Weak block symmetric term operations in finite Taylor algebras

Dmitriy Zhuk
zhuk.dmitriy@gmail.com

Charles University
Prague

The 106th Workshop on General Algebra
Olomouc, Czech Republic
February 7-9, 2025

Funded by the European Union (ERC, POCOCOP, 101071674) and (ERC, CoCoSym, 771005).
Views and opinions expressed are however those of the author(s) only and do not necessarily reflect those of the European Union or the European Research Council Executive Agency. Neither the European Union nor the granting authority can be held responsible for them.

$\mathbf{A} = (A; f_1, f_2, \ldots)$ is an algebra

$\mathbf{A} = (A; f_1, f_2, \ldots)$ is an algebra

1. A is finite

$\mathbf{A} = (A; f_1, f_2, \dots)$ is an algebra

1. A is finite
2. $\mathbf{A}$ is idempotent, i.e. $f_i(x, \dots, x) = x$

$\mathbf{A} = (A; f_1, f_2, \dots)$ is an algebra

1. A is finite
2. $\mathbf{A}$ is idempotent, i.e. $f_i(x, \dots, x) = x$
3. $\mathbf{A}$ is Taylor

$\mathbf{A} = (A; f_1, f_2, \dots)$ is an algebra

1. A is finite
2. $\mathbf{A}$ is idempotent, i.e. $f_i(x, \dots, x) = x$
3. $\mathbf{A}$ is Taylor
   TFAE
   - $\mathbf{A}$ has a Taylor term operation.

$\mathbf{A} = (A; f_1, f_2, \dots)$ is an algebra

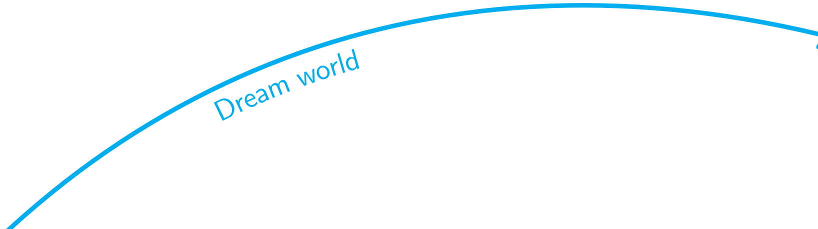
1. A is finite
2. $\mathbf{A}$ is idempotent, i.e. $f_i(x, \dots, x) = x$
3. $\mathbf{A}$ is Taylor
   TFAE
   - $\mathbf{A}$ has a Taylor term operation.
   - $\mathbf{A}$ satisfies a nontrivial identity.

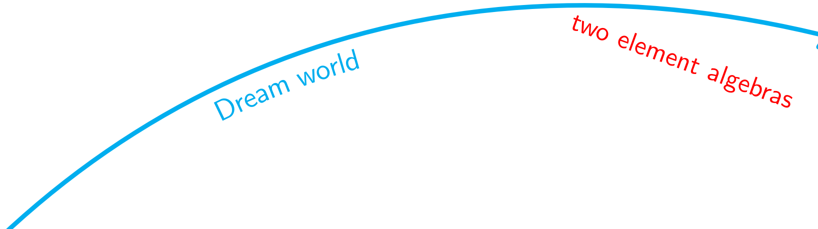
$\mathbf{A} = (A; f_1, f_2, \dots)$ is an algebra

1. A is finite
2. $\mathbf{A}$ is idempotent, i.e. $f_i(x, \dots, x) = x$
3. $\mathbf{A}$ is Taylor
   TFAE
   - $\mathbf{A}$ has a Taylor term operation.
   - $\mathbf{A}$ satisfies a nontrivial identity.
   - $\mathbf{A}$ satisfies a nontrivial h1-identity.

$\mathbf{A} = (A; f_1, f_2, \dots)$ is an algebra

1. A is finite
2. $\mathbf{A}$ is idempotent, i.e. $f_i(x, \dots, x) = x$
3. $\mathbf{A}$ is Taylor
   TFAE
   - $\mathbf{A}$ has a Taylor term operation.
   - $\mathbf{A}$ satisfies a nontrivial identity.
   - $\mathbf{A}$ satisfies a nontrivial h1-identity.
   - there doesn't exist essentially unary algebra $\mathbf{B} \in \mathrm{HS}(\mathbf{A})$.


Dream world



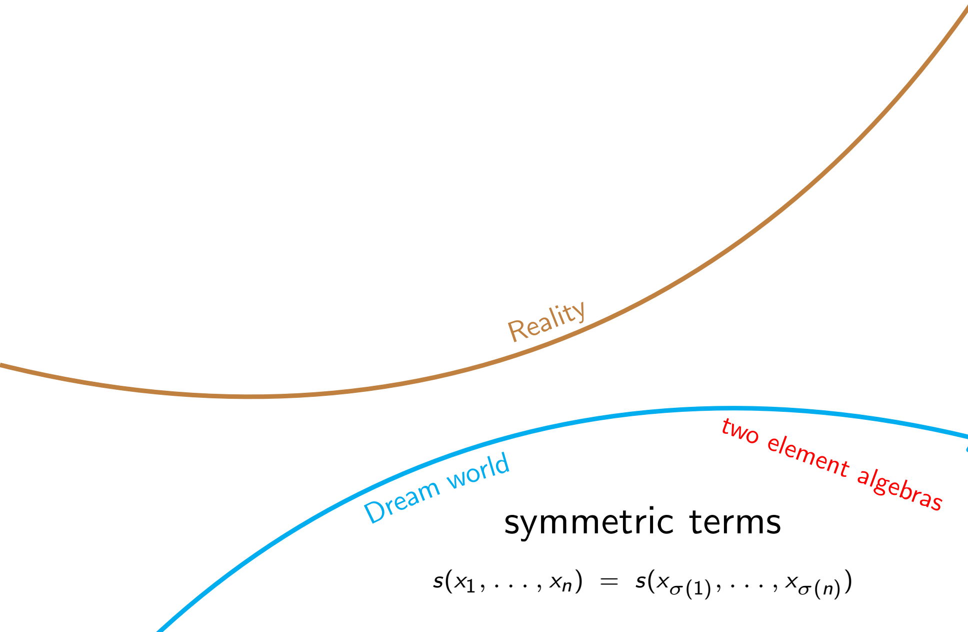
Dream world
two element algebras

Dream world

two element algebras

symmetric terms

$$s(x_1, \ldots, x_n) = s(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$$


Reality
Dream world
two element algebras
symmetric terms
$s(x_1, \ldots, x_n) = s(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$

Siggers term

$$s(y, x, y, z) = s(x, y, z, x)$$

Reality

Dream world

two element algebras

symmetric terms

$$s(x_1, \dots, x_n) = s(x_{\sigma(1)}, \dots, x_{\sigma(n)})$$

Siggers term

$$s(y, x, y, z) = s(x, y, z, x)$$

Olšák term

$$t(x, y, y, y, x, x) = t(y, x, y, x, y, x) = t(y, y, x, x, x, y)$$

Reality

Dream world

two element algebras

symmetric terms

$$s(x_1, \ldots, x_n) = s(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$$

Siggers term

$s(y, x, y, z) = s(x, y, z, x)$

Olšák term

$t(x, y, y, y, x, x) = t(y, x, y, x, y, x) = t(y, y, x, x, x, y)$

Reality

Dream world

two element algebras

symmetric terms

$$s(x_1, \dots, x_n) = s(x_{\sigma(1)}, \dots, x_{\sigma(n)})$$

Siggers term

$s(y, x, y, z) = s(x, y, z, x)$

Olšák term

$t(x, y, y, y, x, x) = t(y, x, y, x, y, x) = t(y, y, x, x, x, y)$

cyclic term

$c(x_1, x_2, \ldots, x_n) = c(x_2, x_3, \ldots, x_n, x_1)$

Reality

Dream world

two element algebras

symmetric terms

$s(x_1, \ldots, x_n) = s(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$

Siggers term

$$s(y, x, y, z) = s(x, y, z, x)$$

Olšák term

$$t(x, y, y, y, x, x) = t(y, x, y, x, y, x) = t(y, y, x, x, x, y)$$

cyclic term

$$c(x_1, x_2, \dots, x_n) = c(x_2, x_3, \dots, x_n, x_1)$$

WNU term

$$w(x, y, \dots, y) = w(y, x, y, \dots, y) = \cdots = w(y, y, \dots, y, x)$$

Reality

Dream world

two element algebras

symmetric terms

$$s(x_1, \dots, x_n) = s(x_{\sigma(1)}, \dots, x_{\sigma(n)})$$

Siggers term

$$s(y, x, y, z) = s(x, y, z, x)$$

Olšák term

$$t(x, y, y, y, x, x) = t(y, x, y, x, y, x) = t(y, y, x, x, x, y)$$

cyclic term

$$c(x_1, x_2, \dots, x_n) = c(x_2, x_3, \dots, x_n, x_1)$$

WNU term

$$w(x, y, \dots, y) = w(y, x, y, \dots, y) = \cdots = w(y, y, \dots, y, x)$$

XY-symmetric term

$$w(\underbrace{x, \dots, x}_{m}, y, \dots, y) = w(\underbrace{\dots x, \dots, y, \dots}_{\text{any tuple with } m \text{ xs}})$$

Reality

Dream world

two element algebras

symmetric terms

$$s(x_1, \dots, x_n) = s(x_{\sigma(1)}, \dots, x_{\sigma(n)})$$

Motivation

Motivation

Why symmetric?

Motivation

Why symmetric?

- They are cool!

Motivation

Why symmetric?

- They are cool!
- Symmetric operations characterize the algebra

Why symmetric?

- They are cool!
- Symmetric operations characterize the algebra

  For an XY-symmetric operation f

*	
$f(a, b, b \ldots, b, b, b)$	
$f(a, a, b \ldots, b, b, b)$	
$f(a, a, a \ldots, b, b, b)$	
$f(a, a, a \ldots, a, b, b)$	
$f(a, a, a \ldots, a, a, b)$	

Why symmetric?

- They are cool!
- Symmetric operations characterize the algebra
  For an XY-symmetric operation f

*	$b \to a$
$f(a, b, b \ldots, b, b, b)$	a
$f(a, a, b \ldots, b, b, b)$	a
$f(a, a, a \ldots, b, b, b)$	a
$f(a, a, a \ldots, a, b, b)$	a
$f(a, a, a \ldots, a, a, b)$	a

Why symmetric?

- They are cool!
- Symmetric operations characterize the algebra

  For an XY-symmetric operation f

*	$b \to a$	maj
$f(a, b, b \ldots, b, b, b)$	a	b
$f(a, a, b \ldots, b, b, b)$	a	b
$f(a, a, a \ldots, b, b, b)$	a	...
$f(a, a, a \ldots, a, b, b)$	a	a
$f(a, a, a \ldots, a, a, b)$	a	a

Why symmetric?

- They are cool!
- Symmetric operations characterize the algebra
  For an XY-symmetric operation f

*	$b \to a$	maj	affine
$f(a, b, b \ldots, b, b, b)$	a	b	a
$f(a, a, b \ldots, b, b, b)$	a	b	b
$f(a, a, a \ldots, b, b, b)$	a	$\ldots$	$\ldots$
$f(a, a, a \ldots, a, b, b)$	a	a	a
$f(a, a, a \ldots, a, a, b)$	a	a	b

Motivation

Why symmetric?

- They are cool!
- Symmetric operations characterize the algebra

  For an XY-symmetric operation f

*	$b \to a$	maj	affine	$a \to b$
$f(a, b, b \ldots, b, b, b)$	a	b	a	b
$f(a, a, b \ldots, b, b, b)$	a	b	b	b
$f(a, a, a \ldots, b, b, b)$	a	...	...	b
$f(a, a, a \ldots, a, b, b)$	a	a	a	b
$f(a, a, a \ldots, a, a, b)$	a	a	b	b

Why symmetric?

- They are cool!
- Symmetric operations characterize the algebra

  For an XY-symmetric operation f

*	$b \to a$	maj	affine	$a \to b$	$a \to c \leftarrow b$
$f(a, b, b \ldots, b, b, b)$	a	b	a	b	c
$f(a, a, b \ldots, b, b, b)$	a	b	b	b	c
$f(a, a, a \ldots, b, b, b)$	a	$\ldots$	$\ldots$	b	c
$f(a, a, a \ldots, a, b, b)$	a	a	a	b	c
$f(a, a, a \ldots, a, a, b)$	a	a	b	b	c

Why symmetric?

- They are cool!
- Symmetric operations characterize the algebra

  For an XY-symmetric operation f

*	$b \to a$	maj	affine	$a \to b$	$a \to c \leftarrow b$
$f(a, b, b \ldots, b, b, b)$	a	b	a	b	c
$f(a, a, b \ldots, b, b, b)$	a	b	b	b	c
$f(a, a, a \ldots, b, b, b)$	a	…	…	b	c
$f(a, a, a \ldots, a, b, b)$	a	a	a	b	c
$f(a, a, a \ldots, a, a, b)$	a	a	b	b	c

- Important for the complexity of CSP and Promise CSP.

Motivation

Why symmetric?

