Toward the Algorithm

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CSP

Aim: P-time algorithm for CSPs, which is ... (to be specified)

CSP - basic version

INPUT
- V finite set of variables
- for each $x \in V$ its domain D_x
- finite list of constraints of the form

$$\underbrace{x_1 x_2 \dots x_n}_{\text{scope}} \in \underset{\text{constraint relation}}{R}, \quad \text{where} \quad \begin{array}{l} x_i \in V \\ R \subseteq D_{x_1} \times D_{x_2} \times \dots \times D_{x_n} \end{array}$$

OUTPUT $\exists$ solution ? $(d_x)_{x \in V}$, $d_x \in D_x$, satisfies all constraints

Fixed-template CSP : fix allowed domains & allowed constraint relations

AIM

2

P-time algorithm for CSPs, which is

one motivation

- uniform – really polynomial-time
- powerful – benchmark: fixed-finite-template CSPs, Promise CSPs, ...
- beautiful – simple, natural, straightforward, no tricks

Fixed-template algorithms of Bulatov, Zhuk

- uniform ✗
- powerful ✓ but

Uniform algorithms

AC, BLP, SDP, AIP

boosting:
- combinations BLP+AIP
- singleton versions singl (BLP+AIP)
- k-th level k-th level AC

best explained on LC

- uniform ✓
- powerful ?
- beauty not perfect

LABEL COVER

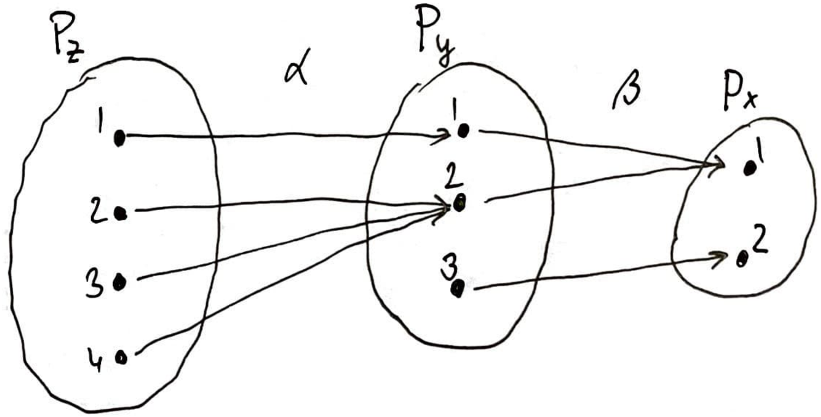
Label Cover (LC) instance: CSP instance such that every constraint is of the form $x = \alpha(y)$ where $\alpha : D_y \to D_x$

Example

- $V = \{x, y, z\}$
- $D_z = \{1,2,3,4\}$, $D_y = \{1,2,3\}$, $D_x = \{1,2\}$
- $y = \alpha(z)$, $x = \beta(y)$

$\alpha : 1 \mapsto 1$, $2,3,4 \mapsto 2$

$\beta : 1,2 \mapsto 1$, $3 \mapsto 2$



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point : pair xa $x \in V$, $a \in D_x$

P : all points $(= \coprod D_x)$

P_x : $\{xa ; a \in D_x\}$ .. a potato

Ad: cleanest def. see
M. Hadek, T. Jakl, J. Opršal:
A categorical perspective on constraint satisfaction: The wonderland of adjunctions

CSP → LC

given CSP instance, create LC instance

- keep variables and their domains
- for each constraint $x_1 x_2 \cdots x_n \in R$

