Taking Model-Complete Cores

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Finitely Homogeneous Structures

Definition. A relational structure $\mathfrak{B}$ is called

- homogeneous if every isomorphism between finite substructures of $\mathfrak{B}$ can be extended to an automorphism of $\mathfrak{B}$.
- finitely homogeneous if it is homogeneous in a finite signature.

Examples.

- $(\mathbb{Q}; <)$
- The Rado graph (aka the Random graph)
- The binary branching countable universal homogeneous C-relation

Facts.

- Finitely homogeneous structures are ω-categorical.
- Henson'1972: there are 2^ω many homogeneous digraphs.
- Cherlin'1998: classification of homogeneous digraphs.

Long-term goal. Classify countable homogeneous structures in a finite relational signature that are NIP (i.e., have a dependent first-order theory).

Motivation

Many open questions for finitely homogeneous structures $\mathfrak{B}$:

1. **Thomas' conjecture (1991):** $\mathfrak{B}$ has only finitely many first-order reducts, up to interdefinability.
   Equivalently: $\text{Sym}(B)$ has only finitely many closed subgroups that contain $\text{Aut}(\mathfrak{B})$.
2. **Macpherson's question 1 (2011):** Does $\mathfrak{B}$ have the small index property: every subgroup of $\text{Aut}(\mathfrak{B})$ of index $< 2^\omega$ is open?
3. **Macpherson's question 2 (2011):** Does $\text{Aut}(\mathfrak{B})$ have only finitely many closed normal subgroups?
4. **Ramsey expansion conjecture**: $\mathfrak{B}$ has an expansion by finitely many relations which is homogeneous and additionally Ramsey (B.+Pinsker+Tsankov'2011).
5. **CSP dichotomy conjecture:** If $\text{CSP}(\mathfrak{B}) := \{\mathfrak{A} \text{ finite} \mid \mathfrak{A} \to \mathfrak{B}\}$ is in NP, then $\text{CSP}(\mathfrak{B})$ is in P or NP-complete.
   Strengthening of conjecture of B.+Pinsker'2011.
   Conjectured for finite $\mathfrak{B}$ by Feder+Vardi'1993.
   Verified for finite $\mathfrak{B}$ by Bulatov and by Zhuk in 2017.

How to classify 1: first-order reducts

Definition. $\mathfrak{A}$ is called first-order reduct of $\mathfrak{B}$ if $\mathfrak{A}$ is a reduct of the expansion of $\mathfrak{B}$ by all first-order definable relations.

Example. $(\mathbb{Q}; \{(x, y, z) \mid x < y < z \vee z < y < x\})$ is first-order reduct of $(\mathbb{Q}; <)$.

Definition. Two structures are first-order interdefinable if they are first-order reducts of each other.

Example. $(\mathbb{Q}; \{(x, y, z) \mid x < y \vee x < z\})$ is interdefinable with $(\mathbb{Q}; <)$.

Fact. If $\mathfrak{A}, \mathfrak{B}$ are ω-categorical, then:
$\mathfrak{A}$ and $\mathfrak{B}$ are interdefinable if and only if $\text{Aut}(\mathfrak{A}) = \text{Aut}(\mathfrak{B})$
(follows from theorem of Ryll-Nardzewski, Engeler, Svenonius).

Warning. Finite homogeneity not preserved by first-order reducts!

How to classify 2: first-order interpretations

Definition. A (d-dimensional, first-order) interpretation of $\mathfrak{A}$ in $\mathfrak{B}$ is a partial surjective map $I \colon A^d \twoheadrightarrow B$ such that the pre-images of definable sets in $\mathfrak{B}$ are definable in $\mathfrak{A}$.

