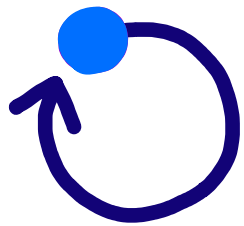
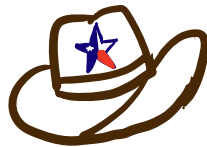


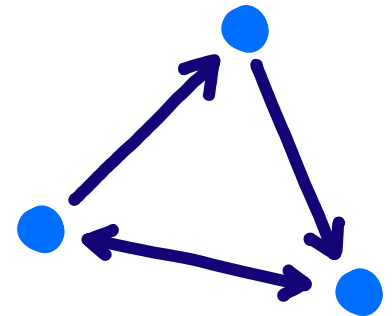
FROM FINITE TO INFINITE : IDENTITIES IN ω -CAT. STRUCTURES



Thomas Bruner



TU Wien



with M. K~~zik~~ik, T. N~~ayy~~ayy, M. Pinsker, M. Sch^ubi



ÖAW

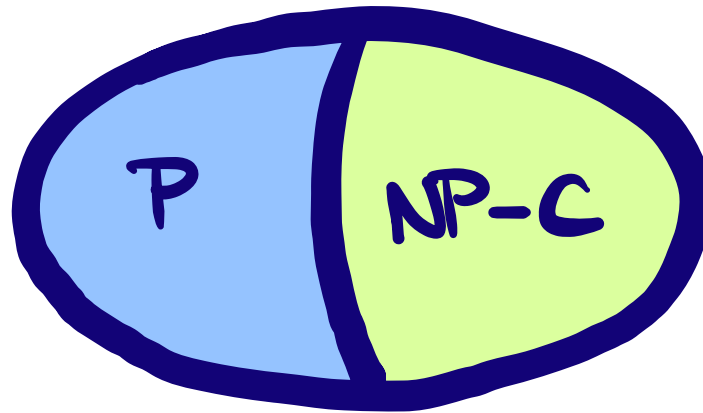
ÖSTERREICHISCHE
AKADEMIE DER
WISSENSCHAFTEN

FINITE CSP-DICHOTOMY

BULATOV,
ZHUK '17

A FINITE

$\text{CSP}(A)$

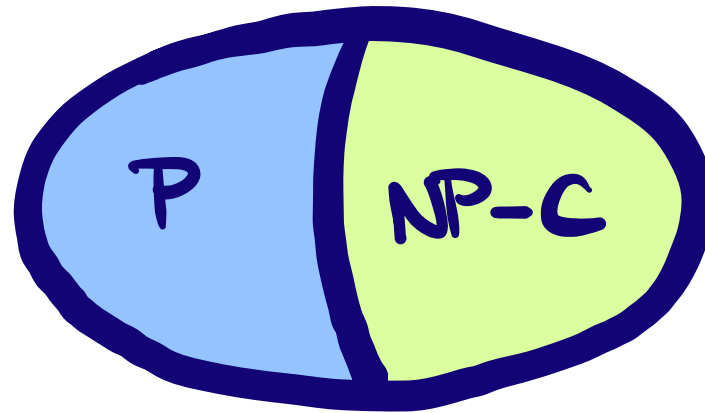


FINITE CSP-DICHOTOMY

BULATOV,
ZHUK '17

A FINITE

$\text{CSP}(A)$



A OMNI-
EXPRESSIVE



PP-CONSTRUCT
ALL FINITE
STRUCTURES



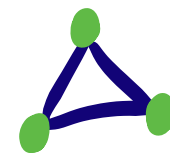
$\text{EQ}'(P|A)$
TRIVIAL

FINITE CSP-DICHOTOMY

BULATOV,
ZHUK '17

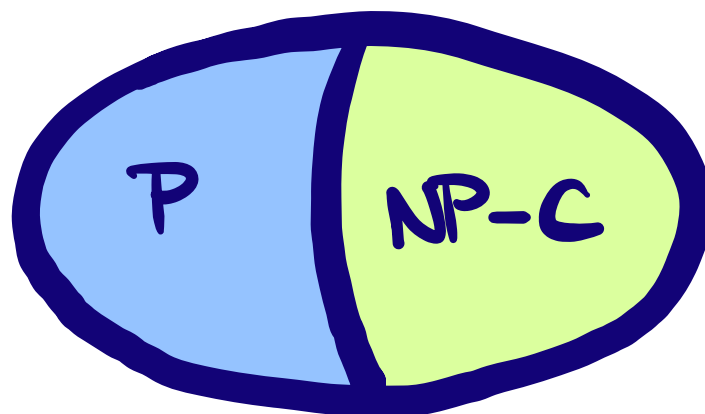