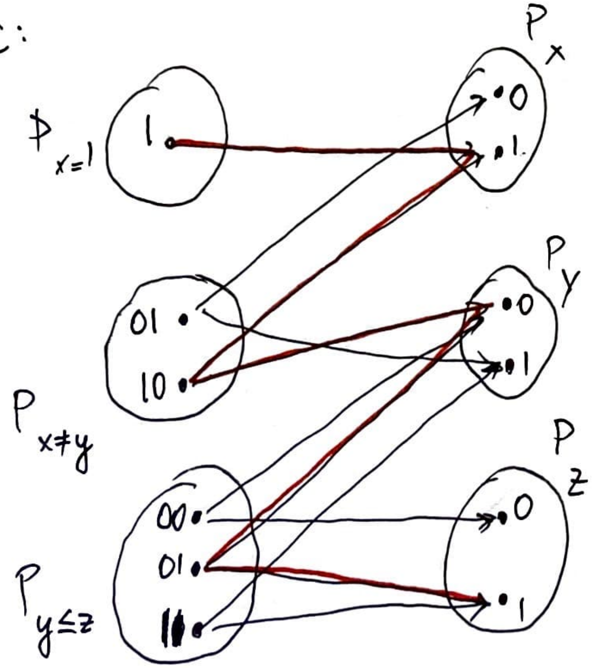
  add LC variable c domain R

  & constraints $x_1 = \pi_1(c)$, $x_2 = \pi_2(c)$, …

  projection maps

CSP: $x=1$, $x \neq y$, $y \leq z$ domains $\{0,1\}$

LC:



solution of the LC instance

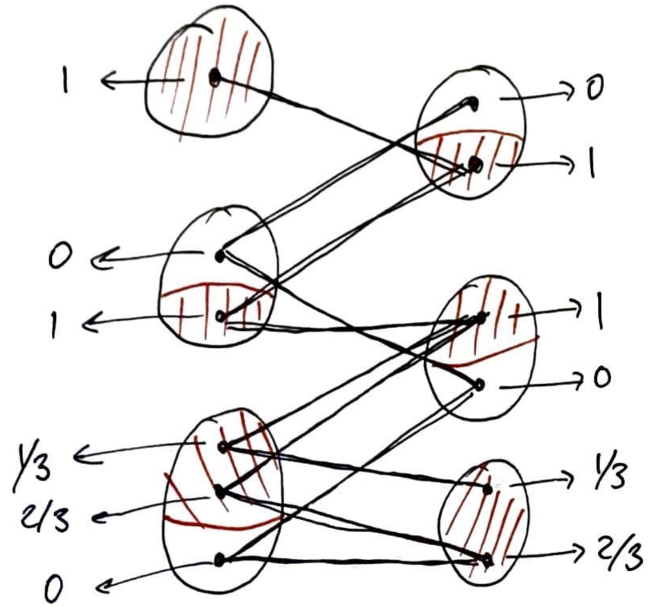
= solution of the CSP instance

+ witnesses that constraints are satisfied

~~We~~ will only look at LC instances

Note: LC instances coming from CSP inst.
have a special shape -
irrelevant for us

RELAXATIONS



strength: AC < BLP < SDP

$\mathbb{Z}_p < \mathbb{Z}$

Note: $\{0,1\}$ version = solving it

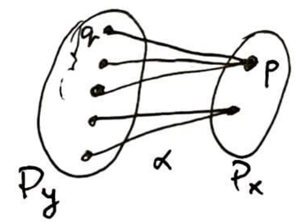
* J. Brakensiek, V. Guruswami, S. Sandeep
L. Ciardo, S. Živný

AC accepts if $\exists\, \emptyset \neq D'_x \subseteq D_x$ such that

- for each constraint $x = \alpha(y)$, $D'_x = \alpha(D'_y)$

BLP accepts if $\exists f : P \to [0,1]$ such that

- $\forall$ constraint $x = \alpha(y)$, $\forall p \in P_x$



$$f(p) = \sum_{q \in \alpha^{-1}(p)} f(q)$$

$\boxed{\text{Summing condition}}$

- $\forall x \in V \quad \sum_{p \in P_x} f(p) = 1$

AIP ditto $\quad f : P \to \mathbb{Z}$

$\mathbb{Z}_{pm}$ ↑ prime $\quad f : P \to \mathbb{Z}_{\cancel{p}m}$

SDP * $\quad f : P \to \mathbb{R}^N$ (for some N)

- summing condition
- $\forall x \quad \sum_{P_x} f(p) = \vec{i}$ unit vector (independent on x)
- $\forall x \; \forall p \neq q \in P_x \quad f(p) \cdot f(q) = 0$