- They are cool!
- Symmetric operations characterize the algebra
  For an XY-symmetric operation f

*	$b \to a$	maj	affine	$a \to b$	$a \to c \leftarrow b$
$f(a, b, b \dots, b, b, b)$	a	b	a	b	c
$f(a, a, b \dots, b, b, b)$	a	b	b	b	c
$f(a, a, a \dots, b, b, b)$	a	...	...	b	c
$f(a, a, a \dots, a, b, b)$	a	a	a	b	c
$f(a, a, a \dots, a, a, b)$	a	a	b	b	c

- Important for the complexity of CSP and Promise CSP.

Linear programming algorithms solve CSP if there are symmetric polymorphisms of the constraint language.

Why symmetric?

- They are cool!
- Symmetric operations characterize the algebra
  For an XY-symmetric operation f

*	$b \to a$	maj	affine	$a \to b$	$a \to c \leftarrow b$
$f(a, b, b \dots, b, b, b)$	a	b	a	b	c
$f(a, a, b \dots, b, b, b)$	a	b	b	b	c
$f(a, a, a \dots, b, b, b)$	a	$\dots$	$\dots$	b	c
$f(a, a, a \dots, a, b, b)$	a	a	a	b	c
$f(a, a, a \dots, a, a, b)$	a	a	b	b	c

- Important for the complexity of CSP and Promise CSP.

Linear programming algorithms solve CSP if there are symmetric polymorphisms of the constraint language.

More symmetries $\Rightarrow$ easier algorithm works.

Why don't we have symmetric term operations?

Why don't we have symmetric term operations?

Cycle

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \mathsf{Inv}(\mathbf{A})$$

Why don't we have symmetric term operations?

Cycle

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \mathrm{Inv}(\mathbf{A})$$

A has no symmetric term operations of even arities

Why don't we have symmetric term operations?

Cycle

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \mathsf{Inv}(\mathbf{A})$$

A has no symmetric term operations of even arities

$$f \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} b \\ b \end{pmatrix} \notin \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Why don't we have symmetric term operations?

Cycle

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \mathrm{Inv}(\mathbf{A})$$

A has no symmetric term operations of even arities

$$f\begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} b \\ b \end{pmatrix} \notin \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Observations for $A = \{0, 1\}$

Why don't we have symmetric term operations?

Cycle

$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \mathsf{Inv}(\mathbf{A})$

A has no symmetric term operations of even arities

$$f \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} b \\ b \end{pmatrix} \notin \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Observations for $A = \{0, 1\}$

- $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \notin \mathsf{Inv}(\mathbf{A}) \Leftrightarrow \mathbf{A}$ has symmetric term operations of all arities.

Why don't we have symmetric term operations?

Cycle

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \mathrm{Inv}(\mathbf{A})$$

A has no symmetric term operations of even arities

$$f\begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} b \\ b \end{pmatrix} \notin \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Observations for $A = \{0, 1\}$

- $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \notin \mathrm{Inv}(\mathbf{A}) \Leftrightarrow$ **A** has symmetric term operations of all arities.
- **A** has symmetric term operations of all odd arities.

Why don't we have symmetric term operations?

Cycle

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \mathsf{Inv}(\mathbf{A})$$

A has no symmetric term operations of even arities

$$f \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} b \\ b \end{pmatrix} \notin \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Observations for $A = \{0, 1\}$

- $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \notin \mathsf{Inv}(\mathbf{A}) \Leftrightarrow$ **A** has symmetric term operations of all arities.
- **A** has symmetric term operations of all odd arities.

What to do:

Why don't we have symmetric term operations?

Cycle

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \mathrm{Inv}(\mathbf{A})$$

A has no symmetric term operations of even arities

$$f \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} b \\ b \end{pmatrix} \notin \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Observations for $A = \{0, 1\}$

- $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \notin \mathrm{Inv}(\mathbf{A}) \Leftrightarrow$ **A** has symmetric term operations of all arities.
- **A** has symmetric term operations of all odd arities.

What to do:

- Consider only terms of odd arities

Why don't we have symmetric term operations?

Cycle

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \mathrm{Inv}(\mathbf{A})$$

A has no symmetric term operations of even arities

$$f \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} b \\ b \end{pmatrix} \notin \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Observations for $A = \{0, 1\}$

- $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \notin \mathrm{Inv}(\mathbf{A}) \Leftrightarrow \mathbf{A}$ has symmetric term operations of all arities.
- **A** has symmetric term operations of all odd arities.

What to do:

- Consider only terms of odd arities
- Don't require symmetricity on tuples with equal number of 0s and 1s.

Why don’t we have symmetric term operations?

Why don't we have symmetric term operations?

$\mathbb{Z}_p$

$\mathbf{A} = (\mathbb{Z}_p; x - y + z)$

Why don't we have symmetric term operations?

$\mathbb{Z}_p$

$\mathbf{A} = (\mathbb{Z}_p; x - y + z)$

$\mathbf{A}$ has no symmetric term operations of arities divisible by p

Why don't we have symmetric term operations?

$\mathbb{Z}_p$

$\mathbf{A} = (\mathbb{Z}_p; x - y + z)$

$\mathbf{A}$ has no symmetric term operations of arities divisible by p

---

Linear idempotent operations: $a_1x_1 + \cdots + a_nx_n$, where $a_1 + \cdots + a_n = 1$.
$a_1 = \cdots = a_n \Rightarrow n$ is coprime with p.

Why don't we have symmetric term operations?

$\mathbb{Z}_p$

$\mathbf{A} = (\mathbb{Z}_p; x - y + z)$

$\mathbf{A}$ has no symmetric term operations of arities divisible by p

Linear idempotent operations: $a_1x_1 + \cdots + a_nx_n$, where $a_1 + \cdots + a_n = 1$.
$a_1 = \cdots = a_n \Rightarrow n$ is coprime with p.

Observation

- $(\mathbb{Z}_p; x - y + z)$ has symmetric operations of all arities coprime with p.

Why don't we have symmetric term operations?

$\mathbb{Z}_p$

$\mathbf{A} = (\mathbb{Z}_p; x - y + z)$

$\mathbf{A}$ has no symmetric term operations of arities divisible by p

Linear idempotent operations: $a_1x_1 + \cdots + a_nx_n$, where $a_1 + \cdots + a_n = 1$.
$a_1 = \cdots = a_n \Rightarrow n$ is coprime with p.

Observation

- $(\mathbb{Z}_p; x - y + z)$ has symmetric operations of all arities coprime with p.
- Symmetric on some non-constant tuple $\Rightarrow$ symmetric.

Why don't we have symmetric term operations?

$\mathbb{Z}_p$

$\mathbf{A} = (\mathbb{Z}_p; x - y + z)$

$\mathbf{A}$ has no symmetric term operations of arities divisible by p

Linear idempotent operations: $a_1x_1 + \cdots + a_nx_n$, where $a_1 + \cdots + a_n = 1$.
$a_1 = \cdots = a_n \Rightarrow n$ is coprime with p.

Observation

- $(\mathbb{Z}_p; x - y + z)$ has symmetric operations of all arities coprime with p.
- Symmetric on some non-constant tuple $\Rightarrow$ symmetric.

What to do:

- Avoid arities divisible by p.

Why don't we have symmetric term operations?

Why don't we have symmetric term operations?

Two Cycles

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 0 & 3 & 4 & 2 \end{pmatrix} \in \mathsf{Inv}(\mathbf{A})$$

Why don't we have symmetric term operations?

Two Cycles

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 0 & 3 & 4 & 2 \end{pmatrix} \in \mathsf{Inv}(\mathbf{A})$$

A has no symmetric term operations.

Why don't we have symmetric term operations?

Two Cycles

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 0 & 3 & 4 & 2 \end{pmatrix} \in \mathsf{Inv}(\mathbf{A})$$

A has no symmetric term operations.

$$f \begin{pmatrix} 0 & 1 & 0 & 1 & 2 & 3 & 4 \\ 1 & 0 & 1 & 0 & 3 & 4 & 2 \end{pmatrix} = \begin{pmatrix} b \\ b \end{pmatrix} \notin \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 0 & 3 & 4 & 2 \end{pmatrix}$$

Why don't we have symmetric term operations?

Two Cycles

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 0 & 3 & 4 & 2 \end{pmatrix} \in \mathrm{Inv}(\mathbf{A})$$

A has no symmetric term operations.

$$f \begin{pmatrix} 0 & 1 & 0 & 1 & 2 & 3 & 4 \\ 1 & 0 & 1 & 0 & 3 & 4 & 2 \end{pmatrix} = \begin{pmatrix} b \\ b \end{pmatrix} \notin \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 0 & 3 & 4 & 2 \end{pmatrix}$$

What to do:

- Require symmetricity only on good tuples: having different numbers of 0s and 1s or different numbers of 2s, 3s, and 4s.

Why don't we have symmetric term operations?

Why don't we have symmetric term operations?

Absorption to cycle

$\begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \end{pmatrix} \in \mathrm{Inv}(\mathbf{A})$ and $\{0,1\}$ strongly absorbs $\{0,1,2\}$

Why don't we have symmetric term operations?

Absorption to cycle

$\begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \end{pmatrix} \in \mathrm{Inv}(\mathbf{A})$ and $\{0,1\}$ strongly absorbs $\{0,1,2\}$

B strongly absorbs A (denote $B \lhd A$) if
$\forall i, j\colon f_i^{\mathbf{A}}(\underbrace{A \ldots, A, B}_{j}, A, \ldots, A) \subseteq B$

Why don't we have symmetric term operations?

Absorption to cycle

$\begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \end{pmatrix} \in \mathrm{Inv}(\mathbf{A})$ and $\{0,1\}$ strongly absorbs $\{0,1,2\}$

B strongly absorbs A (denote $B \lhd A$) if
$\forall i,j\colon f_i^{\mathbf{A}}(\underbrace{A\ldots,A,B}_{j},A,\ldots,A) \subseteq B$

A has no symmetric term operations.

Why don't we have symmetric term operations?

Absorption to cycle

$\begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \end{pmatrix} \in \mathrm{Inv}(\mathbf{A})$ and $\{0,1\}$ strongly absorbs $\{0,1,2\}$

B strongly absorbs A (denote $B \lhd A$) if

$$\forall i,j\colon f_i^{\mathbf{A}}(\underbrace{A \ldots, A, B}_{j}, A, \ldots, A) \subseteq B$$

A has no symmetric term operations.

$$f \begin{pmatrix} 0 & 0 & 1 & 1 & 2 & 2 & 2 \\ 1 & 1 & 0 & 0 & 2 & 2 & 2 \end{pmatrix} = \begin{pmatrix} b \\ b \end{pmatrix} \notin \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \end{pmatrix} \cap \{0,1\}^2$$

Why don't we have symmetric term operations?

Absorption to cycle

$\begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \end{pmatrix} \in \mathrm{Inv}(\mathbf{A})$ and $\{0,1\}$ strongly absorbs $\{0,1,2\}$

B strongly absorbs A (denote $B \triangleleft A$) if
$\forall i,j\colon f_i^{\mathbf{A}}(\underbrace{A\ldots,A,B}_{j},A,\ldots,A) \subseteq B$

A has no symmetric term operations.

$$f\begin{pmatrix} 0 & 0 & 1 & 1 & 2 & 2 & 2 \\ 1 & 1 & 0 & 0 & 2 & 2 & 2 \end{pmatrix} = \begin{pmatrix} b \\ b \end{pmatrix} \notin \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \end{pmatrix} \cap \{0,1\}^2$$

What to do:

- Require symmetricity only on tuples with different number of 0s and 1s.

Why don't we have symmetric term operations?

Why don't we have symmetric term operations?

Absorption to $\mathbb{Z}_2$

$\mathbf{A} = (\{0,1,2\}; h)$, where

$$h(x,y,z) = \begin{cases} x+y+z, & \text{if } x,y,z \in \{0,1\} \\ 2, & \text{if } x=y=z=2 \\ \text{first non-2}, & \text{otherwise} \end{cases}$$

Why don't we have symmetric term operations?

Absorption to $\mathbb{Z}_2$

$\mathbf{A} = (\{0,1,2\}; h)$, where

$$h(x,y,z) = \begin{cases} x+y+z, & \text{if } x,y,z \in \{0,1\} \\ 2, & \text{if } x=y=z=2 \\ \text{first non-2}, & \text{otherwise} \end{cases}$$

Observations

1. $\{0,1\}$ strongly absorbs $\{0,1,2\}$.

Why don't we have symmetric term operations?

Absorption to $\mathbb{Z}_2$

$\mathbf{A} = (\{0,1,2\}; h)$, where

$$h(x,y,z) = \begin{cases} x+y+z, & \text{if } x,y,z \in \{0,1\} \\ 2, & \text{if } x=y=z=2 \\ \text{first non-2}, & \text{otherwise} \end{cases}$$

Observations

1. $\{0,1\}$ strongly absorbs $\{0,1,2\}$.
2. $f(x_1,\ldots,x_s,2,2,\ldots,2)|_{\{0,1\}}$ is an idempotent linear operation for any $f \in \mathrm{Clo}(\mathbf{A})$

Why don't we have symmetric term operations?

Absorption to $\mathbb{Z}_2$

$\mathbf{A} = (\{0,1,2\}; h)$, where

$$h(x,y,z) = \begin{cases} x+y+z, & \text{if } x,y,z \in \{0,1\} \\ 2, & \text{if } x=y=z=2 \\ \text{first non-2}, & \text{otherwise} \end{cases}$$

Observations

1. $\{0,1\}$ strongly absorbs $\{0,1,2\}$.
2. $f(x_1,\ldots,x_s,2,2,\ldots,2)|_{\{0,1\}}$ is an idempotent linear operation for any $f \in \mathrm{Clo}(\mathbf{A})$

A has no symmetric term operations.