Example. The following structure is first-order definable in $(\mathbb{Q}; <)$:

$$(\mathbb{Q}^2; <_1, <_2) \text{ with } <_i := \{((a_1, a_2), (b_1, b_2)) \mid a_i < b_i\}$$

Definition. $\mathfrak{A}$ and $\mathfrak{B}$ are bi-interpretable if there are interpretations $I \colon \mathfrak{A}^{d_1} \twoheadrightarrow \mathfrak{B}$ and $J \colon \mathfrak{B}^{d_2} \twoheadrightarrow \mathfrak{A}$ such that $I \circ J$ is definable in $\mathfrak{B}$ and $J \circ I$ is definable in $\mathfrak{A}$.

Example. The structure $(\mathbb{Q}; =)$ is bi-interpretable with

$$(\mathbb{Q}^2; \{((a_1, a_2), (b_1, b_2)) \mid a_2 = b_1\})$$

$$\text{and with } \left(\binom{\mathbb{Q}}{2}; \{(A, B) : |A \cap B| = 1\}\right)$$

Fact (Ahlbrand-Ziegler'86, Coquand): ω-categorical structures $\mathfrak{A}$ and $\mathfrak{B}$ are bi-interpretable if and only if $\mathrm{Aut}(\mathfrak{A})$ and $\mathrm{Aut}(\mathfrak{B})$ are topologically isomorphic.

The Stable Case

Stability: Shelah 1969.

Definition. An ω-categorical structure $\mathfrak{M}$ is stable if and only if there is no formula $\phi(\bar{x}, \bar{y})$ and parameters $a_0, a_1, \dots \in M^{|x|}$ and $b_0, b_1, \dots \in M^{|y|}$ such that $\mathfrak{M} \models \phi(a_i, b_j)$ if and only if $i < j$.

Examples. $(\mathbb{N}; =)$.
The countable vector space over a finite field (not finitely homogeneous)

Non-examples.

- $(\mathbb{Q}; <)$
- the Rado graph

Fact. Stability (and ω-categoricity) preserved by first-order interpretations.

Examples. $\big(\mathbb{Q}^2; \{((a_1, a_2), (b_1, b_2)) \mid a_2 = b_1\}\big)$
and $\big(\binom{\mathbb{Q}}{2}; \{(A, B) : |A \cap B| = 1\}\big)$ are stable and ω-categorical.

Lachlan's Theorem

Theorem (Lachlan'1986) The following are equivalent.

- $\mathfrak{B}$ is stable and a first-order reduct of a finitely homogeneous structure;
- $\mathfrak{B}$ is stable and interpretable over $(\mathbb{Q}; <)$.

'Lachlan's class'.

Picture:

First-order reducts of finitely homogeneous structures

| B+Bodor+Marimon'25

Structures interpretable over (Q,<)

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Stable first-order reducts of finitely homogeneous structures
= stable structures interpretable over (Q,<)

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Structures interpretable over (Q,=)

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Finite structures

Contributions, Part 1

B.+Bodor+Marimon'25:

1. Conjecture: Lachlan's class equals closure of class of structures interpretable over $(\mathbb{Q}; =)$ under taking model-complete cores.
2. Taking model-complete cores preserves most model-theoretic properties (e.g., stability).
3. Show that class of structures interpretable over $(\mathbb{Q}; =)$ is not closed under taking model-complete cores.
4. Consequence (was known before, but not explicitly in the literature): Lachlan's class is distinct from class of structures interpretable over $(\mathbb{Q}; =)$.

How to classify 3: model-complete cores

An ω-categorical structure $\mathfrak{B}$ is called a

- core if every endomorphism $\mathfrak{B} \rightarrow_{\text{hom}} \mathfrak{B}$ is an embedding,
- model-complete if every embedding $\mathfrak{B} \hookrightarrow \mathfrak{B}$ preserves all first-order formulas.

Fact (B.+Junker'09). $\mathfrak{B}$ is a model-complete core
if and only if $\overline{\text{Aut}(\mathfrak{B})} = \text{End}(\mathfrak{B})$.

$\mathfrak{A}$ and $\mathfrak{B}$ are homomorphically equivalent if $\mathfrak{A} \rightarrow_{\text{hom}} \mathfrak{B}$ and $\mathfrak{B} \rightarrow_{\text{hom}} \mathfrak{A}$.