SINGLETON VERSIONS & HIGHER LEVELS

singleton AIP

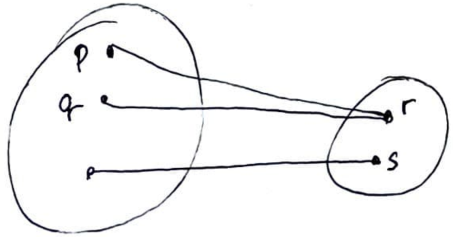
- $\emptyset \neq D_x' \subseteq D_x$ ~~the corresponding subset of the~~
- $\forall x \ \forall a \in D_x'$ AIP solution $f: P \longrightarrow \mathbb{Z}$ which is 0 outside ~~the~~ the D_x' and $f(xa) = 1$

2nd level AIP

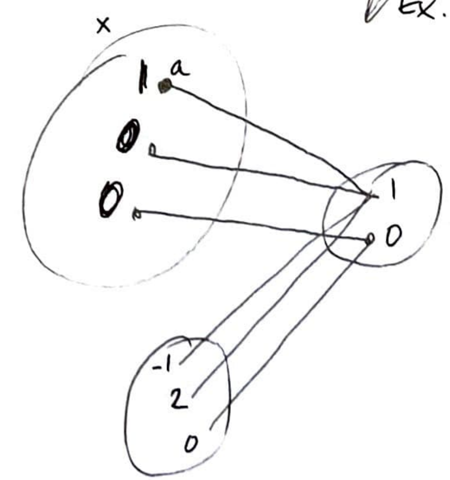
- $f: P \times P \longrightarrow \mathbb{Z}$ such that ...