Why don't we have symmetric term operations?

Absorption to $\mathbb{Z}_2$

$\mathbf{A} = (\{0,1,2\}; h)$, where

$$h(x,y,z) = \begin{cases} x+y+z, & \text{if } x,y,z \in \{0,1\} \\ 2, & \text{if } x=y=z=2 \\ \text{first non-2}, & \text{otherwise} \end{cases}$$

Observations

1. $\{0,1\}$ strongly absorbs $\{0,1,2\}$.
2. $f(x_1,\dots,x_s,2,2,\dots,2)|_{\{0,1\}}$ is an idempotent linear operation for any $f \in \mathrm{Clo}(\mathbf{A})$

A has no symmetric term operations.

$f(x_1,x_2,2,2,2,2,2)|_{\{0,1\}}$ is linear, binary, and idempotent on $\mathbb{Z}_2$.

Why don't we have symmetric term operations?

Absorption to $\mathbb{Z}_2$

$\mathbf{A} = (\{0,1,2\}; h)$, where

$$h(x,y,z) = \begin{cases} x+y+z, & \text{if } x,y,z \in \{0,1\} \\ 2, & \text{if } x=y=z=2 \\ \text{first non-2}, & \text{otherwise} \end{cases}$$

Observations

1. $\{0,1\}$ strongly absorbs $\{0,1,2\}$.
2. $f(x_1,\ldots,x_s,2,2,\ldots,2)|_{\{0,1\}}$ is an idempotent linear operation for any $f \in \mathrm{Clo}(\mathbf{A})$

$\mathbf{A}$ has no symmetric term operations.

$f(x_1,x_2,2,2,2,2,2)|_{\{0,1\}}$ is linear, binary, and idempotent on $\mathbb{Z}_2$.

What to do:

- Consider only tuples with odd number of elements from $\{0,1\}$.

Why don't we have symmetric term operations?

Why don't we have symmetric term operations?

$\mathbb{Z}_2 \cup \mathbb{Z}_3$

$\mathbf{A} = (\mathbb{Z}_2 \cup \mathbb{Z}_3'; m)$,
where $m|_{\mathbb{Z}_2}$, $m|_{\mathbb{Z}_3'}$ are $x - y + z$, $m/_{\mathbb{Z}_2|\mathbb{Z}_3'}$ is minority.

Why don't we have symmetric term operations?

$\mathbb{Z}_2 \cup \mathbb{Z}_3$

$\mathbf{A} = (\mathbb{Z}_2 \cup \mathbb{Z}_3'; m)$,
where $m|_{\mathbb{Z}_2}$, $m|_{\mathbb{Z}_3'}$ are $x - y + z$, $m/_{\mathbb{Z}_2|\mathbb{Z}_3'}$ is minority.

$\mathbf{A}$ has no symmetric term operations.

Why don't we have symmetric term operations?

$\mathbb{Z}_2 \cup \mathbb{Z}_3$

$\mathbf{A} = (\mathbb{Z}_2 \cup \mathbb{Z}_3'; m)$,
where $m|_{\mathbb{Z}_2}$, $m|_{\mathbb{Z}_3'}$ are $x - y + z$, $m/_{\mathbb{Z}_2|\mathbb{Z}_3'}$ is minority.

$\mathbf{A}$ has no symmetric term operations.

$f(0,0,0,0,x_1,x_2,x_3)|_{\{0',1',2'\}}$ is linear and idempotent on $\mathbb{Z}_3'$.

Why don't we have symmetric term operations?

$\mathbb{Z}_2 \cup \mathbb{Z}_3$

$\mathbf{A} = (\mathbb{Z}_2 \cup \mathbb{Z}_3'; m)$,
where $m|_{\mathbb{Z}_2}$, $m|_{\mathbb{Z}_3'}$ are $x - y + z$, $m/_{\mathbb{Z}_2|\mathbb{Z}_3'}$ is minority.

A has no symmetric term operations.

$f(0,0,0,0,x_1,x_2,x_3)|_{\{0',1',2'\}}$ is linear and idempotent on $\mathbb{Z}_3'$.

What to do:

- Require symmetricity only on very special tuples.

Why don't we have symmetric term operations?

Why don't we have symmetric term operations?

Disconnectedness!

Why don't we have symmetric term operations?

Disconnectedness!

Two Cycles

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 0 & 3 & 4 & 2 \end{pmatrix} \in \mathsf{Inv}(\mathbf{A})$$

Why don't we have symmetric term operations?

Disconnectedness!

Two Cycles

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 0 & 3 & 4 & 2 \end{pmatrix} \in \mathsf{Inv}(\mathbf{A})$$

$\{0, 1\}$ and $\{2, 3, 4\}$ are disconnected.

Why don't we have symmetric term operations?

Disconnectedness!

Two Cycles

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 0 & 3 & 4 & 2 \end{pmatrix} \in \mathsf{Inv}(\mathbf{A})$$

$\{0, 1\}$ and $\{2, 3, 4\}$ are disconnected.

Absorption to cycle

$\begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \end{pmatrix} \in \mathsf{Inv}(\mathbf{A})$ and $\{0, 1\}$ strongly absorbs $\{0, 1, 2\}$

Why don't we have symmetric term operations?

Disconnectedness!

Two Cycles

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 0 & 3 & 4 & 2 \end{pmatrix} \in \mathsf{Inv}(\mathbf{A})$$

$\{0, 1\}$ and $\{2, 3, 4\}$ are disconnected.

Absorption to cycle

$\begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \end{pmatrix} \in \mathsf{Inv}(\mathbf{A})$ and $\{0, 1\}$ strongly absorbs $\{0, 1, 2\}$

$\{0, 1\}$ and $\{2\}$ are disconnected.

Why don't we have symmetric term operations?

Disconnectedness!

Two Cycles

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 0 & 3 & 4 & 2 \end{pmatrix} \in \mathrm{Inv}(\mathbf{A})$$

$\{0,1\}$ and $\{2,3,4\}$ are disconnected.

Absorption to cycle

$\begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \end{pmatrix} \in \mathrm{Inv}(\mathbf{A})$ and $\{0,1\}$ strongly absorbs $\{0,1,2\}$

$\{0,1\}$ and $\{2\}$ are disconnected.

$\mathbb{Z}_2 \cup \mathbb{Z}_3$

$\mathbf{A} = (\mathbb{Z}_2 \cup \mathbb{Z}_3'; m)$,
where $m|_{\mathbb{Z}_2}$, $m|_{\mathbb{Z}_3'}$ are $x - y + z$, $m/_{\mathbb{Z}_2|\mathbb{Z}_3'}$ is minority.

Why don't we have symmetric term operations?

Disconnectedness!

Two Cycles

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 0 & 3 & 4 & 2 \end{pmatrix} \in \mathrm{Inv}(\mathbf{A})$$

$\{0, 1\}$ and $\{2, 3, 4\}$ are disconnected.

Absorption to cycle

$\begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \end{pmatrix} \in \mathrm{Inv}(\mathbf{A})$ and $\{0, 1\}$ strongly absorbs $\{0, 1, 2\}$

$\{0, 1\}$ and $\{2\}$ are disconnected.

$\mathbb{Z}_2 \cup \mathbb{Z}_3$

$\mathbf{A} = (\mathbb{Z}_2 \cup \mathbb{Z}_3'; m)$,
where $m|_{\mathbb{Z}_2}$, $m|_{\mathbb{Z}_3'}$ are $x - y + z$, $m/_{\mathbb{Z}_2|\mathbb{Z}_3'}$ is minority.

$\mathbb{Z}_2$ and $\mathbb{Z}_3'$ are disconnected.

Block Symmetric Operations

Block Symmetric Operations

w is (m, n, k)-block symmetric if for all permutations σ, δ, ω

$$w(x_1, \ldots, x_m, y_1, \ldots, y_n, z_1, \ldots, z_k) = w(x_{\sigma(1)}, \ldots, x_{\sigma(m)}, y_{\delta(1)}, \ldots, y_{\delta(n)}, z_{\omega(1)}, \ldots, z_{\omega(k)})$$

Block Symmetric Operations

w is (m, n, k)-block symmetric if for all permutations σ, δ, ω

$$w(x_1, \ldots, x_m, y_1, \ldots, y_n, z_1, \ldots, z_k) = w(x_{\sigma(1)}, \ldots, x_{\sigma(m)}, y_{\delta(1)}, \ldots, y_{\delta(n)}, z_{\omega(1)}, \ldots, z_{\omega(k)})$$

Definition

$k_1, k_2, \ldots, k_n \in \mathbb{N}$,

Block Symmetric Operations

w is (m, n, k)-block symmetric if for all permutations σ, δ, ω

$$w(x_1, \ldots, x_m, y_1, \ldots, y_n, z_1, \ldots, z_k) = w(x_{\sigma(1)}, \ldots, x_{\sigma(m)}, y_{\delta(1)}, \ldots, y_{\delta(n)}, z_{\omega(1)}, \ldots, z_{\omega(k)})$$

Definition

$k_1, k_2, \ldots, k_n \in \mathbb{N}$,
$f : A^{k_1+\cdots+k_n} \to A$ is $(k_1, \ldots, k_n)$-block symmetric if
$f(\mathbf{x}_1, \ldots, \mathbf{x}_n) = f(\mathbf{x}_1^{\sigma_1}, \ldots, \mathbf{x}_n^{\sigma_n})$ for any permutations $\sigma_1, \ldots, \sigma_n$.

Block Symmetric Operations

w is (m, n, k)-block symmetric if for all permutations σ, δ, ω

$$w(x_1, \dots, x_m, y_1, \dots, y_n, z_1, \dots, z_k) = \\ w(x_{\sigma(1)}, \dots, x_{\sigma(m)}, y_{\delta(1)}, \dots, y_{\delta(n)}, z_{\omega(1)}, \dots, z_{\omega(k)})$$

Definition

$k_1, k_2, \dots, k_n \in \mathbb{N}$,
$f : A^{k_1+\cdots+k_n} \to A$ is $(k_1, \dots, k_n)$-block symmetric if
$f(\mathbf{x}_1, \dots, \mathbf{x}_n) = f(\mathbf{x}_1^{\sigma_1}, \dots, \mathbf{x}_n^{\sigma_n})$ for any permutations $\sigma_1, \dots, \sigma_n$.

Observation

None of $\mathrm{Pol}\left(\begin{smallmatrix}0&1&2&3&4\\1&0&3&4&2\end{smallmatrix}\right)$, $\mathrm{Pol}\left(\begin{smallmatrix}0&1&2\\1&0&2\end{smallmatrix}\right) + \{0,1\} \lhd \{0,1,2\}$, $\mathbb{Z}_2 \cup \mathbb{Z}_3$ have $(k_1, \dots, k_n)$-block symmetric term operations for any $k_1, \dots, k_n \geq 2$.

Block Symmetric Operations

Block Symmetric Operations

Observation [Brakensiek, Guruswami, Wrochna, Živný]

TFAE:

1. $\forall m$ **A** has a $(k_1, \ldots, k_n)$-block symmetric term operation with $k_1, \ldots, k_n \geq m$
2. **A** has infinitely many $(m+1, m)$-block symmetric term operations.

Block Symmetric Operations

Observation [Brakensiek, Guruswami, Wrochna, Živný]

TFAE:

1. $\forall m$ $\mathbf{A}$ has a $(k_1, \ldots, k_n)$-block symmetric term operation with $k_1, \ldots, k_n \geq m$
2. $\mathbf{A}$ has infinitely many $(m+1, m)$-block symmetric term operations.

$(\mathbb{Z}; x - y + z)$ has $(m+1, m)$-block symmetric term operation for every m but does not have symmetric term operations.

Block Symmetric Operations

Observation [Brakensiek, Guruswami, Wrochna, Živný]

TFAE:

1. $\forall m$ **A** has a $(k_1, \dots, k_n)$-block symmetric term operation with $k_1, \dots, k_n \geq m$
2. **A** has infinitely many $(m+1, m)$-block symmetric term operations.

$(\mathbb{Z}; x - y + z)$ has $(m+1, m)$-block symmetric term operation for every m but does not have symmetric term operations.

Question

Block Symmetric Operations

Observation [Brakensiek, Guruswami, Wrochna, Živný]

TFAE:

1. $\forall m$ $\mathbf{A}$ has a $(k_1, \ldots, k_n)$-block symmetric term operation with $k_1, \ldots, k_n \geq m$
2. $\mathbf{A}$ has infinitely many $(m+1, m)$-block symmetric term operations.

$(\mathbb{Z}; x - y + z)$ has $(m+1, m)$-block symmetric term operation for every m but does not have symmetric term operations.

Question

Are the following conditions equivalent? (for finite algebras $\mathbf{A}$)

Block Symmetric Operations

Observation [Brakensiek, Guruswami, Wrochna, Živný]

TFAE:

1. $\forall m$ **A** has a $(k_1, \ldots, k_n)$-block symmetric term operation with $k_1, \ldots, k_n \geq m$
2. **A** has infinitely many $(m+1, m)$-block symmetric term operations.

