NPA Hierarchy for Quantum Isomorphism and Homomorphism Indistinguishability

Peter Zeman
https://zemanpeter.matfyz.cz

Department of Algebra, MFF UK

Algebra colloquium

Joint work with Prem Nigam Kar, David E. Roberson, and Tim Seppelt
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Graph isomorphisms and homomorphisms

Definition. A map $\varphi\colon V(G) \to V(H)$ is an **isomorphism** if $\varphi(u)\varphi(v) \in E(H) \Leftrightarrow uv \in E(G)$. We write $G \cong H$.

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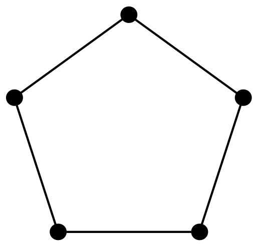

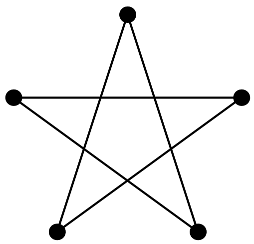

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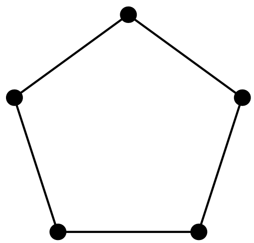

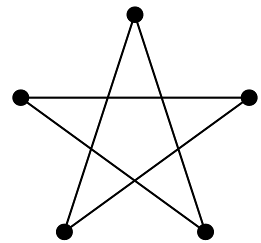

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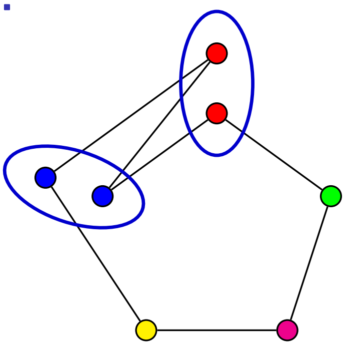
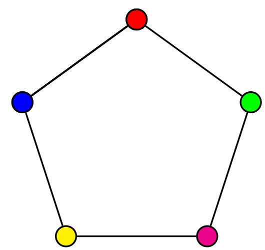


$C_7 \to C_5$



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Thm (Lovász). $G \cong H \Leftrightarrow \forall\, F \;\; \hom(F, G) = \hom(F, H)$, where $\mathbf{hom}(F, G) := \#$ of homomorphisms from F to G.

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All graphs	Isomorphism [Lovász 1967]
Cycles	Cospectrality
Cycles & paths	Cospectral & cospectral complements
Trees	Fractional isomorphism [Dvořák 2010]
Treewidth $\leqslant k$	Indistinguishable by k-WL [Dvořák 2010]
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$\mathcal{P}_k$	k^{th} level of NPA is feasible for (G, H)-iso. game

Benefits. (1) randomized poly-time algorithm for k^{th} level
(2) more elementary proof avoiding quantum groups.

Main theorem

The k^{th} level of the NPA hierarchy for the (G, H)-isomorphism game is feasible

$\Updownarrow$

There exists a level-k quantum isomorphism map from G to H

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G and H are homomorphism indistinguishable over $\mathcal{P}_k$

Nonlocal games (Bell games, Bell inequalities)

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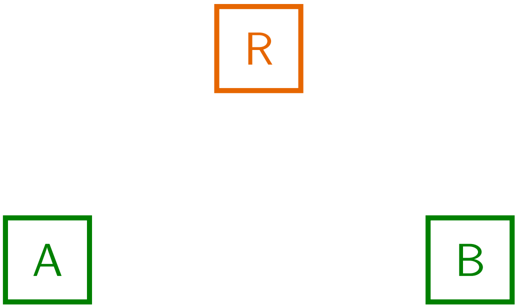

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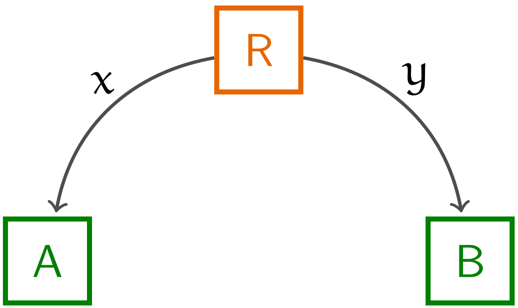

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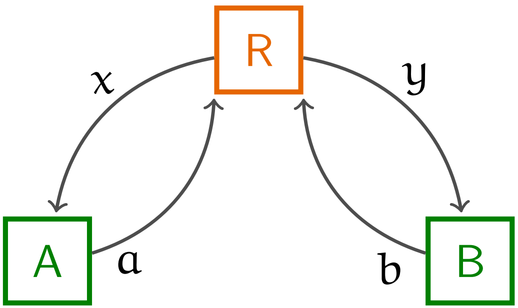