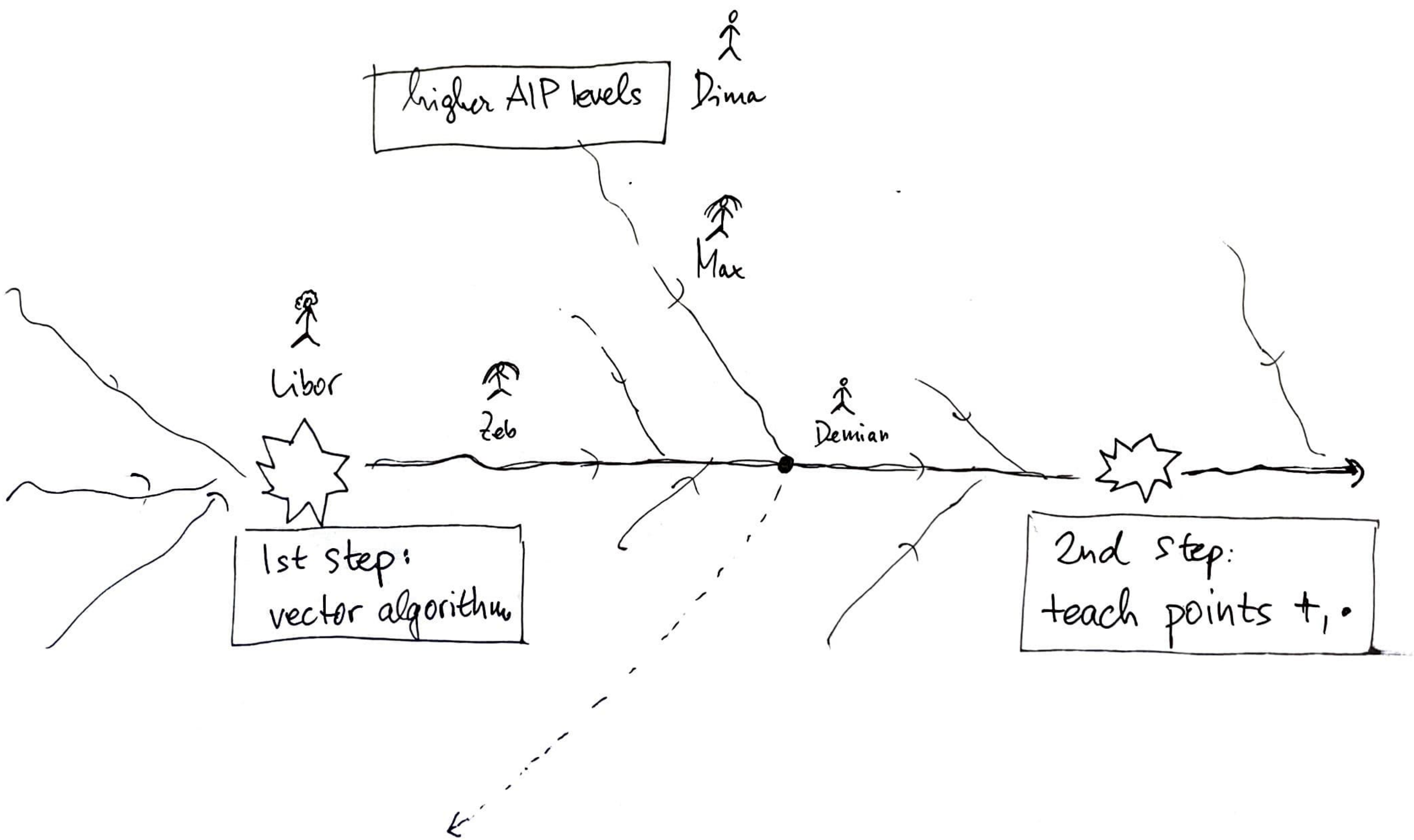
e.g. $f(p,q) = 0$

$f(p,s) + f(q,s) = p(r,s)$



Ex.





ad: L.B., M. Hadek, D. Zhuk: Toward a Uniform Algorithm and Uniform Reduction for Constraint Problems

on arXiv

VECTOR ALGORITHM

VP-$\mathbb{Z}_{pm}$ accepts if $\exists f: P \longrightarrow \mathbb{Z}_{pm}^N$

- summing condition
- $\forall x \quad \sum_{p \in P_x} f(p) = \vec{\imath}$ $\quad$ weight$(\vec{\imath}) = 1$
- $\forall x \quad \forall p \neq q \in P_x \quad f(p) \cdot f(q) = 0$

+ combines consistency info (SDP) and algebraic info ($\mathbb{Z}_{pm}$) in one object
 - compare to Bulatov & Zhuk alg., BLP+AIP, descriptive complexity, approx. of satisfiable inst.

\+ solves the smallest (8-element) fixed-template CSP not solvable by Singleton (BLP+AIP)

± equal strength to 2nd level $\mathbb{Z}_m$ (+ D. Zhuk)

VP-$\mathbb{Z}$

\+ stronger than SDP & AIP

\- in P?

\- not powerful enoug (Z. Brady)

\- still a bit ad hoc

VP-$\mathbb{Z}$

version 1

INPUT: LC instance

OUTPUT YES: $\exists$ VP-$\mathbb{Z}$ solution
NO: $\neg\exists$ VP-$\mathbb{Z}$ solution

version 2

INPUT: LC instance

OUTPUT YES: $\exists$ solution
NO: $\neg\exists$ VP-$\mathbb{Z}$ solution

Question: Is version 1/2 in P?

Note: For CSP purposes enough ~~that~~ if version 2 $\in$ P

2nd step

Teach points $+, \cdot$

(Recall we want $p \mapsto 0$ or 1 such that... anyway)

→ algorithm can **gain info** instead of loosing info (AIP)