$(\mathbb{Z}; x - y + z)$ has $(m+1, m)$-block symmetric term operation for every m but does not have symmetric term operations.

Question

Are the following conditions equivalent? (for finite algebras **A**)

1. **A** has infinitely many symmetric term operations.

Block Symmetric Operations

Observation [Brakensiek, Guruswami, Wrochna, Živný]

TFAE:

1. $\forall m$ **A** has a $(k_1, \ldots, k_n)$-block symmetric term operation with $k_1, \ldots, k_n \geq m$
2. **A** has infinitely many $(m+1, m)$-block symmetric term operations.

$(\mathbb{Z}; x - y + z)$ has $(m+1, m)$-block symmetric term operation for every m but does not have symmetric term operations.

Question

Are the following conditions equivalent? (for finite algebras **A**)

1. **A** has infinitely many symmetric term operations.
2. **A** has infinitely many $(m+1, m)$-block symmetric term operations.

Block Symmetric Operations

Observation [Brakensiek, Guruswami, Wrochna, Živný]

TFAE:

1. $\forall m$ $\mathbf{A}$ has a $(k_1, \ldots, k_n)$-block symmetric term operation with $k_1, \ldots, k_n \geq m$
2. $\mathbf{A}$ has infinitely many $(m+1, m)$-block symmetric term operations.

$(\mathbb{Z}; x - y + z)$ has $(m+1, m)$-block symmetric term operation for every m but does not have symmetric term operations.

Question

Are the following conditions equivalent? (for finite algebras $\mathbf{A}$)

1. $\mathbf{A}$ has infinitely many symmetric term operations.
2. $\mathbf{A}$ has infinitely many $(m+1, m)$-block symmetric term operations.
3. $\mathbf{A}$ has a $(m+1, m)$-block symmetric term operation for every $m \in \mathbb{N}$.

Weak Block Symmetric Operations

Weak Block Symmetric Operations

Definition

$k_1, k_2, \ldots, k_n \in \mathbb{N}$, $\mathbf{a}_1 \in A^{k_1}, \ldots, \mathbf{a}_n \in A^{k_n}$

Weak Block Symmetric Operations

Definition

$k_1, k_2, \ldots, k_n \in \mathbb{N}$, $\mathbf{a}_1 \in A^{k_1}, \ldots, \mathbf{a}_n \in A^{k_n}$
$f : A^{k_1+\cdots+k_n} \to A$ is $(k_1, \ldots, k_n)$-block symmetric on $(\mathbf{a}_1, \ldots, \mathbf{a}_n)$
if $f(\mathbf{a}_1, \ldots, \mathbf{a}_n) = f(\mathbf{a}_1^{\sigma_1}, \ldots, \mathbf{a}_n^{\sigma_n})$ for any permutations $\sigma_1, \ldots, \sigma_n$.

Weak Block Symmetric Operations

Definition

$k_1, k_2, \ldots, k_n \in \mathbb{N}$, $\mathbf{a}_1 \in A^{k_1}, \ldots, \mathbf{a}_n \in A^{k_n}$
$f : A^{k_1+\cdots+k_n} \to A$ is $(k_1, \ldots, k_n)$-block symmetric on $(\mathbf{a}_1, \ldots, \mathbf{a}_n)$
if $f(\mathbf{a}_1, \ldots, \mathbf{a}_n) = f(\mathbf{a}_1^{\sigma_1}, \ldots, \mathbf{a}_n^{\sigma_n})$ for any permutations $\sigma_1, \ldots, \sigma_n$.

An operation is $(k_1, \ldots, k_n)$-*block symmetric* if it is $(k_1, \ldots, k_n)$-block symmetric on all tuples.

Weak Block Symmetric Operations

Definition

$k_1, k_2, \ldots, k_n \in \mathbb{N}$, $\mathbf{a}_1 \in A^{k_1}, \ldots, \mathbf{a}_n \in A^{k_n}$
$f : A^{k_1+\cdots+k_n} \to A$ is $(k_1, \ldots, k_n)$-block symmetric on $(\mathbf{a}_1, \ldots, \mathbf{a}_n)$
if $f(\mathbf{a}_1, \ldots, \mathbf{a}_n) = f(\mathbf{a}_1^{\sigma_1}, \ldots, \mathbf{a}_n^{\sigma_n})$ for any permutations $\sigma_1, \ldots, \sigma_n$.

$(\mathbf{a}_1, \ldots, \mathbf{a}_n)$ is conservative if $\forall j\ \forall c \in \mathbf{a}_j\ \ \exists i\ \ \mathbf{a}_i = (c, c, \ldots, c)$.

An operation is $(k_1, \ldots, k_n)$-*block symmetric* if it is $(k_1, \ldots, k_n)$-block symmetric on all tuples.

Weak Block Symmetric Operations

Definition

$k_1, k_2, \ldots, k_n \in \mathbb{N}$, $\mathbf{a}_1 \in A^{k_1}, \ldots, \mathbf{a}_n \in A^{k_n}$
$f : A^{k_1+\cdots+k_n} \to A$ is $(k_1, \ldots, k_n)$-block symmetric on $(\mathbf{a}_1, \ldots, \mathbf{a}_n)$
if $f(\mathbf{a}_1, \ldots, \mathbf{a}_n) = f(\mathbf{a}_1^{\sigma_1}, \ldots, \mathbf{a}_n^{\sigma_n})$ for any permutations $\sigma_1, \ldots, \sigma_n$.

$(\mathbf{a}_1, \ldots, \mathbf{a}_n)$ is conservative if $\forall j \; \forall c \in \mathbf{a}_j \;\; \exists i \;\; \mathbf{a}_i = (c, c, \ldots, c)$.

An operation is $(k_1, \ldots, k_n)$-*block symmetric* if it is $(k_1, \ldots, k_n)$-block symmetric on all tuples.

Conservative tuple:

$$(\underbrace{2,2,0},\underbrace{2,2,2}_{2},\underbrace{1,2,0},\underbrace{1,2,0},\underbrace{0,0,0}_{0},\underbrace{0,0,1},\underbrace{1,1,1}_{1},\underbrace{1,2,0},\underbrace{1,2,2})$$

Weak Block Symmetric Operations

Definition

$k_1, k_2, \ldots, k_n \in \mathbb{N}$, $\mathbf{a}_1 \in A^{k_1}, \ldots, \mathbf{a}_n \in A^{k_n}$
$f : A^{k_1+\cdots+k_n} \to A$ is $(k_1, \ldots, k_n)$-block symmetric on $(\mathbf{a}_1, \ldots, \mathbf{a}_n)$
if $f(\mathbf{a}_1, \ldots, \mathbf{a}_n) = f(\mathbf{a}_1^{\sigma_1}, \ldots, \mathbf{a}_n^{\sigma_n})$ for any permutations $\sigma_1, \ldots, \sigma_n$.

$(\mathbf{a}_1, \ldots, \mathbf{a}_n)$ is conservative if $\forall j\ \forall c \in \mathbf{a}_j\ \ \exists i\ \ \mathbf{a}_i = (c, c, \ldots, c)$.

An operation is $(k_1, \ldots, k_n)$-*block symmetric* if it is $(k_1, \ldots, k_n)$-block symmetric on all tuples.

Conservative tuple:

$$(\underbrace{2,2,0}, \underbrace{2,2,2}_{2}, \underbrace{1,2,0}, \underbrace{1,2,0}, \underbrace{0,0,0}_{0}, \underbrace{0,0,1}, \underbrace{1,1,1}_{1}, \underbrace{1,2,0}, \underbrace{1,2,2})$$

Weak Block Symmetric Operations

Definition

$k_1, k_2, \ldots, k_n \in \mathbb{N}$, $\mathbf{a}_1 \in A^{k_1}, \ldots, \mathbf{a}_n \in A^{k_n}$
$f : A^{k_1+\cdots+k_n} \to A$ is $(k_1, \ldots, k_n)$-block symmetric on $(\mathbf{a}_1, \ldots, \mathbf{a}_n)$ if $f(\mathbf{a}_1, \ldots, \mathbf{a}_n) = f(\mathbf{a}_1^{\sigma_1}, \ldots, \mathbf{a}_n^{\sigma_n})$ for any permutations $\sigma_1, \ldots, \sigma_n$.

$(\mathbf{a}_1, \ldots, \mathbf{a}_n)$ is conservative if $\forall j\ \forall c \in \mathbf{a}_j\ \ \exists i\ \ \mathbf{a}_i = (c, c, \ldots, c)$.

$f : A^{k_1+\cdots+k_n} \to A$ is $(k_1, \ldots, k_n)$-weak block symmetric if it is block symmetric on any conservative tuple.

An operation is $(k_1, \ldots, k_n)$-*block symmetric* if it is $(k_1, \ldots, k_n)$-block symmetric on all tuples.

Conservative tuple:

$$(\underbrace{2,2,0}, \underbrace{2,2,2}_{2}, \underbrace{1,2,0}, \underbrace{1,2,0}, \underbrace{0,0,0}_{0}, \underbrace{0,0,1}, \underbrace{1,1,1}_{1}, \underbrace{1,2,0}, \underbrace{1,2,2})$$

Weak block symmetric term operations

Two Cycles

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 0 & 3 & 4 & 2 \end{pmatrix} \in \mathsf{Inv}(\mathbf{A})$$

Weak block symmetric term operations

Two Cycles

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 0 & 3 & 4 & 2 \end{pmatrix} \in \mathsf{Inv}(\mathbf{A})$$

Observation

$\mathsf{Pol}\left(\begin{smallmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 0 & 3 & 4 & 2 \end{smallmatrix}\right)$ has $(k_1, \ldots, k_n)$-WBS operations for all $k_1, \ldots, k_n$.

Weak block symmetric term operations

Two Cycles

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 0 & 3 & 4 & 2 \end{pmatrix} \in \mathrm{Inv}(\mathbf{A})$$

Observation

$\mathrm{Pol}\left(\begin{smallmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 0 & 3 & 4 & 2 \end{smallmatrix}\right)$ has $(k_1, \ldots, k_n)$-WBS operations for all $k_1, \ldots, k_n$.

$$w(\mathbf{a}_1, \ldots, \mathbf{a}_n) = \begin{cases} c, & \text{if } \mathbf{a}_i = (c, c, \ldots, c) \text{ and} \\ & \mathbf{a}_i \text{ is the first such tuple} \\ \mathbf{a}_1^j & \text{otherwise} \end{cases}$$

$$w(\underbrace{1,2,3},\underbrace{2,2,2},\underbrace{1,2,0},\underbrace{0,0,0},\underbrace{1,1,1},\underbrace{1,2,0},\underbrace{3,3,3},\underbrace{3,2,0}) = 2$$

$$w(\underbrace{1,2,3},\underbrace{0,1,2},\underbrace{1,2,0},\underbrace{0,2,4},\underbrace{3,4,3},\underbrace{2,1,2},\underbrace{1,2,0},\underbrace{4,1,0}) = 1$$

Weak block symmetric term operations

Absorption to cycle

$\begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \end{pmatrix} \in \mathrm{Inv}(\mathbf{A})$ and $\{0,1\}$ strongly absorbs $\{0,1,2\}$

Weak block symmetric term operations

Absorption to cycle

$\begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \end{pmatrix} \in \mathrm{Inv}(\mathbf{A})$ and $\{0,1\}$ strongly absorbs $\{0,1,2\}$

Observation

$\mathrm{Pol}\left(\begin{smallmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \end{smallmatrix}\right) \cap \mathrm{Pol}(\{0,1\} \lhd \{0,1,2\})$ has $(k_1, \ldots, k_n)$-WBS operations for all $k_1, \ldots, k_n$.

Weak block symmetric term operations

Absorption to cycle

$\begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \end{pmatrix} \in \mathrm{Inv}(\mathbf{A})$ and $\{0,1\}$ strongly absorbs $\{0,1,2\}$

Observation

$\mathrm{Pol}\left(\begin{smallmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \end{smallmatrix}\right) \cap \mathrm{Pol}(\{0,1\} \lhd \{0,1,2\})$ has $(k_1,\ldots,k_n)$-WBS operations for all $k_1,\ldots,k_n$.

$$w(\mathbf{a}_1,\ldots,\mathbf{a}_n) = \begin{cases} c, & \text{if } \mathbf{a}_i = (c,c,\ldots,c),\ c \neq 2, \text{ and} \\ & \mathbf{a}_i \text{ is the first such tuple} \\ \mathbf{a}_1^1 & \text{otherwise} \end{cases}$$

$$w(\underbrace{1,2,0},\underbrace{2,2,2},\underbrace{2,1,2},\underbrace{0,0,0},\underbrace{1,1,1},\underbrace{1,2,0},\underbrace{0,2,1},\underbrace{1,2,0}) = 0$$

$$w(\underbrace{1,2,0},\underbrace{0,1,2},\underbrace{1,2,0},\underbrace{0,2,1},\underbrace{0,1,0},\underbrace{2,1,2},\underbrace{1,2,0},\underbrace{1,1,0}) = 1$$

Weak block symmetric term operations

Absorption to $\mathbb{Z}_2$

$\mathbf{A} = (\{0,1,2\}; h)$, where

$$h(x,y,z) = \begin{cases} x+y+z, & \text{if } x,y,z \in \{0,1\} \\ 2, & \text{if } x=y=z=2 \\ \text{first non-2}, & \text{otherwise} \end{cases}$$

Weak block symmetric term operations

Absorption to $\mathbb{Z}_2$

$\mathbf{A} = (\{0,1,2\}; h)$, where

$$h(x,y,z) = \begin{cases} x+y+z, & \text{if } x,y,z \in \{0,1\} \\ 2, & \text{if } x=y=z=2 \\ \text{first non-2}, & \text{otherwise} \end{cases}$$

Observation

$\mathbf{A}$ has $(k_1,\ldots,k_n)$-WBS terms for all odd $k_1,\ldots,k_n$.