A FINITE

$CSP(A)$



A OMNI-
EXPRESSIVE

$EQ'(BI(A))$
NON-TRIVIAL



PP-CONSTRUCT
ALL FINITE
STRUCTURES



$EQ'(BI(A))$
TRIVIAL

- $S(x, y, x, z, y, z) \approx S(y, x, z, x, z, y)$

SIGGERS '10

- $S(a, r, e, a) \approx S(r, a, r, e)$

KEARNES + MARKOVIĆ + MCKENZIE '15

- $C(x_1 \dots x_p) \approx C(x_2 \dots x_p, x_1)$

BARTO + KOZIK '12

- ...

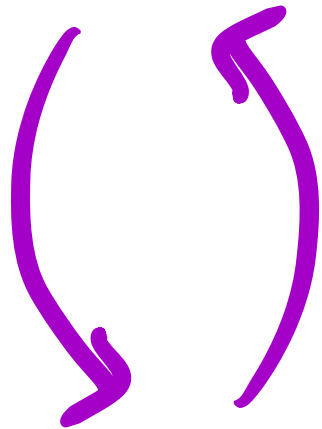
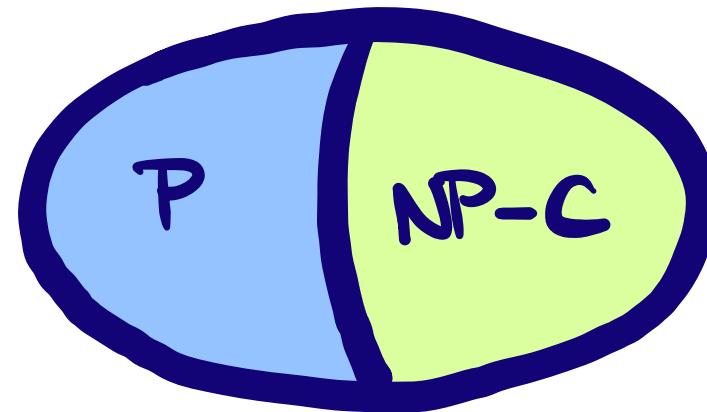
ALGEBRAIC APPROACH

JEAVONS '38
BULATOV+
JEAVONS+
KROKHIN '05

BARTO+OPRŠAL+
PINSKER '18

$\mathcal{A}$ FINITE
 $\text{CSP}(\mathcal{A})$

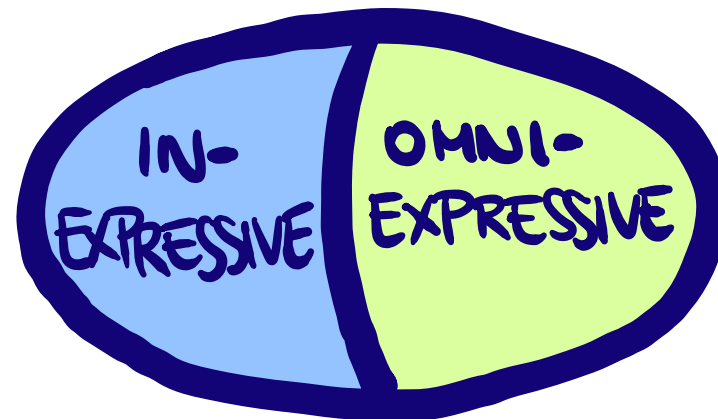
...



$\mathcal{A}$

SYMMETRIES
LOOPS

...



EXPRESS
STH HARD



LOOP LEMMATA

$\mathbb{Q}$ -COLOURING

Thm: (HELL+NEŠETŘIL '80)
(BULATOV '05)