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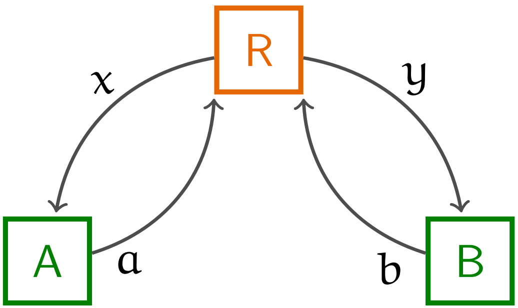

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- Players win if $V(a, b|x, y) = 1$.

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Example: Clauser, Horne, Shimony, Holt (CHSH) game

- $X = Y = A = Y = \{0, 1\}$,
- π is uniform,
- $V(a, b|x, y) = \begin{cases} 1 & \text{if } a \oplus b = x \wedge y, \\ 0 & \text{if } a \oplus b \neq x \wedge y. \end{cases}$

Strategies

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Quantum. Players share a state $|\Psi\rangle \in \mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}$.

- A POVM $\mathcal{E}_x = \{E_{xa} \in \mathbb{C}^{d_A \times d_A} : a \in A\}$, for each $x \in X$.
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The **quantum value** $\omega^*(\mathcal{G})$ of a game $\mathcal{G}$ is the supermum of

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Thm. $\omega(\mathsf{CHSH}) = 3/4 < \cos^2(\pi/8) = \omega^*(\mathsf{CHSH})$.

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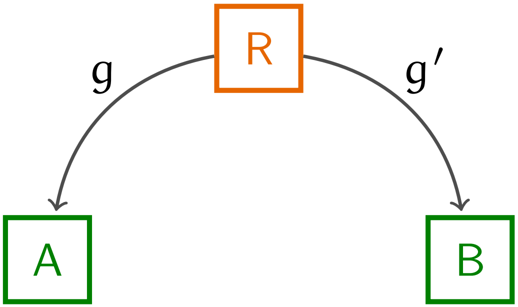

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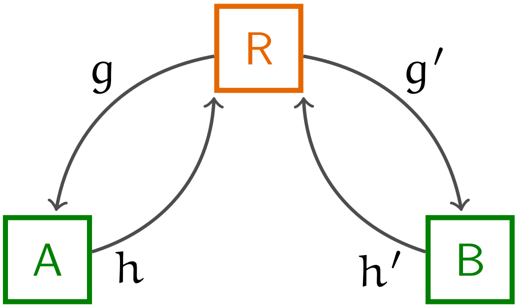

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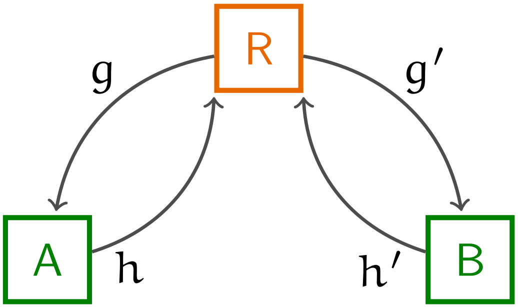

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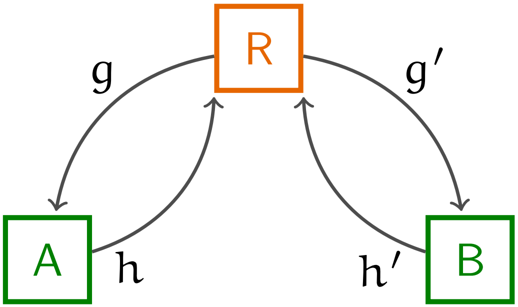

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Proposition. $G \cong H \Leftrightarrow$ Classical players can **win** the game.

Quantum isomorphism

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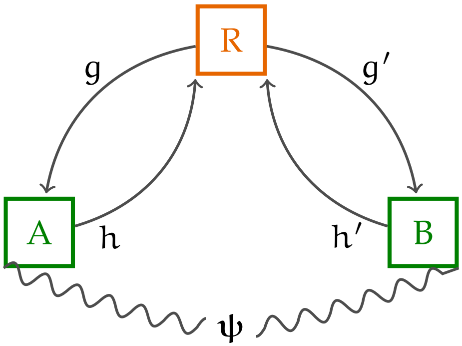
Quantum strategies