→ $\exists$ **obvious** iterative improvement

→ there are levels, but better

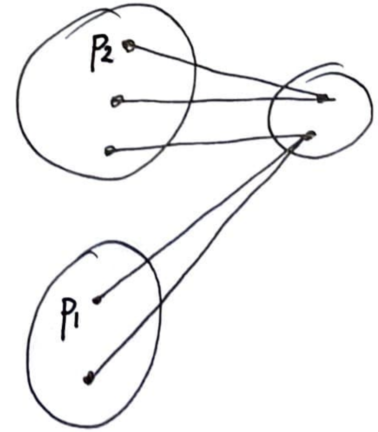
will discuss $\mathbb{Z}_5$

$\mathbb{Z}$ slightly more complicated

Teaching +

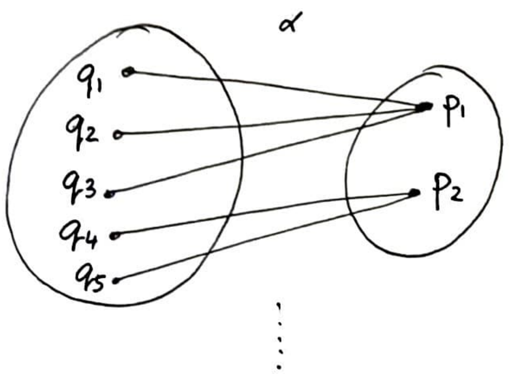
$P \subseteq \mathbf{R}_1 :=$ polynomials of degree ≤ 1 over $\mathbb{Z}$ or $\mathbb{Z}_m$

~~it is a vector~~ variables $p \in P$



- it is a vector space over $\mathbb{Z}_5$ with basis $1, p_1, p_2, \ldots$
- now $2 + 3p_1 + p_2$ makes sense
- points **are** vectors

WHAT IS A $\mathbb{Z}_5$ SOLUTION



mapping $P \xrightarrow{f} \mathbb{Z}_5$ such that ...

$\sim$ linear map $R_1 \xrightarrow{\varphi} \mathbb{Z}_5$ with $\varphi(1) = 1$ such that

$$\begin{aligned} \varphi(p_1) &= \varphi(q_1) + \varphi(q_2) + \varphi(q_3) \\ \varphi(p_2) &= \varphi(q_4) + \varphi(q_5) \\ &\cdots \\ \varphi(p_1) + \varphi(p_2) &= 1 \end{aligned} \quad \Longleftrightarrow \quad \begin{aligned} \varphi(p_1 - (q_1+q_2+q_3)) &= 0 \\ \varphi(p_2 - (q_4+q_5)) &= 0 \\ &\cdots \\ \varphi(p_1 + p_2 - 1) &= 0 \end{aligned}$$

$\sim$ denoting $N_1 := \mathrm{Span}\{ p_1 - (q_1+q_2+q_3),\ p_2 - (q_4+q_5), \dots, p_1 + p_2 - 1 \}$

linear map $R_1 \xrightarrow{\varphi} \mathbb{Z}_5$ such that $\varphi(1) = 1$ & $\varphi(N_1) = 0$

$\sim$ linear map $R_1/N_1 \xrightarrow{\bar{\varphi}} \mathbb{Z}_5$ such that $\varphi(\bar{1}) = 1$

$$P \subseteq R_1 \xrightarrow{/N_1} R_1/N_1 \longrightarrow \mathbb{Z}_5$$

1st level RP-$\mathbb{Z}_5$

$\mathbb{Z}_5$ solution if $\bar{1} \mapsto 1$

~~(∃ if $N_1 \neq R_1$)~~

(∃ if $1 \notin N_1$)

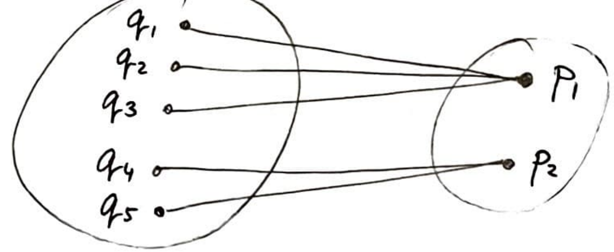
- we taught P how to +
- then gained info: each $v \in N_1$ must be 0 in any $\{0,1\}$ solution

TEACHING •

$R :=$ polynomials over $\mathbb{Z}_5$ in variables $p \in P \Big/ \, p^2 = p, \; p_1 p_2 = 0$ if $p_1 \neq p_2 \in P_x$ for some x

potato ring

$R_k \subseteq R$ ... degree $\leq k$



now $q_1 p_2 + 3q_4 + 1$ makes sense

N_1 generated by

$p_1 - q_1 - q_2 - q_3$

$p_2 - q_4 - q_5$

...