Theorem (B.'2005, B.+Hils+Martin'2010). Every ω-categorical $\mathfrak{B}$ is homomorphically equivalent to a model-complete core $\mathfrak{C}$, which is unique up to isomorphism, and ω-categorical.

Theorem (B.+Bodor+Marimon'26). If $\mathfrak{B}$ is stable, then so is $\mathfrak{B}$.
Similarly, taking model-complete cores preserves NIP, NSOP, simplicity, superstability, monadic stability, monadic NIP, strong minimality, etc.

Examples

- The model-complete core of $(\mathbb{Q}; \leq)$ is a single point.
- The model-complete core of the Random Graph is K_ω.
- The model-complete core of $(\mathbb{Q} \cap [0,1]; <)$ is $(\mathbb{Q}; <)$.
- The model-complete core of $(\mathbb{Q}^2; <_1, <_2)$ is the countable universal homogeneous permutation structure (aka generic permutation structure).
- Similarly: Cherlin's tournament $S(2)$ is a model-complete core of a structure interpretable over $(\mathbb{Q}; <)$.

The Challenge

Goal: find a structure with is interpretable over $(\mathbb{Q};=)$, but whose model-complete core is not interpretable over $(\mathbb{Q};=)$.

Typical approach: To prove that a structure $\mathfrak{A}$ is not interpretable over $\mathfrak{B}$:

- Identify a property P that is preserved by first-order interpretations,
- Show that $\mathfrak{B}$ has property P,
- Show that $\mathfrak{A}$ does not have property P.

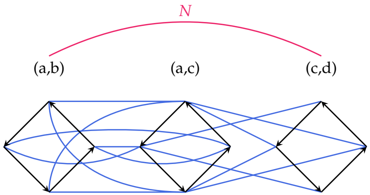
Challenge: almost all model-theoretic properties are preserved by taking model-complete cores!

The Counterexample

$X := \{(a,b,m) : a,b \in \mathbb{Q}, m \in \mathbb{Z}_4, a \neq b\}$.

$\mathfrak{X}$: structure with domain X and relations

- $R := \big\{((a,b,m),(a,b,m+1)) \mid a,b \in \mathbb{Q}, m \in \mathbb{Z}_4\big\}$,
- $E \subseteq X^2$, the binary relation given by



$$\begin{aligned} &\{((a,b,2m),(a,c,2n)) : n,m \in \mathbb{Z}_4, b \neq c\} \\ \cup\, &\{((a,b,2m),(c,a,2n+1)) : n,m \in \mathbb{Z}_4, b \neq c\} \\ \cup\, &\{((b,a,2m+1),(a,c,2n)) : n,m \in \mathbb{Z}_4, b \neq c\} \\ \cup\, &\{((b,a,2m+1),(c,a,2n+1)) : n,m \in \mathbb{Z}_4, b \neq c\}. \end{aligned}$$

- the binary relation
  $N := \{((a,b,m),(c,d,n)) : n,m \in \mathbb{Z}_4, \{a,b\} \cap \{c,d\} = \emptyset\}$.

Verifying the counterexample

1. $\mathfrak{X}$ is interpretable over $(\mathbb{Q}; =)$.
2. the substructure $\mathfrak{Y}$ induced by $Y := \{(a, b, m) : a, b \in \mathbb{Q}, m \in \mathbb{Z}_4, a < b\}$ is homomorphically equivalent to $\mathfrak{X}$.
3. $\mathfrak{Y}$ is interpretable over $(\mathbb{Q}; <)$.
4. the structure $\mathfrak{Y}$ is a model-complete core.
5. $\mathfrak{Y}$ is not interpretable over $(\mathbb{Q}; =)$:
   - every structure $\mathfrak{A}$ with a first-order interpretation in $(\mathbb{Q}; =)$ is finite or $\mathrm{Aut}(\mathfrak{A})$ has a subgroup isomorphic to $\mathrm{Sym}(\mathbb{Q})$.
   - The subgroup of involutions of $\mathrm{Aut}(\mathfrak{Y})$ is abelian.