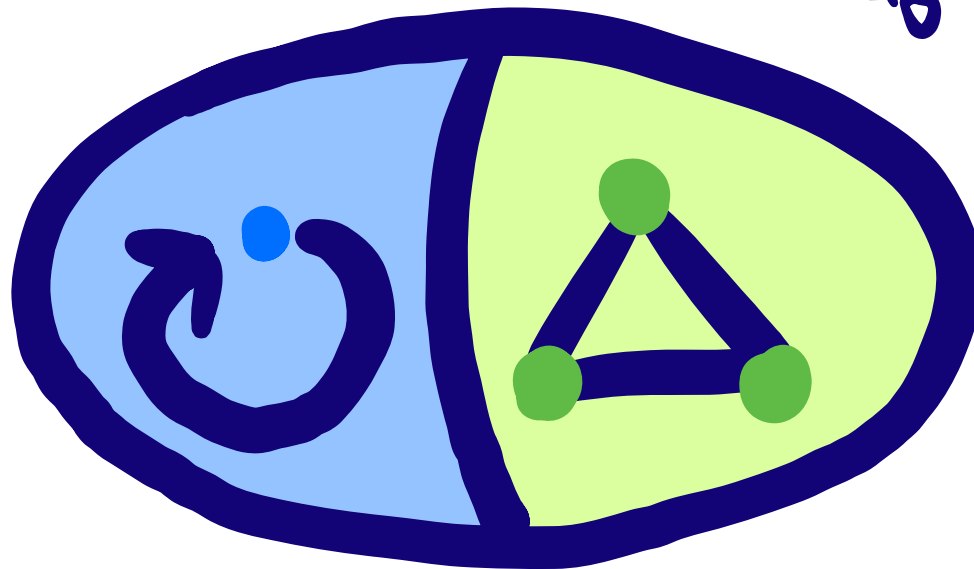
$\mathbb{Q}$ FINITE **UNDIRECTED**
NON-BIPARTITE.

Thm: (BARTO+KEŽIK+NIVEN '03)

$\mathbb{Q}$ FINITE **DIRECTED** SMOOTH
ALGEBRAIC LENGTH 1.

$a_0 \xrightarrow{+1} a_1 \xleftarrow{-1} a_2 \xrightarrow{+1} a_3$

$\mathbb{Q}$ HAS
LOOP

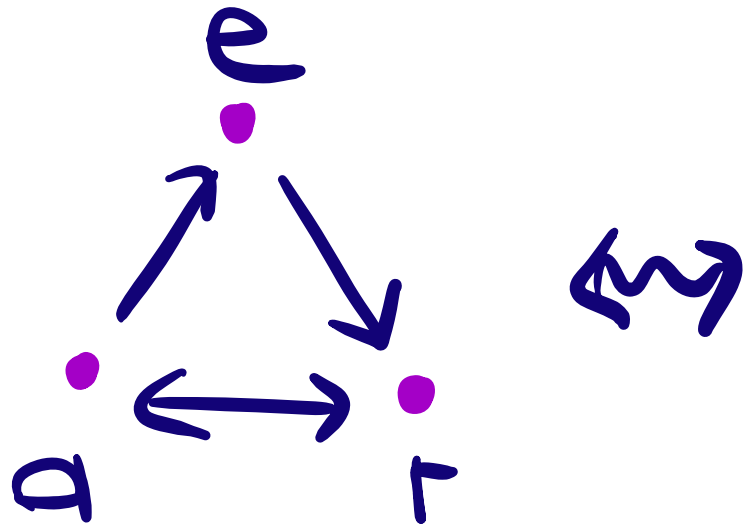


$\mathbb{Q}$ OMNI-
EXPRESSIVE

LOOPS $\Leftrightarrow$ IDENTITIES

SIGGERS '10

BARTO+KOZIK+
NIVEN '03



$s(a, r, e, a)$

$s(r, a, r, e)$

KEARNES+
MARKOVIC+
MCKENZIE '15

$$\mathbb{F} = \left\langle \begin{pmatrix} a \\ r \end{pmatrix} \begin{pmatrix} r \\ a \end{pmatrix} \begin{pmatrix} e \\ r \end{pmatrix} \begin{pmatrix} a \\ e \end{pmatrix} \right\rangle_{\text{POL}(\Delta)} \ni \begin{pmatrix} e \\ e \end{pmatrix}$$

IS OMNI-EXPRESSIVE OR HAS LOOP.

FROM FINITE
TO INFINITE

FROM FINITE TO INFINITE

(BODIRSKY+PINSKER '15)

$\mathcal{A}$ W-CAT. $\leadsto$ 'LOCAL' EQUATIONAL
STRUCTURE OF $\text{Pol}(\mathcal{A})$.

FROM FINITE TO INFINITE

(BODIRSKY+PINSKER '15)

$\mathcal{A}$ ω -CAT. $\leadsto$ 'LOCAL' EQUATIONAL
STRUCTURE OF $\text{POL}(\mathcal{A})$.

Thm (BARTO+PINSKER '21)

$\mathcal{A}$ ω -CAT, NOT OMNI-EXPRESSIVE. THEN
 $\text{POL}(\mathcal{A}) \models u \circ s(x, y, x, z, y, z) \approx v \circ s(y, x, z, x, z, y)$

FROM FINITE TO INFINITE

(BODIRSKY+PINSKER '15)

A ω -CAT. $\leadsto$ 'LOCAL' EQUATIONAL
STRUCTURE OF $\text{POL}(A)$.

