da Silva, and Teixeira is a *Datalog program*.
- **Split graphs**. Algorithm from Golumbic, Kaplan, and Shamir is a *reduction to the finite*.
- **Threshold graphs**. Tractability also explained by the *algebraic approach* to CSPs.
- **Comparability graphs**. Hardness follows from the classifications of CSPs of reducts of the random poset (Kompatscher and van Pham, 2018).
- **Generalized split graphs**. The P vs. NP-complete classification of the sandwich problem for (p, q)-split graphs (Dantas, Figueiredo, da Silva, and Teixeira) recovered in terms of pp-constructions.
- **Permutation graphs**. Hardness proof of Golumbic, Kaplan and Shamir is a pp-construction of $(\mathbb{Q}, Betw)$.

The approach

Something new?

- For $\{P_4, K_4\}$-free graphs is NP-complete

The approach

Something new?

- For $\{P_4, K_4\}$-free graphs is NP-complete
- For K_k-free perfect graphs is NP-complete for $k \geq 4$

The approach

Something new?

- For $\{P_4, K_4\}$-free graphs is NP-complete
- For K_k-free perfect graphs is NP-complete for $k \geq 4$
- For line graphs of bipartite multigraphs is NP-complete

The approach

Something new?

- For $\{P_4, K_4\}$-free graphs is NP-complete
- For K_k-free perfect graphs is NP-complete for $k \geq 4$
- For line graphs of bipartite multigraphs is NP-complete
- For line graphs of multigraphs is NP-complete

The approach

Something new?

- For $\{P_4, K_4\}$-free graphs is NP-complete
- For K_k-free perfect graphs is NP-complete for $k \geq 4$
- For line graphs of bipartite multigraphs is NP-complete
- For line graphs of multigraphs is NP-complete
- There is a hereditary class $\mathcal{C}$ such that SP($\mathcal{C}$) is coNP-intermediate

The approach

Something open?

- Is there an ω-categorical perfect graph?

The approach

Something open?

- Is there an ω-categorical perfect graph?
- The *Gyárfás–Sumner Sandwich Problem Conjecture*: The $\mathcal{T} \cup \{K_k\}$ is NP-hard for every set of non-star trees $\mathcal{T}$ and $k \geq 4$.

The approach

Something open?

- Is there an ω-categorical perfect graph?
- The *Gyárfás–Sumner Sandwich Problem Conjecture*: The $\mathcal{T} \cup \{K_k\}$ is NP-hard for every set of non-star trees $\mathcal{T}$ and $k \geq 4$.
  - *Why*: Gyárfás–Sumner + Brakensiek–Gurusuwami Conjectures imply it.

The approach

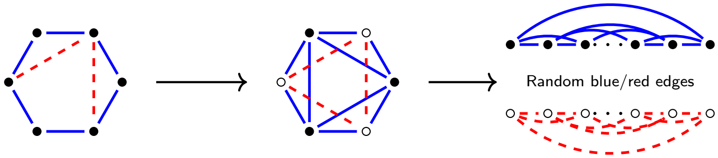
Something open?

- Is there an ω-categorical perfect graph?
- The *Gyárfás–Sumner Sandwich Problem Conjecture*: The $\mathcal{T} \cup \{K_k\}$ is NP-hard for every set of non-star trees $\mathcal{T}$ and $k \geq 4$.
  - *Why*: Gyárfás–Sumner + Brakensiek–Gurusuwami Conjectures imply it.
  - *Known cases*: True for $\{P_n, K_k\}$-free graphs with $n, k \geq 4$.

Something open?

- Is there an ω-categorical perfect graph?
- The *Gyárfás–Sumner Sandwich Problem Conjecture*: The $\mathcal{T} \cup \{K_k\}$ is NP-hard for every set of non-star trees $\mathcal{T}$ and $k \geq 4$.
  - *Why*: Gyárfás–Sumner + Brakensiek–Gurusuwami Conjectures imply it.
  - *Known cases*: True for $\{P_n, K_k\}$-free graphs with $n, k \geq 4$.
- Is there a hereditary class $\mathcal{C}$ such that SP($\mathcal{C}$) is NP-intermediate?
  - ... and such that SP($\mathcal{C}$) is a CSP?
  - ... and such that $\mathcal{C} = \mathcal{F}$-free graphs for finite $\mathcal{F}$?

Thank you for your attention!



Views and opinions expressed are however those of the authors only and do not necessarily reflect those of the European Union or the European Research Council Executive Agency. Neither the European Union nor the granting authority can be held responsible for them.