Weak block symmetric term operations

Absorption to $\mathbb{Z}_2$

$\mathbf{A} = (\{0,1,2\}; h)$, where

$$h(x,y,z) = \begin{cases} x+y+z, & \text{if } x,y,z \in \{0,1\} \\ 2, & \text{if } x = y = z = 2 \\ \text{first non-2}, & \text{otherwise} \end{cases}$$

Observation

$\mathbf{A}$ has $(k_1,\dots,k_n)$-WBS terms for all odd $k_1,\dots,k_n$.

$$w(\mathbf{a}_1,\dots,\mathbf{a}_n) = \begin{cases} \mathbf{a}_i^1 + \cdots + \mathbf{a}_i^{k_i}, & \text{if } \mathbf{a}_i \in \{0,1\}^{k_i} \text{ and} \\ & \mathbf{a}_i \text{ is the first such tuple} \\ \mathbf{a}_1^1 & \text{otherwise} \end{cases}$$

$$w(\underbrace{1,2,0},\underbrace{2,2,2},\underbrace{1,2,0},\underbrace{0,1,0},\underbrace{1,1,1},\underbrace{1,2,0},\underbrace{1,0,1},\underbrace{0,2,0}) = 1$$

$$w(\underbrace{0,2,1},\underbrace{0,1,2},\underbrace{1,2,0},\underbrace{0,2,1},\underbrace{1,0,2},\underbrace{2,1,2},\underbrace{1,2,0},\underbrace{2,1,0}) = 0$$

Weak block symmetric term operations

$\mathbb{Z}_2 \cup \mathbb{Z}_3$

$\mathbf{A} = (\mathbb{Z}_2 \cup \mathbb{Z}_3'; m)$,
where $m|_{\mathbb{Z}_2}$, $m|_{\mathbb{Z}_3'}$ are $x - y + z$, $m/_{\mathbb{Z}_2|\mathbb{Z}_3'}$ is minority.

Weak block symmetric term operations

$\mathbb{Z}_2 \cup \mathbb{Z}_3$

$\mathbf{A} = (\mathbb{Z}_2 \cup \mathbb{Z}_3'; m)$,
where $m|_{\mathbb{Z}_2}$, $m|_{\mathbb{Z}_3'}$ are $x - y + z$, $m/_{\mathbb{Z}_2|\mathbb{Z}_3'}$ is minority.

Observation

$\mathbf{A}$ has $(k_1, \ldots, k_n)$-WBS terms for all $k_1, \ldots, k_n$ coprime with 2 and 3.

Weak block symmetric term operations

$\mathbb{Z}_2 \cup \mathbb{Z}_3$

$\mathbf{A} = (\mathbb{Z}_2 \cup \mathbb{Z}_3'; m)$,
where $m|_{\mathbb{Z}_2}$, $m|_{\mathbb{Z}_3'}$ are $x - y + z$, $m/_{\mathbb{Z}_2|\mathbb{Z}_3'}$ is minority.

Observation

$\mathbf{A}$ has $(k_1, \ldots, k_n)$-WBS terms for all $k_1, \ldots, k_n$ coprime with 2 and 3.

---

1. Calculate symmetric operation on the first block modulo $\{\mathbb{Z}_2 \mid \mathbb{Z}_3'\}$.

Weak block symmetric term operations

$\mathbb{Z}_2 \cup \mathbb{Z}_3$

$\mathbf{A} = (\mathbb{Z}_2 \cup \mathbb{Z}_3'; m)$,
where $m|_{\mathbb{Z}_2}$, $m|_{\mathbb{Z}_3'}$ are $x - y + z$, $m/_{\mathbb{Z}_2|\mathbb{Z}_3'}$ is minority.

Observation

$\mathbf{A}$ has $(k_1, \ldots, k_n)$-WBS terms for all $k_1, \ldots, k_n$ coprime with 2 and 3.

---

1. Calculate symmetric operation on the first block modulo $\{\mathbb{Z}_2 \mid \mathbb{Z}_3'\}$.

$$w(\underbrace{1, 2', 0', 0', 1}, \underbrace{2', 1', 2', 2', 2'}, \underbrace{1, 1, 1, 1, 1}, \underbrace{0', 1, 0, 0, 1}) =$$

$$w(\underbrace{0, 2', 0', 1, 0}, \underbrace{1', 2', 1'.2', 1'}, \underbrace{0, 1, 1, 1, 1}, \underbrace{0, 0, 0, 0, 0}) =$$

Weak block symmetric term operations

$\mathbb{Z}_2 \cup \mathbb{Z}_3$

$\mathbf{A} = (\mathbb{Z}_2 \cup \mathbb{Z}'_3; m)$,
where $m|_{\mathbb{Z}_2}$, $m|_{\mathbb{Z}'_3}$ are $x - y + z$, $m/_{\mathbb{Z}_2|\mathbb{Z}'_3}$ is minority.

Observation

$\mathbf{A}$ has $(k_1, \ldots, k_n)$-WBS terms for all $k_1, \ldots, k_n$ coprime with 2 and 3.

---

1. Calculate symmetric operation on the first block modulo $\{\mathbb{Z}_2 \mid \mathbb{Z}'_3\}$.

$$w(\underbrace{1, 2', 0', 0', 1}, \underbrace{2', 1', 2', 2', 2'}, \underbrace{1, 1, 1, 1, 1}, \underbrace{0', 1, 0, 0, 1}) =$$

$$w(\underbrace{0, 2', 0', 1, 0}, \underbrace{1', 2', 1'.2', 1'}, \underbrace{0, 1, 1, 1, 1}, \underbrace{0, 0, 0, 0, 0}) =$$

Weak block symmetric term operations

$\mathbb{Z}_2 \cup \mathbb{Z}_3$

$\mathbf{A} = (\mathbb{Z}_2 \cup \mathbb{Z}_3'; m)$,
where $m|_{\mathbb{Z}_2}$, $m|_{\mathbb{Z}_3'}$ are $x - y + z$, $m/_{\mathbb{Z}_2|\mathbb{Z}_3'}$ is minority.

Observation

$\mathbf{A}$ has $(k_1, \ldots, k_n)$-WBS terms for all $k_1, \ldots, k_n$ coprime with 2 and 3.

---

1. Calculate symmetric operation on the first block modulo $\{\mathbb{Z}_2 \mid \mathbb{Z}_3'\}$.

$$w(\underbrace{1, 2', 0', 0', 1}, \underbrace{2', 1', 2', 2', 2'}, \underbrace{1, 1, 1, 1, 1}, \underbrace{0', 1, 0, 0, 1}) =$$

$$w(\underbrace{0, 2', 0', 1, 0}, \underbrace{1', 2', 1'.2', 1'}, \underbrace{0, 1, 1, 1, 1}, \underbrace{0, 0, 0, 0, 0}) =$$

Weak block symmetric term operations

$\mathbb{Z}_2 \cup \mathbb{Z}_3$

$\mathbf{A} = (\mathbb{Z}_2 \cup \mathbb{Z}_3'; m)$,
where $m|_{\mathbb{Z}_2}$, $m|_{\mathbb{Z}_3'}$ are $x - y + z$, $m/_{\mathbb{Z}_2|\mathbb{Z}_3'}$ is minority.

Observation

$\mathbf{A}$ has $(k_1, \ldots, k_n)$-WBS terms for all $k_1, \ldots, k_n$ coprime with 2 and 3.

---

1. Calculate symmetric operation on the first block modulo $\{\mathbb{Z}_2 \mid \mathbb{Z}_3'\}$.

$$w(\underbrace{1, 2', 0', 0', 1}, \underbrace{2', 1', 2', 2', 2'}, \underbrace{1, 1, 1, 1, 1}, \underbrace{0', 1, 0, 0, 1}) = \ '$$

$$w(\underbrace{0, 2', 0', 1, 0}, \underbrace{1', 2', 1'.2', 1'}, \underbrace{0, 1, 1, 1, 1}, \underbrace{0, 0, 0, 0, 0}) =$$

Weak block symmetric term operations

$\mathbb{Z}_2 \cup \mathbb{Z}_3$

$\mathbf{A} = (\mathbb{Z}_2 \cup \mathbb{Z}_3'; m)$,
where $m|_{\mathbb{Z}_2}$, $m|_{\mathbb{Z}_3'}$ are $x - y + z$, $m/_{\mathbb{Z}_2|\mathbb{Z}_3'}$ is minority.

Observation

$\mathbf{A}$ has $(k_1, \ldots, k_n)$-WBS terms for all $k_1, \ldots, k_n$ coprime with 2 and 3.

---

1. Calculate symmetric operation on the first block modulo $\{\mathbb{Z}_2 \mid \mathbb{Z}_3'\}$.
2. Choose the first block with elements from the corresponding $\mathbb{Z}_2$ or $\mathbb{Z}_3'$.

$$w(\underbrace{1, 2', 0', 0', 1}, \underbrace{2', 1', 2', 2', 2'}, \underbrace{1, 1, 1, 1, 1}, \underbrace{0', 1, 0, 0, 1}) = \ '$$

$$w(\underbrace{0, 2', 0', 1, 0}, \underbrace{1', 2', 1'.2', 1'}, \underbrace{0, 1, 1, 1, 1}, \underbrace{0, 0, 0, 0, 0}) =$$

Weak block symmetric term operations

$\mathbb{Z}_2 \cup \mathbb{Z}_3$

$\mathbf{A} = (\mathbb{Z}_2 \cup \mathbb{Z}_3'; m)$,
where $m|_{\mathbb{Z}_2}$, $m|_{\mathbb{Z}_3'}$ are $x - y + z$, $m/_{\mathbb{Z}_2|\mathbb{Z}_3'}$ is minority.

Observation

A has $(k_1, \ldots, k_n)$-WBS terms for all $k_1, \ldots, k_n$ coprime with 2 and 3.

---

1. Calculate symmetric operation on the first block modulo $\{\mathbb{Z}_2 \mid \mathbb{Z}_3'\}$.
2. Choose the first block with elements from the corresponding $\mathbb{Z}_2$ or $\mathbb{Z}_3'$.

$$w(\underbrace{1, 2', 0', 0', 1}, \underbrace{2', 1', 2', 2', 2'}, \underbrace{1, 1, 1, 1, 1}, \underbrace{0', 1, 0, 0, 1}) = \ '$$

$$w(\underbrace{0, 2', 0', 1, 0}, \underbrace{1', 2', 1'.2', 1'}, \underbrace{0, 1, 1, 1, 1}, \underbrace{0, 0, 0, 0, 0}) =$$

Weak block symmetric term operations

$\mathbb{Z}_2 \cup \mathbb{Z}_3$

$\mathbf{A} = (\mathbb{Z}_2 \cup \mathbb{Z}_3'; m)$,
where $m|_{\mathbb{Z}_2}$, $m|_{\mathbb{Z}_3'}$ are $x - y + z$, $m/_{\mathbb{Z}_2|\mathbb{Z}_3'}$ is minority.

Observation

$\mathbf{A}$ has $(k_1, \ldots, k_n)$-WBS terms for all $k_1, \ldots, k_n$ coprime with 2 and 3.

---

1. Calculate symmetric operation on the first block modulo $\{\mathbb{Z}_2 \mid \mathbb{Z}_3'\}$.
2. Choose the first block with elements from the corresponding $\mathbb{Z}_2$ or $\mathbb{Z}_3'$.
3. Calculate symmetric operation for the corresponding block.

$$w(\underbrace{1, 2', 0', 0', 1}, \underbrace{2', 1', 2', 2', 2'}, \underbrace{1, 1, 1, 1, 1}, \underbrace{0', 1, 0, 0, 1}) = \ '$$

$$w(\underbrace{0, 2', 0', 1, 0}, \underbrace{1', 2', 1'.2', 1'}, \underbrace{0, 1, 1, 1, 1}, \underbrace{0, 0, 0, 0, 0}) =$$

Weak block symmetric term operations

$\mathbb{Z}_2 \cup \mathbb{Z}_3$

$\mathbf{A} = (\mathbb{Z}_2 \cup \mathbb{Z}_3'; m)$,
where $m|_{\mathbb{Z}_2}$, $m|_{\mathbb{Z}_3'}$ are $x - y + z$, $m/_{\mathbb{Z}_2|\mathbb{Z}_3'}$ is minority.

Observation

$\mathbf{A}$ has $(k_1, \dots, k_n)$-WBS terms for all $k_1, \dots, k_n$ coprime with 2 and 3.