Thm (BARTO+PINSKER '21)

A ω -CAT, NOT OMNI-EXPRESSIVE. THEN
 $\text{POL}(A) \models u \circ s(x, y, x, z, y, z) \approx v \circ s(y, x, z, x, z, y)$

WHAT ELSE?

(PSEUDO-) IDENTITIES

Λ NOT OMNI-EXPRESSIVE

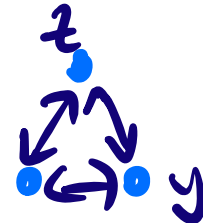
Λ FINITE	Λ ω-CAT.
$s(x, y, x, z, y, z) \approx s(y, x, z, x, z, y)$	$u \circ s(\dots) \approx v \circ s(\dots)$
$s(\triangleright, r, e, \triangleright) \approx s(r, \triangleright, r, e)$	?
$c(x_1 \dots x_p) \approx c(x_2 \dots x_p x_1)$	×
$w(x \dots xy) \approx w(x \dots yx) \dots \approx w(yx \dots x)$	×

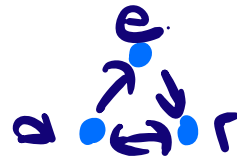
(PSEUDO-) IDENTITIES

Λ NOT OMNI-EXPRESSIVE

Λ FINITE

Λ ω-CAT.

$$s(x, y, x, z, y, z) \approx s(y, x, z, x, z, y)$$


$$s(\triangleright, r, e, \triangleright) \approx s(r, \triangleright, r, e)$$


$$c(x_1 \dots x_p) \approx c(x_2 \dots x_p x_1)$$

FOR SOME p !

$$w(x \dots xy) \approx w(x \dots yx) \dots \approx w(yx \dots x)$$

OF SOME ARITY!

$$u \circ s(\dots) \approx v \circ s(\dots)$$

?

X

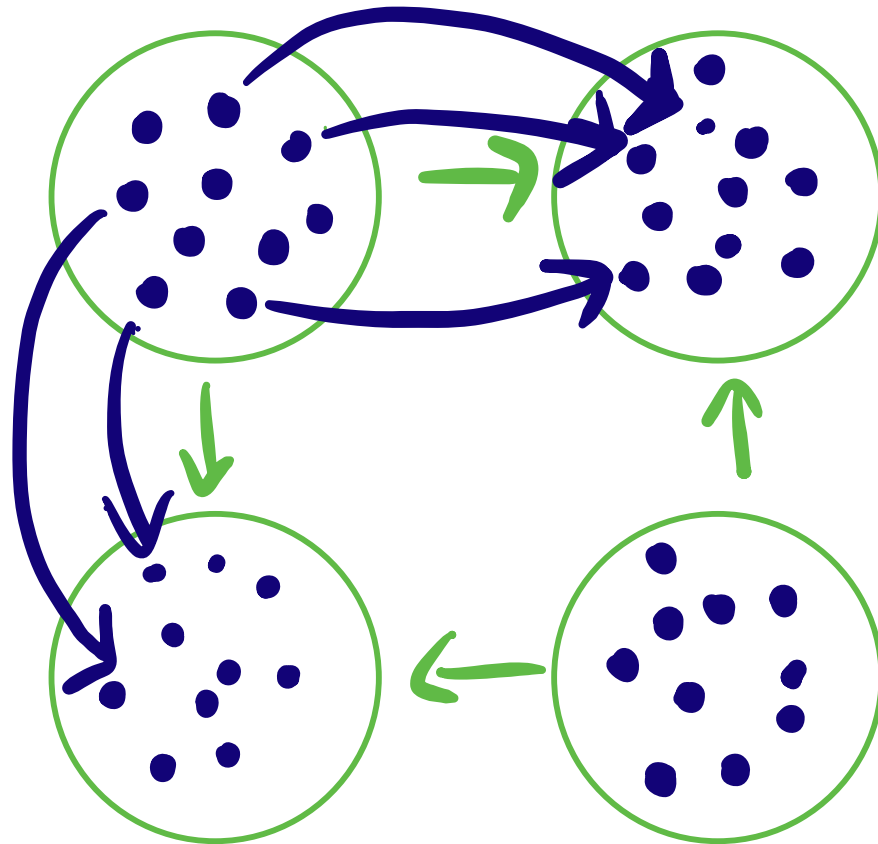
X

INFINITE LOOP LEMMA

$\mathbb{E}$ ω -CATEGORICAL DIGRAPH

FINITE FACTOR:

$\mathbb{E}/\text{Aut}(\mathbb{E})$



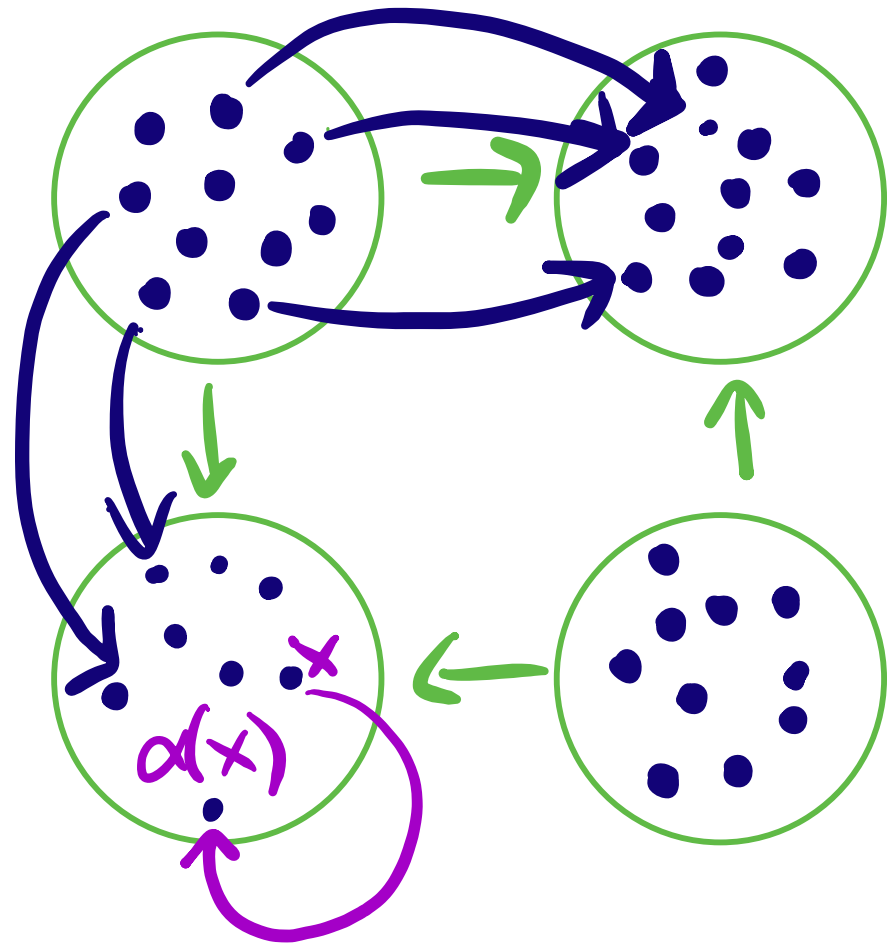
INFINITE LOOP LEMMA

$\mathbb{E}$ ω -CATEGORICAL DIGRAPH

FINITE FACTOR:

$\mathbb{E}/\text{Aut}(\mathbb{E})$

PSEUDO-LOOP



INFINITE HELL-NESETRIL

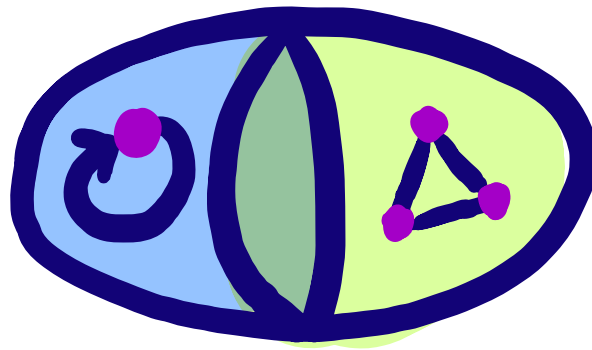
Thm (BARTO + BODOR + KOZIK + MOTTET + PINSKER '23)

$\mathbb{Q}$ SMOOTH + ω -CATEGORICAL

$\mathbb{Q}/\text{Aut}(\mathbb{Q})$ SYMMETRIC + NON-BIPARTITE.