Beyond Stability

$(\mathbb{Q}; <)$ is not stable, but NIP.

Definition. An ω-categorical structure $\mathfrak{B}$ has the Independence Property (IP) if there exists a formula $\phi(\bar{x}, \bar{y})$ and an infinite $S \subseteq B^{|x|}$ and $b_S \in B^{|y|}$ for every finite $I \subseteq S$ such that

$$\mathfrak{B} \models \phi(a, b_I) \text{ if and only if } a \in I.$$

Fact (Shelah): every ω-categorical unstable structure $\mathfrak{B}$ has IP or the SOP: there exists a formula $\phi(\bar{x}, \bar{y})$ and $a_0, a_1, \dots \in B^{|y|}$ such that

$$\phi(\bar{x}, b_0)^{\mathfrak{B}} \subsetneq \phi(\bar{x}, b_1)^{\mathfrak{B}} \subsetneq \cdots .$$

Examples:

- $(\mathbb{Q}; <)$, $S(2)$, and the generic permutation have NIP and SOP.
- The Rado graph and the Henson digraphs have IP and NSOP.

Structures Interpretable over $(\mathbb{Q}; <)$

Definition. binary finitely homogeneous structure $\mathfrak{B}$:
all relations of $\mathfrak{B}$ have arity at most 2.

Example. The countable universal homogeneous binary branching C-relation

- is finitely homogeneous and NIP,
- but is not first-order reduct of binary finitely homogeneous structure.

Fact (B+Bodor+Marimon'25). Every structure with a first-order interpretation in $(\mathbb{Q}; <)$ is

- the reduct of a binary finitely homogeneous (Ramsey) structure,
- NIP.

Note. All the mentioned conjectures are already open for NIP structures that are reducts of binary finitely homogeneous structures.

Binary finitely homogeneous NIP structures

Theorem (B+Bodor+Marimon'25). The class $\mathcal{I}$ of structures with a first-order interpretation in $(\mathbb{Q};<)$ is not closed under taking model-complete cores:

- the countable universal homogeneous permutation is not in $\mathcal{I}$
- $S(2)$ is not in $\mathcal{I}$.

Conjecture (B+Bodor+Marimon'25). Every NIP structure which is a first-order reduct of a binary finitely homogeneous structure is interdefinable with the model-complete core of a structure interpretable over $(\mathbb{Q};<)$.

Fuller Picture

First-order reducts of finitely homogeneous structures

| ≠

NIP first-order reducts of binary finitely homogeneous structures

| =? (B+Bodor+Marimon'25)

Interdefinable with model-complete cores of structures interpretable over (Q,<)

| ≠ (B+Bodor+Marimon'25)

Structures interpretable over (Q,<)

| ≠

Stable first-order reducts of finitely homogeneous structures
= stable structures interpretable over (Q,<)

| =? (B+Bodor+Marimon'25)

Interdefinable with model-complete cores of structures interpretable over (Q,=)

| ≠ (B+Bodor+Marimon'25)

Structures interpretable over (Q,=)

| ≠

Finite structures

Concluding Remarks

- Two types of results in B+Bodor+Marimon'25:
  - Preservation of model-theoretic properties under taking model-complete cores.
  - Techniques to show that certain structures do *not* have a first-order interpretation in another structure.
  - We believe that model-complete cores are a useful concept in classifications of finitely homogeneous structures.
- Most definitions and results about model-complete cores do not rely on ω-categoricity.
- Related work:
  - positive model theory (see, e.g., Ben-Yaacov'2003, Dmitrieva+Gallinaro+Kamsma'23)
  - trace definability by Walsberg'2025.

    Fact. The model-complete core of an ω-categorial structure $\mathfrak{A}$ has a trace definition in $\mathfrak{A}$.