---

1. Calculate symmetric operation on the first block modulo $\{\mathbb{Z}_2 \mid \mathbb{Z}_3'\}$.
2. Choose the first block with elements from the corresponding $\mathbb{Z}_2$ or $\mathbb{Z}_3'$.
3. Calculate symmetric operation for the corresponding block.

$$w(\underbrace{1, 2', 0', 0', 1}, \underbrace{2', 1', 2', 2', 2'}, \underbrace{1, 1, 1, 1, 1}, \underbrace{0', 1, 0, 0, 1}) = 1'$$

$$w(\underbrace{0, 2', 0', 1, 0}, \underbrace{1', 2', 1'.2', 1'}, \underbrace{0, 1, 1, 1, 1}, \underbrace{0, 0, 0, 0, 0}) = 0$$

Some results

Some results

Theorem (Bounded width $\Rightarrow$ WBS)

A *has WNU of all arities*

$\Rightarrow$ **A** *has* $(k_1, \ldots, k_n)$*-WBS for all* $k_1, \ldots, k_n$.

Some results

Theorem (Bounded width ⇒ WBS)

$\mathbf{A}$ *has WNU of all arities*

$\Rightarrow \mathbf{A}$ *has* $(k_1, \ldots, k_n)$*-WBS for all* $k_1, \ldots, k_n$.

Theorem (small algebras have WBS)

$\mathbf{A} = \mathbf{B}_1 \times \cdots \times \mathbf{B}_s$, *where* $|B_i| \leq 5$ *for all* i

$\Rightarrow \mathbf{A}$ *has* $(\underbrace{p, \ldots, p}_{n})$*-WBS for all primes* $p > 5$ *and* $n \in \mathbb{N}$.

Some results

Theorem (Bounded width $\Rightarrow$ WBS)

$\mathbf{A}$ *has WNU of all arities*

$$\Rightarrow \mathbf{A} \text{ has } (k_1, \dots, k_n)\text{-WBS for all } k_1, \dots, k_n.$$

Theorem (small algebras have WBS)

$\mathbf{A} = \mathbf{B}_1 \times \cdots \times \mathbf{B}_s$, *where* $|B_i| \leq 5$ *for all* i

$$\Rightarrow \mathbf{A} \text{ has } (\underbrace{p, \dots, p}_{n})\text{-WBS for all primes } p > 5 \text{ and } n \in \mathbb{N}.$$

Conjecture

Every finite idempotent Taylor algebra $\mathbf{A}$ *has* $(p, p, \dots, p)$*-WBS term operation for any prime* $p > |A|$.

Constraint Satisfaction Problem

Γ is a set of relation on A.

CSP(Γ)

Given: a conjunction of relations, i.e. a formula

$$R_1(x_{i_{1,1}}, \dots, x_{i_{1,n_1}}) \wedge \cdots \wedge R_s(x_{i_{s,1}}, \dots, x_{i_{s,n_s}}),$$

where $R_1, \dots, R_s \in \Gamma$.
Decide: whether the formula is satisfiable.

Constraint Satisfaction Problem

Γ is a set of relation on A.

CSP(Γ)

Given: a conjunction of relations, i.e. a formula

$$R_1(x_{i_{1,1}}, \dots, x_{i_{1,n_1}}) \wedge \dots \wedge R_s(x_{i_{s,1}}, \dots, x_{i_{s,n_s}}),$$

where $R_1, \dots, R_s \in \Gamma$.
Decide: whether the formula is satisfiable.

Theorem (Bulatov, Zhuk, 2017)

CSP(Γ) *is solvable in polynomial time if* Pol(Γ) *has a WNU operation;* CSP(Γ) *is NP-complete otherwise.*

Constraint Satisfaction Problem

Γ is a set of relation on A.

CSP(Γ)

Given: a conjunction of relations, i.e. a formula

$$R_1(x_{i_{1,1}},\dots,x_{i_{1,n_1}}) \wedge \dots \wedge R_s(x_{i_{s,1}},\dots,x_{i_{s,n_s}}),$$

where $R_1,\dots,R_s \in \Gamma$.
Decide: whether the formula is satisfiable.

Theorem (Bulatov, Zhuk, 2017)

CSP(Γ) *is solvable in polynomial time if* Pol(Γ) *has a WNU operation;* CSP(Γ) *is NP-complete otherwise.*

Is it the end of the story?

Constraint Satisfaction Problem

Γ is a set of relation on A.

CSP(Γ)

Given: a conjunction of relations, i.e. a formula

$$R_1(x_{i_{1,1}}, \dots, x_{i_{1,n_1}}) \wedge \dots \wedge R_s(x_{i_{s,1}}, \dots, x_{i_{s,n_s}}),$$

where $R_1, \dots, R_s \in \Gamma$.
Decide: whether the formula is satisfiable.

Theorem (Bulatov, Zhuk, 2017)

CSP(Γ) *is solvable in polynomial time if* Pol(Γ) *has a WNU operation;* CSP(Γ) *is NP-complete otherwise.*

Is it the end of the story? NO!

Constraint Satisfaction Problem

Γ is a set of relation on A.

CSP(Γ)

Given: a conjunction of relations, i.e. a formula

$$R_1(x_{i_{1,1}}, \ldots, x_{i_{1,n_1}}) \wedge \cdots \wedge R_s(x_{i_{s,1}}, \ldots, x_{i_{s,n_s}}),$$

where $R_1, \ldots, R_s \in \Gamma$.
Decide: whether the formula is satisfiable.

Theorem (Bulatov, Zhuk, 2017)

CSP(Γ) *is solvable in polynomial time if* Pol(Γ) *has a WNU operation;* CSP(Γ) *is NP-complete otherwise.*

Is it the end of the story? NO!

- both algorithm are not universal (work only for a fixed domain). We are looking for the universal algorithm!

Constraint Satisfaction Problem

Γ is a set of relation on A.

CSP(Γ)

Given: a conjunction of relations, i.e. a formula

$$R_1(x_{i_{1,1}}, \dots, x_{i_{1,n_1}}) \wedge \dots \wedge R_s(x_{i_{s,1}}, \dots, x_{i_{s,n_s}}),$$

where $R_1, \dots, R_s \in \Gamma$.
Decide: whether the formula is satisfiable.

Theorem (Bulatov, Zhuk, 2017)

CSP(Γ) *is solvable in polynomial time if* Pol(Γ) *has a WNU operation;* CSP(Γ) *is NP-complete otherwise.*

Is it the end of the story? NO!

- both algorithm are not universal (work only for a fixed domain). We are looking for the universal algorithm!
- the complexity of Promise CSP is widely open.

Linear Programming

Linear Programming

- Take an instance $\mathcal{I}$ with variables $x_1, \ldots, x_n$, their respective domains $D_1, \ldots, D_n$, and constraints $C_1, \ldots, C_t$.

Linear Programming

- Take an instance $\mathcal{I}$ with variables $x_1, \ldots, x_n$, their respective domains $D_1, \ldots, D_n$, and constraints $C_1, \ldots, C_t$.
- Build a new instance

Linear Programming

- Take an instance $\mathcal{I}$ with variables $x_1, \ldots, x_n$, their respective domains $D_1, \ldots, D_n$, and constraints $C_1, \ldots, C_t$.
- Build a new instance with variables:

Linear Programming

- Take an instance $\mathcal{I}$ with variables $x_1, \dots, x_n$, their respective domains $D_1, \dots, D_n$, and constraints $C_1, \dots, C_t$.
- Build a new instance with variables:
  x_i^a for every i and $a \in D_i$,

Linear Programming

- Take an instance $\mathcal{I}$ with variables $x_1, \ldots, x_n$, their respective domains $D_1, \ldots, D_n$, and constraints $C_1, \ldots, C_t$.
- Build a new instance with variables:
  x_i^a for every i and $a \in D_i$,
  C_j^α for every j and a tuple α.

Linear Programming

- Take an instance $\mathcal{I}$ with variables $x_1, \ldots, x_n$, their respective domains $D_1, \ldots, D_n$, and constraints $C_1, \ldots, C_t$.
- Build a new instance with variables:
  x_i^a for every i and $a \in D_i$,
  C_j^α for every j and a tuple α.
  and constraints:

Linear Programming

- Take an instance $\mathcal{I}$ with variables $x_1, \ldots, x_n$, their respective domains $D_1, \ldots, D_n$, and constraints $C_1, \ldots, C_t$.
- Build a new instance with variables:
  x_i^a for every i and $a \in D_i$,
  C_j^α for every j and a tuple α.
  and constraints:
  $\sum_\alpha C_j^\alpha = 1$ for every j

Linear Programming

- Take an instance $\mathcal{I}$ with variables $x_1, \ldots, x_n$, their respective domains $D_1, \ldots, D_n$, and constraints $C_1, \ldots, C_t$.
- Build a new instance with variables:
  x_i^a for every i and $a \in D_i$,
  C_j^α for every j and a tuple α.
  and constraints:
  $\sum_\alpha C_j^\alpha = 1$ for every j
  $\sum_{\alpha\colon \alpha(k)=a} C_j^\alpha = x_i^a$ for every $C_j = R(\underbrace{\ldots, x_i}_{k}, \ldots)$ and $a \in D_i$

Linear Programming

- Take an instance $\mathcal{I}$ with variables $x_1, \ldots, x_n$, their respective domains $D_1, \ldots, D_n$, and constraints $C_1, \ldots, C_t$.
- Build a new instance with variables:
  x_i^a for every i and $a \in D_i$,
  C_j^α for every j and a tuple α.
  and constraints:
  $\sum_\alpha C_j^\alpha = 1$ for every j
  $\sum_{\alpha\colon\, \alpha(k)=a} C_j^\alpha = x_i^a$ for every $C_j = R(\underbrace{\ldots, x_i}_{k}, \ldots)$ and $a \in D_i$

Solve in $\{0,1\}$: is equivalent to the original instance (usually NP-hard)

Linear Programming

- Take an instance $\mathcal{I}$ with variables $x_1, \ldots, x_n$, their respective domains $D_1, \ldots, D_n$, and constraints $C_1, \ldots, C_t$.
- Build a new instance with variables:
  x_i^a for every i and $a \in D_i$,
  C_j^α for every j and a tuple α.
  and constraints:
  $\sum_{\alpha} C_j^\alpha = 1$ for every j
  $\sum_{\alpha\colon\, \alpha(k)=a} C_j^\alpha = x_i^a$ for every $C_j = R(\underbrace{\ldots, x_i}_{k}, \ldots)$ and $a \in D_i$

Solve in $\{0,1\}$: is equivalent to the original instance (usually NP-hard)

Solve in $\mathbb{Q} \cap [0,1]$: Basic Linear Programming (BLP) (tractable)

Linear Programming

- Take an instance $\mathcal{I}$ with variables $x_1, \ldots, x_n$, their respective domains $D_1, \ldots, D_n$, and constraints $C_1, \ldots, C_t$.
- Build a new instance with variables:
  x_i^a for every i and $a \in D_i$,
  C_j^α for every j and a tuple α.
  and constraints:
  $\sum_\alpha C_j^\alpha = 1$ for every j
  $\sum_{\alpha:\ \alpha(k)=a} C_j^\alpha = x_i^a$ for every $C_j = R(\underbrace{\ldots, x_i}_{k}, \ldots)$ and $a \in D_i$

Solve in $\{0,1\}$: is equivalent to the original instance (usually NP-hard)

Solve in $\mathbb{Q} \cap [0,1]$: Basic Linear Programming (BLP) (tractable)

Solve in $\mathbb{Z}$: Affine Integer Programming (AIP) (tractable)

Linear Programming

- Take an instance $\mathcal{I}$ with variables $x_1, \dots, x_n$, their respective domains $D_1, \dots, D_n$, and constraints $C_1, \dots, C_t$.
- Build a new instance with variables:
  x_i^a for every i and $a \in D_i$,
  C_j^α for every j and a tuple α.
  and constraints:
  $\sum_\alpha C_j^\alpha = 1$ for every j
  $\sum_{\alpha:\, \alpha(k)=a} C_j^\alpha = x_i^a$ for every $C_j = R(\underbrace{\dots, x_i}_{k}, \dots)$ and $a \in D_i$

Solve in $\{0,1\}$: is equivalent to the original instance (usually NP-hard)

Solve in $\mathbb{Q} \cap [0,1]$: Basic Linear Programming (BLP) (tractable)

Solve in $\mathbb{Z}$: Affine Integer Programming (AIP) (tractable)

BLP solves CSP(Γ) $\Leftrightarrow$ Pol(Γ) has all symmetric operations

Linear Programming

- Take an instance $\mathcal{I}$ with variables $x_1, \ldots, x_n$, their respective domains $D_1, \ldots, D_n$, and constraints $C_1, \ldots, C_t$.
- Build a new instance with variables:
  x_i^a for every i and $a \in D_i$,
  C_j^α for every j and a tuple α.
  and constraints:
  $\sum_\alpha C_j^\alpha = 1$ for every j
  $\sum_{\alpha:\ \alpha(k)=a} C_j^\alpha = x_i^a$ for every $C_j = R(\underbrace{\ldots, x_i}_{k}, \ldots)$ and $a \in D_i$

Solve in $\{0,1\}$: is equivalent to the original instance (usually NP-hard)

Solve in $\mathbb{Q} \cap [0,1]$: Basic Linear Programming (BLP) (tractable)

Solve in $\mathbb{Z}$: Affine Integer Programming (AIP) (tractable)

BLP solves CSP(Γ) $\Leftrightarrow$ Pol(Γ) has all symmetric operations
AIP solves CSP(Γ) $\Leftrightarrow$ Pol(Γ) has alternating operations

Linear Programming

- Take an instance $\mathcal{I}$ with variables $x_1, \ldots, x_n$, their respective domains $D_1, \ldots, D_n$, and constraints $C_1, \ldots, C_t$.
- Build a new instance with variables:
  x_i^a for every i and $a \in D_i$,
  C_j^α for every j and a tuple α.
  and constraints:
  $\sum_\alpha C_j^\alpha = 1$ for every j
  $\sum_{\alpha:\ \alpha(k)=a} C_j^\alpha = x_i^a$ for every $C_j = R(\underbrace{\ldots, x_i}_{k}, \ldots)$ and $a \in D_i$

Solve in $\{0,1\}$: is equivalent to the original instance (usually NP-hard)

Solve in $\mathbb{Q} \cap [0,1]$: Basic Linear Programming (BLP) (tractable)

Solve in $\mathbb{Z}$: Affine Integer Programming (AIP) (tractable)

BLP solves CSP(Γ) $\Leftrightarrow$ Pol(Γ) has all symmetric operations
AIP solves CSP(Γ) $\Leftrightarrow$ Pol(Γ) has alternating operations
BLP+AIP solves CSP(Γ) $\Leftrightarrow$ Pol(Γ) has block symmetric operations with any minimal block size.