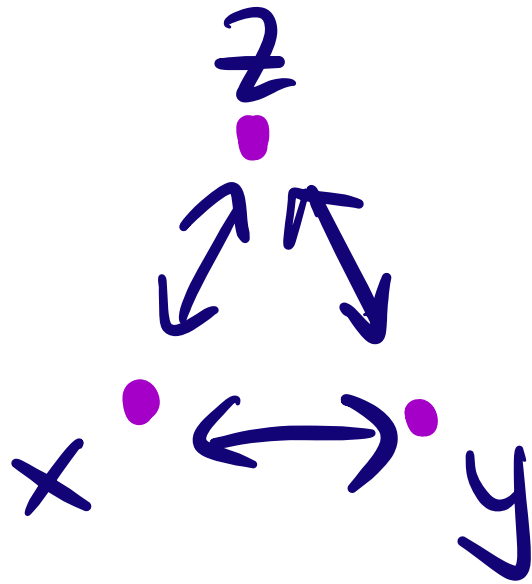
$\mathbb{Q}/\text{Aut}(\mathbb{Q})$
HAS LOOP

"PSEUDO-
LOOP"



$\mathbb{Q}$ OMNI-
EXPRESSIVE

PSEUDO-LOOPS $\Leftrightarrow$ PSEUDO-IDENTITIES



$\Leftrightarrow$

$$\cup \circ S(x, y, x, z, y, z)$$

$$\vee \circ S(y, x, z, x, z, y)$$

$$\mathbb{A} = \langle \binom{x}{y} \binom{y}{x} \binom{x}{z} \binom{z}{x} \binom{y}{z} \binom{z}{y} \rangle_{P_{\geq 1}(\mathbb{A})}$$

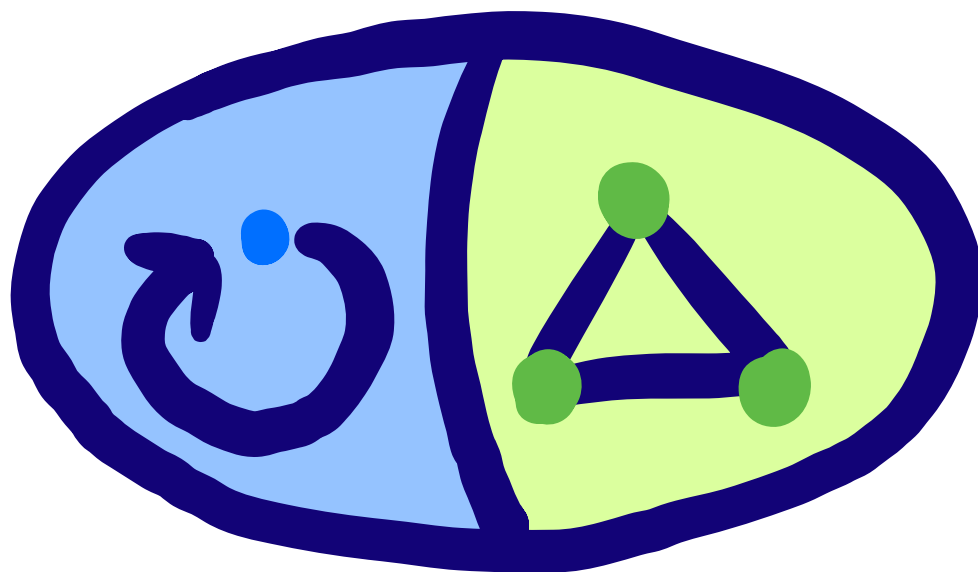
OMNI-EXPRESSIVE OR HAS PSEUDO-LOOP.

OUR DREAM

Thm (BARTO + KOZIK + NIVEN '03)

① FINITE DIGRAPH, SMOOTH,
ALGEBRAIC LENGTH 1.

② HAS
LOOP



③ OMNI-
EXPRESSIVE

CAN WE LIFT THIS TO THE INFINITE?

REDUCTION TO FINITE

BODIRSKY + MOTTET '16

CANONICAL POLYMORPHISMS

↖ ACT ON ORBITS

→ REDUCE TO $\neq$ FINITE $\neq$ FACTOR

→ TRACTABILITY CONDITION

TROUBLEMAKER #1

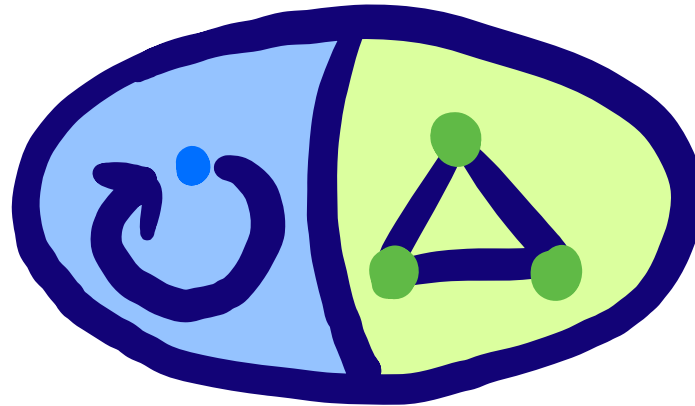
CLASSIFICATION:
BODIRSKY + KÁRA'10

Thm (B. + PINSKER + SCHÖBI '25)