$p_1 + p_2 - 1$

$\{0,1\}$-solution

$=$ ring homomorphism $R \xrightarrow{\varphi} \mathbb{Z}_5$ such that $\varphi(N_1) = 0$

enough information for ↑ already in R_2

WHAT IS A 2nd LEVEL $\mathbb{Z}_5$ SOLUTION

mapping $\quad P \times P \longrightarrow \mathbb{Z}_5$ such that .....

~ ...

~ linear map $\quad R_2/N_2 \longrightarrow \mathbb{Z}_5$ with $T \mapsto 1$

where $\quad N_1 \leqslant R_1$ as before

$N_2 := $ Span of $N_1 \cdot R_1$

$$
\begin{array}{ccccc}
P & \subseteq & R_1 & \leqslant & R_2 \\
 & & \text{VI} & & \text{VI} \\
 & & N_1 & & N_2
\end{array}
$$

- k-th level $\mathbb{Z}_5$ similar
- no such level solves HORN-SAT (☺ Z. Brady)

OBVIOUS IMPROVEMENT

$$\begin{array}{ccc} P \subseteq R_1 & \leq & R_2 \\ \text{VI} & & \text{VI} \\ N_1 & & N_2 \end{array}$$

$N_2 \cap R_1$ can be larger than N_1

we should obviously use it

k-th level RP-$\mathbb{Z}_5$:

compute basis of $N_1 \leq R_1$, $N_2 \leq R_2$, ..., $N_k \leq R_k$ such that

smallest spaces

- N_1 contains e.g. $p_1 - q_1 - q_2 - q_3$, ... as before
- $i + j \leq k \Rightarrow N_i \cdot R_j \subseteq N_{i+j}$
- $i < j \Rightarrow N_j \cap R_i = N_i$

reject if $1 \in N_1$

2nd LEVEL RP-$\mathbb{Z}_5$

Claim: nonzero points in P_x (mod N_1) are linearly independent

$$p+q+r=0 \mod N_1$$
$$\Rightarrow p\cdot(p+q+r)=0 \mod N_2$$
$$\Rightarrow p^2+pq+pr=0 \mod N_2$$
$$\Rightarrow p=0 \mod N_2$$
$$\Rightarrow p=0 \mod N_1$$

$\Longrightarrow$ 2nd level RP-$\mathbb{Z}_5$ stronger than
singleton $\mathbb{Z}_5$ stronger than
AC

$\Rightarrow$ solves HORN-SAT
does not solve 2-SAT

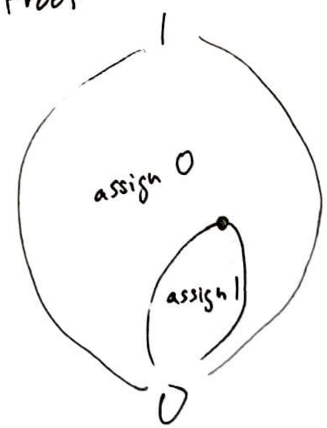
3rd LEVEL RP-$\mathbb{Z}_5$

knows associativity $(p \cdot q) \cdot r = p \cdot (q \cdot r)$

<u>def.</u> $p \leq q$ if $pq = p$

from associativity : it is a partial order!

"Proof"



+ D. Banakh
M. Hadek

$\Longrightarrow$ stronger than singleton AC

M. Kozik $\Longrightarrow$ solves all bounded width fixed-finite-template CSPs

e.g. solves 2-SAT

WRAP UP

uniform ✓
powerful ?
beautiful at least straightforward

TODOs

- make it stronger than BLP/SDP in a natural way

  over $\mathbb{Z}$, achieve $p_1^2 + \dots + p_n^2 = 0 \pmod{N_2} \Rightarrow p_1 = p_2 = \dots = p_n = 0$

- does some level of RP-$\mathbb{Z}_2$ solve $\mathbb{Z}_8$?

  know: even VP-$\mathbb{Z}_2$ solves $\mathbb{Z}_4$ ($\Rightarrow$ 2nd level RP-$\mathbb{Z}_2$ too)

- solve group/Mal'cev CSPs in a uniform way

- find better "minor condition problem"

Thank you!