Algorithm: singleton (BLP + AIP)

Algorithm: singleton (BLP + AIP)

repeat
    for every x_i and $a \in D_i$
        if $\neg(\mathrm{BLP} + \mathrm{AIP})(\mathcal{I} \wedge x_i = a)$
            $D_i := D_i \setminus \{a\}$
        if $D_i = \varnothing$
            return No
until Nothing Changed.
return Yes

Algorithm: singleton (BLP + AIP)

repeat

  for every x_i and $a \in D_i$

    if $\neg(\mathrm{BLP} + \mathrm{AIP})(\mathcal{I} \wedge x_i = a)$

      $D_i := D_i \setminus \{a\}$

    if $D_i = \varnothing$

      return No

until Nothing Changed.

return Yes

Observation

singleton (BLP + AIP) solves CSP(Γ) whenever for every $m \in \mathbb{N}$ Pol(Γ) has a $(k_1, \ldots, k_n)$-WBS, where $k_1, \ldots, k_n \geq m$.

Algorithm: singleton (BLP + AIP)

repeat
  for every x_i and $a \in D_i$
    if $\neg(\mathrm{BLP} + \mathrm{AIP})(\mathcal{I} \wedge x_i = a)$
      $D_i := D_i \setminus \{a\}$
    if $D_i = \varnothing$
      return No
until Nothing Changed.
return Yes

Observation

singleton (BLP + AIP) solves CSP(Γ) whenever for every $m \in \mathbb{N}$ Pol(Γ) has a $(k_1, \ldots, k_n)$-WBS, where $k_1, \ldots, k_n \geq m$.

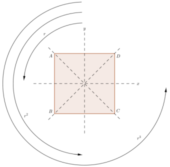
Corollary

Singleton (BLP + AIP) solves all (multi-sorted) tractable CSP for domain of size at most 5.

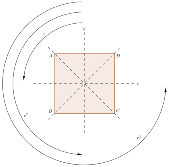
Counter Example provided by Michael Kompatscher

Counter Example provided by Michael Kompatscher

The dihedral group $\mathbf{D}_4$:
symmetry group of a square.



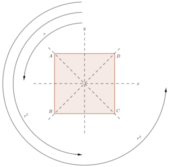
Counter Example provided by Michael Kompatscher



The dihedral group $\mathbf{D}_4$: symmetry group of a square.

representation: $\mathbf{D}_4 = \langle r, s \mid r^4 = s^2 = 1, rs = sr^3 \rangle$

Counter Example provided by Michael Kompatscher



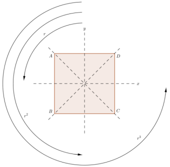
The dihedral group $\mathbf{D}_4$: symmetry group of a square.

representation: $\mathbf{D}_4 = \langle r, s \mid r^4 = s^2 = 1, rs = sr^3 \rangle$

identities: $x^4 \approx 1$, $[x, y]^2 \approx 1$, $[x, y]z = z[x, y]$.

Counter Example provided by Michael Kompatscher

The dihedral group $\mathbf{D}_4$: symmetry group of a square.



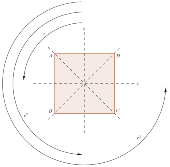
representation: $\mathbf{D}_4 = \langle r, s \mid r^4 = s^2 = 1, rs = sr^3 \rangle$

identities: $x^4 \approx 1$, $[x,y]^2 \approx 1$, $[x,y]z = z[x,y]$.

Observation

Every term operation can be represented in a normal form:
$x_1^{a_1} \dots x_m^{a_m} \cdot \bigwedge_{i,j} [x_i, x_j]^{c_{i,j}}$, where $a_i \in \{0,1,2,3\}$ and $c_{i,j} \in \{0,1\}$

Counter Example provided by Michael Kompatscher



The dihedral group $\mathbf{D}_4$: symmetry group of a square.

representation: $\mathbf{D}_4 = \langle r, s \mid r^4 = s^2 = 1, rs = sr^3 \rangle$

identities: $x^4 \approx 1$, $[x, y]^2 \approx 1$, $[x, y]z = z[x, y]$.

Observation

Every term operation can be represented in a normal form: $x_1^{a_1} \dots x_m^{a_m} \cdot \bigwedge_{i,j} [x_i, x_j]^{c_{i,j}}$, where $a_i \in \{0, 1, 2, 3\}$ and $c_{i,j} \in \{0, 1\}$

Lemma (Kompatscher, 2024)

$\mathbf{D}_4$ has no idempotent $(k_1, \dots, k_n)$-WBS for any $k_1, \dots, k_n \geq 3$ and $n \geq 4$.

Counter Example provided by Michael Kompatscher

Lemma (Kompatscher, 2024)

$\mathbf{D}_4$ *has no idempotent* $(k_1, \ldots, k_n)$*-WBS for any* $k_1, \ldots, k_n \geq 3$ *and* $n \geq 4$.

proof:

Counter Example provided by Michael Kompatscher

Lemma (Kompatscher, 2024)

$\mathbf{D}_4$ *has no idempotent* $(k_1, \ldots, k_n)$*-WBS for any* $k_1, \ldots, k_n \geq 3$ *and* $n \geq 4$.

proof:

Suppose f is $(k_1, \ldots, k_n)$-WBS of arity $m = \sum_{i=1}^{n} k_n$

Counter Example provided by Michael Kompatscher

Lemma (Kompatscher, 2024)

$\mathbf{D}_4$ *has no idempotent* $(k_1, \ldots, k_n)$*-WBS for any* $k_1, \ldots, k_n \geq 3$ *and* $n \geq 4$*.*

proof:

Suppose f is $(k_1, \ldots, k_n)$-WBS of arity $m = \sum_{i=1}^{n} k_n$

$f(x_1, \ldots, x_m) = x_1^{a_1} \ldots x_m^{a_m} \cdot \bigwedge_{i,j} [x_i, x_j]^{c_{i,j}},$

Counter Example provided by Michael Kompatscher

Lemma (Kompatscher, 2024)

$\mathbf{D}_4$ *has no idempotent* $(k_1, \ldots, k_n)$*-WBS for any* $k_1, \ldots, k_n \geq 3$ *and* $n \geq 4$.

proof:

Suppose f is $(k_1, \ldots, k_n)$-WBS of arity $m = \sum_{i=1}^{n} k_n$

$f(x_1, \ldots, x_m) = x_1^{a_1} \ldots x_m^{a_m} \cdot \bigwedge_{i,j} [x_i, x_j]^{c_{i,j}}$,

$\mathbf{D}_4|_{\{1,r,r^2,r^3\}} \cong \mathbf{Z}_4 \Rightarrow a_1 = \cdots = a_{k_1},\ a_{k_1+1} = \cdots = a_{k_1+k_2}, \ldots$

Counter Example provided by Michael Kompatscher

Lemma (Kompatscher, 2024)

$\mathbf{D}_4$ *has no idempotent* $(k_1, \ldots, k_n)$*-WBS for any* $k_1, \ldots, k_n \geq 3$ *and* $n \geq 4$.

proof:

Suppose f is $(k_1, \ldots, k_n)$-WBS of arity $m = \sum_{i=1}^{n} k_n$

$f(x_1, \ldots, x_m) = x_1^{a_1} \ldots x_m^{a_m} \cdot \bigwedge_{i,j} [x_i, x_j]^{c_{i,j}}$,

$\mathbf{D}_4|_{\{1,r,r^2,r^3\}} \cong \mathbf{Z}_4 \Rightarrow a_1 = \cdots = a_{k_1}, a_{k_1+1} = \cdots = a_{k_1+k_2}, \ldots$

f is idempotent $\Rightarrow$ WLOG $a_1 = \cdots = a_{k_1} = 1$

Counter Example provided by Michael Kompatscher

Lemma (Kompatscher, 2024)

$\mathbf{D}_4$ *has no idempotent* $(k_1, \ldots, k_n)$*-WBS for any* $k_1, \ldots, k_n \geq 3$ *and* $n \geq 4$.

proof:

Suppose f is $(k_1, \ldots, k_n)$-WBS of arity $m = \sum_{i=1}^{n} k_n$

$f(x_1, \ldots, x_m) = x_1^{a_1} \ldots x_m^{a_m} \cdot \bigwedge_{i,j} [x_i, x_j]^{c_{i,j}}$,

$\mathbf{D}_4|_{\{1,r,r^2,r^3\}} \cong \mathbf{Z}_4 \Rightarrow a_1 = \cdots = a_{k_1}$, $a_{k_1+1} = \cdots = a_{k_1+k_2}, \ldots$

f is idempotent $\Rightarrow$ WLOG $a_1 = \cdots = a_{k_1} = 1$

$$f(\underbrace{1, \ldots, 1, r, 1, \ldots, 1, s, 1, \ldots, 1}_{k_1}, \underbrace{r, \ldots, r}_{k_2}, \underbrace{s, \ldots, s}_{k_3}, \underbrace{1, \ldots, 1}_{k_4}, \ldots)$$

Counter Example provided by Michael Kompatscher

Lemma (Kompatscher, 2024)

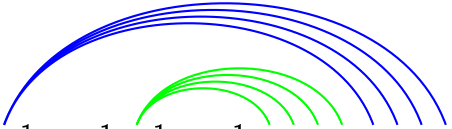
$\mathbf{D}_4$ *has no idempotent* $(k_1, \ldots, k_n)$*-WBS for any* $k_1, \ldots, k_n \geq 3$ *and* $n \geq 4$*.*

proof:

Suppose f is $(k_1, \ldots, k_n)$-WBS of arity $m = \sum_{i=1}^{n} k_n$

$f(x_1, \ldots, x_m) = x_1^{a_1} \ldots x_m^{a_m} \cdot \bigwedge_{i,j} [x_i, x_j]^{c_{i,j}},$

$\mathbf{D}_4|_{\{1,r,r^2,r^3\}} \cong \mathbf{Z}_4 \Rightarrow a_1 = \cdots = a_{k_1},\ a_{k_1+1} = \cdots = a_{k_1+k_2}, \ldots$
f is idempotent $\Rightarrow$ WLOG $a_1 = \cdots = a_{k_1} = 1$



$$f(\underbrace{1, \ldots, 1, r, 1, \ldots, 1, s, 1, \ldots, 1}_{k_1}, \underbrace{r, \ldots, r}_{k_2}, \underbrace{s, \ldots, s}_{k_3}, \underbrace{1, \ldots, 1}_{k_4}, \ldots)$$

Counter Example provided by Michael Kompatscher

Lemma (Kompatscher, 2024)

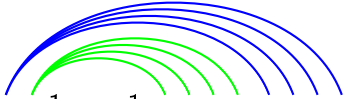
$\mathbf{D}_4$ *has no idempotent* $(k_1, \ldots, k_n)$*-WBS for any* $k_1, \ldots, k_n \geq 3$ *and* $n \geq 4$.

proof:

Suppose f is $(k_1, \ldots, k_n)$-WBS of arity $m = \sum_{i=1}^{n} k_n$

$f(x_1, \ldots, x_m) = x_1^{a_1} \ldots x_m^{a_m} \cdot \bigwedge_{i,j} [x_i, x_j]^{c_{i,j}}$,

$\mathbf{D}_4|_{\{1,r,r^2,r^3\}} \cong \mathbf{Z}_4 \Rightarrow a_1 = \cdots = a_{k_1},\ a_{k_1+1} = \cdots = a_{k_1+k_2}, \ldots$
f is idempotent $\Rightarrow$ WLOG $a_1 = \cdots = a_{k_1} = 1$



$$f(\underbrace{1, \ldots, 1, 1, \ldots, 1, r, s, 1, \ldots, 1}_{k_1}, \underbrace{r, \ldots, r}_{k_2}, \underbrace{s, \ldots, s}_{k_3}, \underbrace{1, \ldots, 1}_{k_4}, \ldots)$$

Counter Example provided by Michael Kompatscher

Lemma (Kompatscher, 2024)

$\mathbf{D}_4$ *has no idempotent* $(k_1, \ldots, k_n)$*-WBS for any* $k_1, \ldots, k_n \geq 3$ *and* $n \geq 4$.