⊆ SMOOTH DIGRAPH OF ALG. LENGTH 1
PP-INTERPRET. **INTEMPORAL CONSTR. LANG.**

TO REDUCT
OF (Q_i)

PSEUDO-
LOOP



OMNI-
EXPRESSIVE

TROUBLEMAKER #2

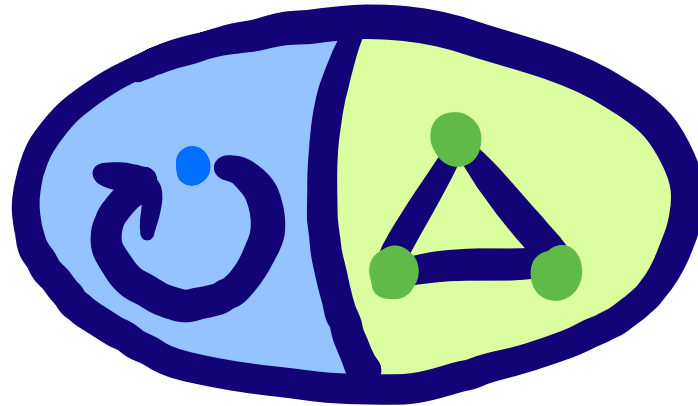
CLASSIFICATION:
BODIRSKY + JONSSON +
PHAM '16

Thm (B. + PINSKER + SCHÖBI '25)

⊆ SMOOTH DIGRAPH OF ALG. LENGTH 1
pp-INTERPRET. IN PHYLOGENY CONSTR. LANG.

↑
FRAÏSSÉ-LIMIT
OF LEAF STRUCT.

PSEUDO-
LOOP



OMNI-
EXPRESSIVE

ON THE ROAD IN THE FINITE...

'93

'03

'17

FEDER+
VARDI

CONJECTURE

BULATOV
CONSERVATIVE



ALL SUBSETS
ARE UNARY RELS

BULATOV,
ZHUK
PROOF

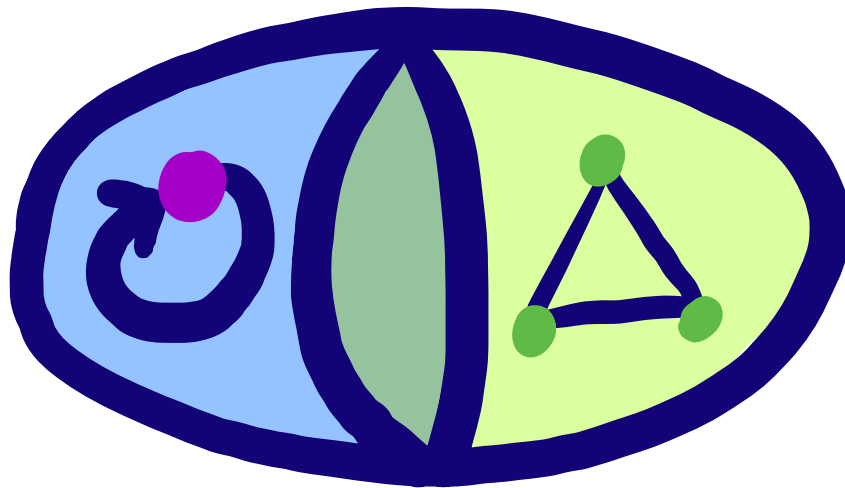
1-ORBITS TAKE ROLE OF ELEMENTS!

CONSERVATIVE DIGRAPHS

Thm (B. + KOZIK + NAGY + PINSKER '25)