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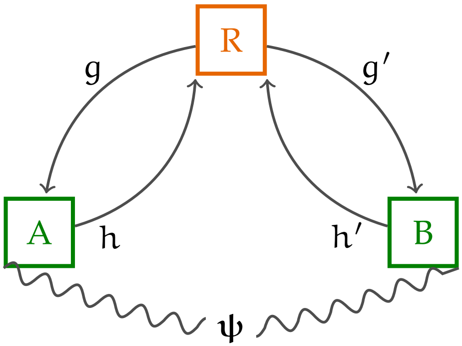

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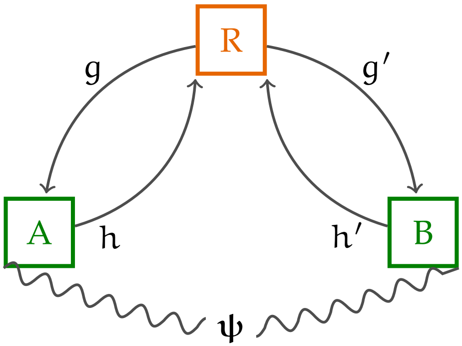

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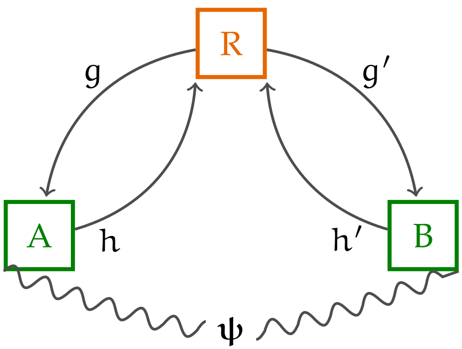

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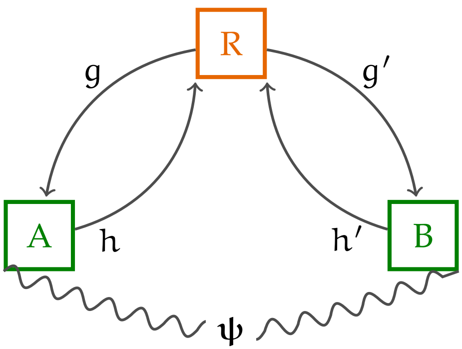

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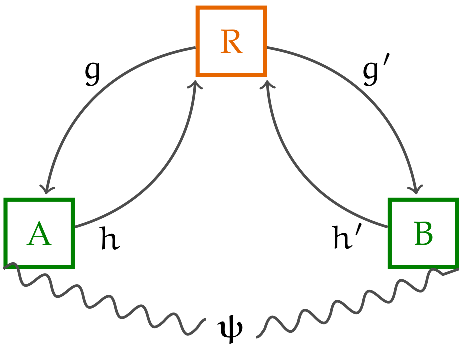

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The probability that players respond with h, h' on questions g, g' is

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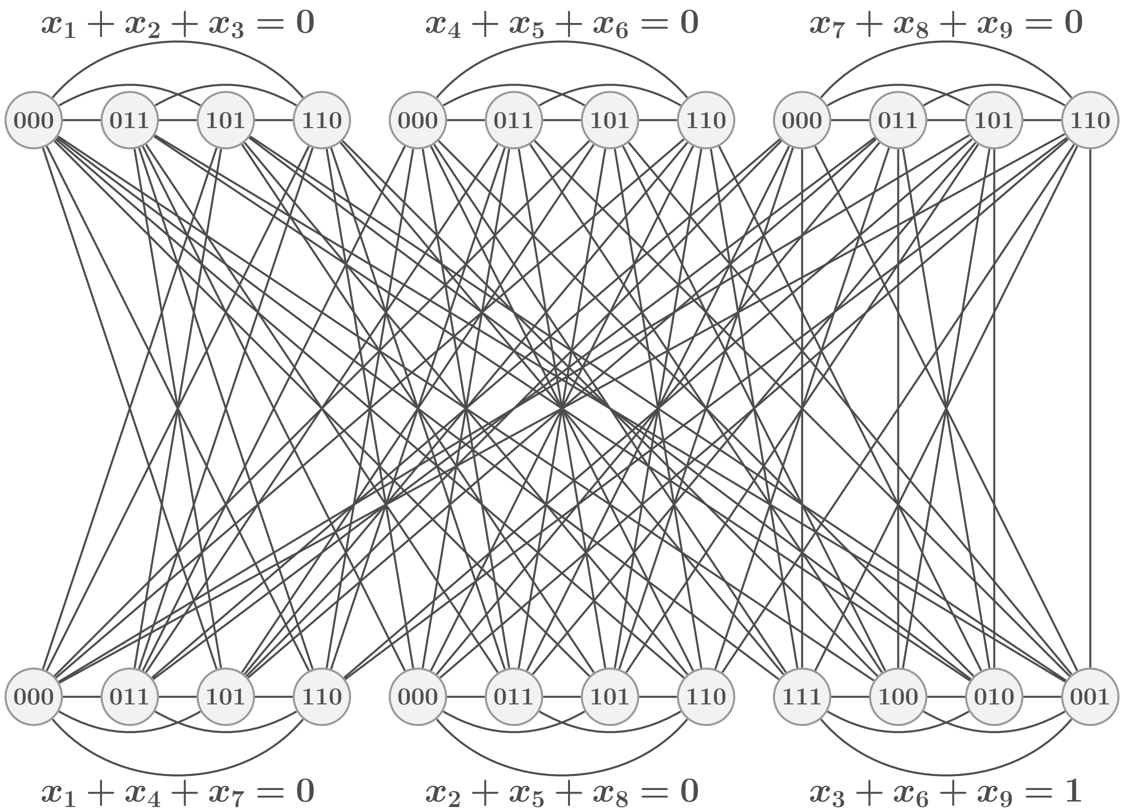
Example: G $\not\cong$ H but G $\cong_{qc}$ H