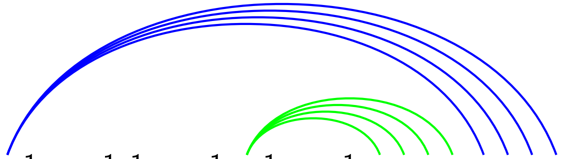
proof:

Suppose f is $(k_1, \ldots, k_n)$-WBS of arity $m = \sum_{i=1}^{n} k_n$

$f(x_1, \ldots, x_m) = x_1^{a_1} \ldots x_m^{a_m} \cdot \bigwedge_{i,j} [x_i, x_j]^{c_{i,j}}$,

$\mathbf{D}_4|_{\{1,r,r^2,r^3\}} \cong \mathbf{Z}_4 \Rightarrow a_1 = \cdots = a_{k_1},\ a_{k_1+1} = \cdots = a_{k_1+k_2}, \ldots$

f is idempotent $\Rightarrow$ WLOG $a_1 = \cdots = a_{k_1} = 1$



$$f(\underbrace{r, 1, \ldots, 1, 1, \ldots, 1, s, 1, \ldots, 1}_{k_1}, \underbrace{r, \ldots, r}_{k_2}, \underbrace{s, \ldots, s}_{k_3}, \underbrace{1, \ldots, 1}_{k_4}, \ldots)$$

Counter Example provided by Michael Kompatscher

Lemma (Kompatscher, 2024)

$\mathbf{D}_4$ *has no idempotent* $(k_1, \ldots, k_n)$*-WBS for any* $k_1, \ldots, k_n \geq 3$ *and* $n \geq 4$.

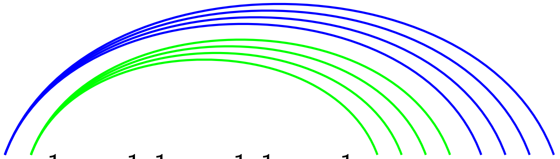
proof:

Suppose f is $(k_1, \ldots, k_n)$-WBS of arity $m = \sum_{i=1}^{n} k_n$

$$f(x_1, \ldots, x_m) = x_1^{a_1} \ldots x_m^{a_m} \cdot \bigwedge_{i,j} [x_i, x_j]^{c_{i,j}},$$

$\mathbf{D}_4|_{\{1,r,r^2,r^3\}} \cong \mathbf{Z}_4 \Rightarrow a_1 = \cdots = a_{k_1},\ a_{k_1+1} = \cdots = a_{k_1+k_2}, \ldots$

f is idempotent $\Rightarrow$ WLOG $a_1 = \cdots = a_{k_1} = 1$



$$f(\underbrace{r, s, 1, \ldots, 1, 1, \ldots, 1, 1, \ldots, 1}_{k_1}, \underbrace{r, \ldots, r}_{k_2}, \underbrace{s, \ldots, s}_{k_3}, \underbrace{1, \ldots, 1}_{k_4}, \ldots)$$

Counter Example provided by Michael Kompatscher

Lemma (Kompatscher, 2024)

$\mathbf{D}_4$ *has no idempotent* $(k_1, \ldots, k_n)$*-WBS for any* $k_1, \ldots, k_n \geq 3$ *and* $n \geq 4$.

proof:

Suppose f is $(k_1, \ldots, k_n)$-WBS of arity $m = \sum_{i=1}^{n} k_n$

$f(x_1, \ldots, x_m) = x_1^{a_1} \ldots x_m^{a_m} \cdot \bigwedge_{i,j} [x_i, x_j]^{c_{i,j}},$

$\mathbf{D}_4|_{\{1,r,r^2,r^3\}} \cong \mathbf{Z}_4 \Rightarrow a_1 = \cdots = a_{k_1},\ a_{k_1+1} = \cdots = a_{k_1+k_2}, \ldots$

f is idempotent $\Rightarrow$ WLOG $a_1 = \cdots = a_{k_1} = 1$

$$f(\underbrace{r, 1, \ldots, 1, 1, \ldots, 1, 1, \ldots, 1, s}_{k_1}, \underbrace{r, \ldots, r}_{k_2}, \underbrace{s, \ldots, s}_{k_3}, \underbrace{1, \ldots, 1}_{k_4}, \ldots)$$

Counter Example provided by Michael Kompatscher

Lemma (Kompatscher, 2024)

$\mathbf{D}_4$ *has no idempotent* $(k_1, \ldots, k_n)$*-WBS for any* $k_1, \ldots, k_n \geq 3$ *and* $n \geq 4$*.*

proof:

Suppose f is $(k_1, \ldots, k_n)$-WBS of arity $m = \sum_{i=1}^{n} k_n$

$f(x_1, \ldots, x_m) = x_1^{a_1} \ldots x_m^{a_m} \cdot \bigwedge_{i,j} [x_i, x_j]^{c_{i,j}}$,

$\mathbf{D}_4|_{\{1,r,r^2,r^3\}} \cong \mathbf{Z}_4 \Rightarrow a_1 = \cdots = a_{k_1},\ a_{k_1+1} = \cdots = a_{k_1+k_2}, \ldots$

f is idempotent $\Rightarrow$ WLOG $a_1 = \cdots = a_{k_1} = 1$

$$f(\underbrace{1, \ldots, 1, 1, \ldots, 1, r, s, 1, \ldots, 1}_{k_1}, \underbrace{r, \ldots, r}_{k_2}, \underbrace{s, \ldots, s}_{k_3}, \underbrace{1, \ldots, 1}_{k_4}, \ldots)$$

Counter Example provided by Michael Kompatscher

Lemma (Kompatscher, 2024)

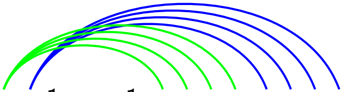
$\mathbf{D}_4$ *has no idempotent* $(k_1, \ldots, k_n)$*-WBS for any* $k_1, \ldots, k_n \geq 3$ *and* $n \geq 4$.

proof:

Suppose f is $(k_1, \ldots, k_n)$-WBS of arity $m = \sum_{i=1}^{n} k_n$

$$f(x_1, \ldots, x_m) = x_1^{a_1} \ldots x_m^{a_m} \cdot \bigwedge_{i,j} [x_i, x_j]^{c_{i,j}},$$

$\mathbf{D}_4|_{\{1,r,r^2,r^3\}} \cong \mathbf{Z}_4 \Rightarrow a_1 = \cdots = a_{k_1},\ a_{k_1+1} = \cdots = a_{k_1+k_2}, \ldots$
f is idempotent $\Rightarrow$ WLOG $a_1 = \cdots = a_{k_1} = 1$



$$f(\underbrace{, 1, \ldots, 1, 1, \ldots, 1, s, r, 1, \ldots, 1}_{k_1}, \underbrace{r, \ldots, r}_{k_2}, \underbrace{s, \ldots, s}_{k_3}, \underbrace{1, \ldots, 1}_{k_4}, \ldots)$$

Counter Example provided by Michael Kompatscher

Lemma (Kompatscher, 2024)

$\mathbf{D}_4$ *has no idempotent* $(k_1, \dots, k_n)$*-WBS for any* $k_1, \dots, k_n \geq 3$ *and* $n \geq 4$*.*

proof:

Suppose f is $(k_1, \dots, k_n)$-WBS of arity $m = \sum_{i=1}^{n} k_n$

$f(x_1, \dots, x_m) = x_1^{a_1} \dots x_m^{a_m} \cdot \bigwedge_{i,j} [x_i, x_j]^{c_{i,j}}$,

$\mathbf{D}_4|_{\{1,r,r^2,r^3\}} \cong \mathbf{Z}_4 \Rightarrow a_1 = \dots = a_{k_1},\ a_{k_1+1} = \dots = a_{k_1+k_2}, \dots$

f is idempotent $\Rightarrow$ WLOG $a_1 = \dots = a_{k_1} = 1$

$\bigwedge_{i,j} [x_i, x_j]^{c_{i,j}}$ is the same!

$$f(\underbrace{,1, \dots, 1, 1, \dots, 1, s, r, 1, \dots, 1}_{k_1}, \underbrace{r, \dots, r}_{k_2}, \underbrace{s, \dots, s}_{k_3}, \underbrace{1, \dots, 1}_{k_4}, \dots)$$

Counter Example provided by Michael Kompatscher

Lemma (Kompatscher, 2024)

$\mathbf{D}_4$ *has no idempotent* $(k_1, \ldots, k_n)$*-WBS for any* $k_1, \ldots, k_n \geq 3$ *and* $n \geq 4$.

proof:

Suppose f is $(k_1, \ldots, k_n)$-WBS of arity $m = \sum_{i=1}^{n} k_n$

$f(x_1, \ldots, x_m) = x_1^{a_1} \ldots x_m^{a_m} \cdot \bigwedge_{i,j} [x_i, x_j]^{c_{i,j}}$,

$\mathbf{D}_4|_{\{1,r,r^2,r^3\}} \cong \mathbf{Z}_4 \Rightarrow a_1 = \cdots = a_{k_1}, \; a_{k_1+1} = \cdots = a_{k_1+k_2}, \ldots$

f is idempotent $\Rightarrow$ WLOG $a_1 = \cdots = a_{k_1} = 1$

$\bigwedge_{i,j} [x_i, x_j]^{c_{i,j}}$ is the same!

$x_1^{a_1} \ldots x_m^{a_m}$ has changed!

$$f(\underbrace{1, \ldots, 1, 1, \ldots, 1, s, r, 1, \ldots, 1}_{k_1}, \underbrace{r, \ldots, r}_{k_2}, \underbrace{s, \ldots, s}_{k_3}, \underbrace{1, \ldots, 1}_{k_4}, \ldots)$$

Counter Example provided by Michael Kompatscher

Lemma (Kompatscher, 2024)

$\mathbf{D}_4$ *has no idempotent* $(k_1, \dots, k_n)$*-WBS for any* $k_1, \dots, k_n \geq 3$ *and* $n \geq 4$.

proof:

Suppose f is $(k_1, \dots, k_n)$-WBS of arity $m = \sum_{i=1}^{n} k_n$

$f(x_1, \dots, x_m) = x_1^{a_1} \dots x_m^{a_m} \cdot \bigwedge_{i,j} [x_i, x_j]^{c_{i,j}}$,

$\mathbf{D}_4|_{\{1,r,r^2,r^3\}} \cong \mathbf{Z}_4 \Rightarrow a_1 = \dots = a_{k_1}, \; a_{k_1+1} = \dots = a_{k_1+k_2}, \dots$

f is idempotent $\Rightarrow$ WLOG $a_1 = \dots = a_{k_1} = 1$

$\bigwedge_{i,j} [x_i, x_j]^{c_{i,j}}$ is the same!

$x_1^{a_1} \dots x_m^{a_m}$ has changed!

Contradiction!

$$f(\underbrace{, 1, \dots, 1, 1, \dots, 1, s, r, 1, \dots, 1}_{k_1}, \underbrace{r, \dots, r}_{k_2}, \underbrace{s, \dots, s}_{k_3}, \underbrace{1, \dots, 1}_{k_4}, \dots)$$

Summary

Summary

Conjecture

$\mathbf{A} = \mathbf{B}_1 \times \cdots \times \mathbf{B}_s$, *where* $|B_i| \leq 7$ *for all* i

$$\Rightarrow \mathbf{A} \text{ has } (\underbrace{p, \ldots, p}_{n})\text{-WBS for all primes } p > 7 \text{ and } n \in \mathbb{N}.$$

Summary

Conjecture

$\mathbf{A} = \mathbf{B}_1 \times \cdots \times \mathbf{B}_s$, *where* $|B_i| \leq 7$ *for all* i

$\Rightarrow \mathbf{A}$ *has* $(\underbrace{p, \ldots, p}_{n})$*-WBS for all primes* $p > 7$ *and* $n \in \mathbb{N}$.

This would imply that all (multi-sorted) CSP for domain of size at most 7 are solvable by singleton (BLP+AIP).

Summary

Conjecture

$\mathbf{A} = \mathbf{B}_1 \times \cdots \times \mathbf{B}_s$, *where* $|B_i| \leq 7$ *for all* i

$$\Rightarrow \mathbf{A} \text{ has } (\underbrace{p, \ldots, p}_{n})\text{-WBS for all primes } p > 7 \text{ and } n \in \mathbb{N}.$$

This would imply that all (multi-sorted) CSP for domain of size at most 7 are solvable by singleton (BLP+AIP).

Theorem

singleton (BLP+AIP) does not solve $\mathrm{CSP}(\mathrm{Inv}(\mathbf{D}_4^{idemp}))$.

Summary

Conjecture

$$\mathbf{A} = \mathbf{B}_1 \times \cdots \times \mathbf{B}_s, \text{ where } |B_i| \leq 7 \text{ for all } i \Rightarrow \mathbf{A} \text{ has } (\underbrace{p, \ldots, p}_{n})\text{-WBS for all primes } p > 7 \text{ and } n \in \mathbb{N}.$$

This would imply that all (multi-sorted) CSP for domain of size at most 7 are solvable by singleton (BLP+AIP).

Theorem

singleton (BLP+AIP) does not solve $\mathrm{CSP}(\mathrm{Inv}(\mathbf{D}_4^{idemp}))$.

Thank you for your attention