$\mathbb{Q}$ ω -CATEGORICAL SMOOTH DIGRAPH
OF ALGEBRAIC LENGTH 1

PSEUDO-
LOOP



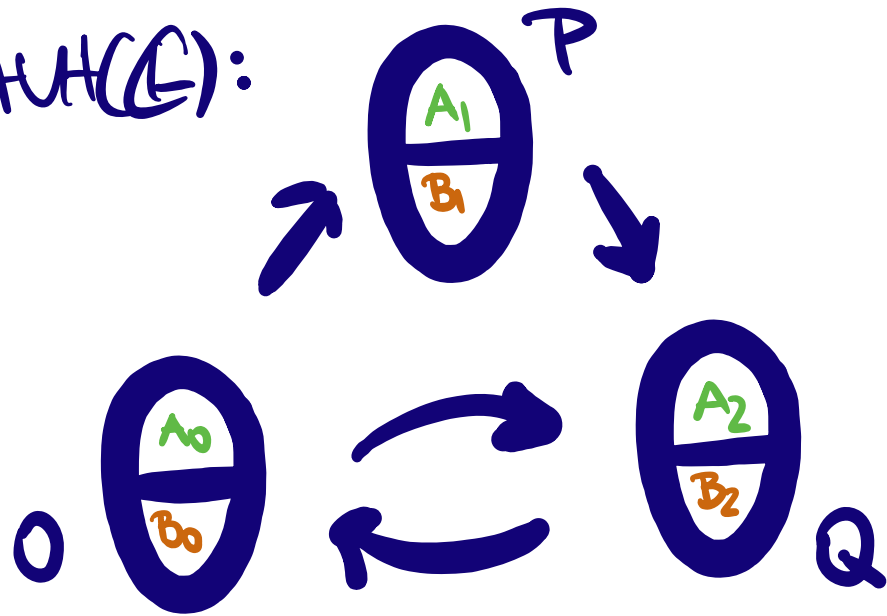
UNIONS OF
PAIRS OF
ORBITS

2-CONSERVATIVE
EXPANSION IS
OMNI-EXPRESSIVE

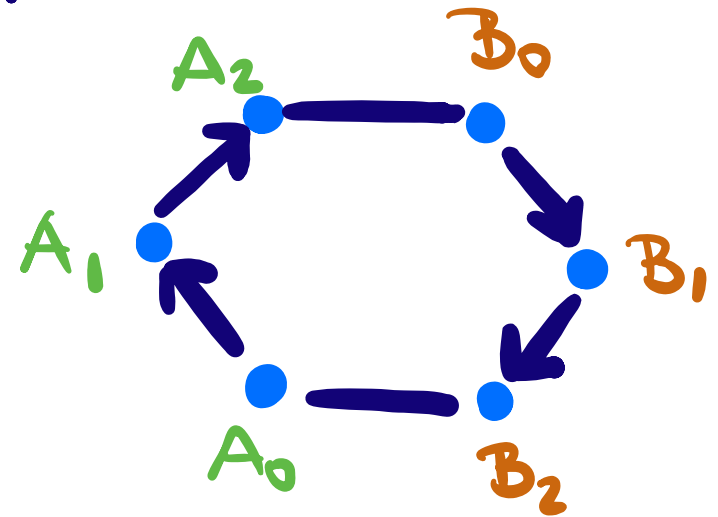
FINITISING $\mathbb{G} \bmod \text{AUT}(\mathbb{G})$

REFINE FACTOR:

$\mathbb{G}/\text{AUT}(\mathbb{G})$:



$\mathbb{G}/\alpha$:



P .. path pattern $(\rightarrow \rightarrow \leftarrow)$

$\alpha_P(x, y) \text{ iff } \exists z: x \overset{P}{\rightsquigarrow} z \overset{P}{\leftarrow} y$

SHOW THAT α
IS PP-DEF. ON
UNIONS OF PAIRS
OF ORBITS
& LIFT

NEW IDENTITIES!

Thm

$\nV A$ NOT OMNI-EXPRESSIVE. IF

- $\nV A$ TEMPORAL

(B. + PINSKER + SCHÖBI '25)

- $\nV A$ PHYLOGENY

(B. + PINSKER + SCHÖBI '25)

- $\nV A$ W-CAT. + CONSERVATIVE

(B. + KOZIK +
NAGY + PINSKER '25)

THEN $POL(\nV A) = \cup \circ S(\nV, \tau, e, \nV) \approx \vee \circ S(\tau, \nV, \tau, e)$



NEW IDENTITIES!

Thm

$\nV A$ NOT OMNI-EXPRESSIVE. IF

- $\nV A$ TEMPORAL

(B.+ PINSKER + SCHÖB) '25)

- $\nV A$ PHYLOGENY

(B.+ PINSKER + SCHÖB) '26)

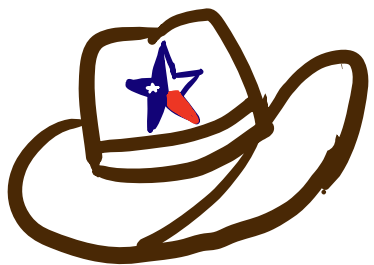
- $\nV A$ W-CAT. + CONSERVATIVE

(B.+ KOZIK +
NAGY + PINSKER '25)

THEN $POL(\nV A) = \cup \circ S(\nV, \tau, e, \nV) \approx \vee \circ S(\tau, \nV, \tau, e)$



WHAT ELSE?



Funded by the EU (ERC, POCOCOP). Views and opinions expressed are those of the author and do not necessarily reflect those of the EU or the ERC grant agency. Neither the EU nor the granting authority can be held responsible for them.