Construction based on reduction from linear system games.

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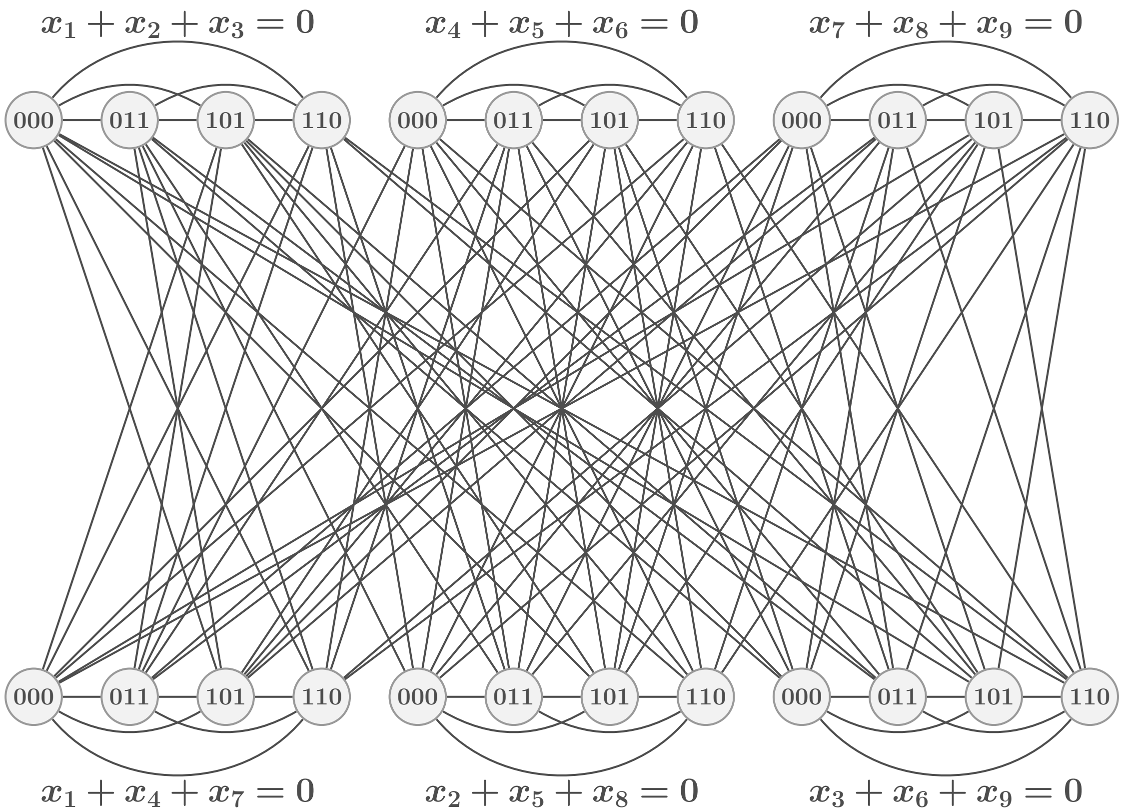

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A matrix $P = (p_{ij})$ is **quantum permutation matrix** if p_{ij} are elements of an C^*-algebra s.t.

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This is similar to

$$G \cong H \Leftrightarrow A_G P = P A_H \quad \text{(isomorphism)},$$
$$G \cong_{\mathcal{T}} H \Leftrightarrow A_G D = D A_H \quad \text{(fractional isomorphism)}.$$

Bilabelled graphs

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Example. $\boldsymbol{F} = \big(K_4, (1,2), (2,2)\big)$.

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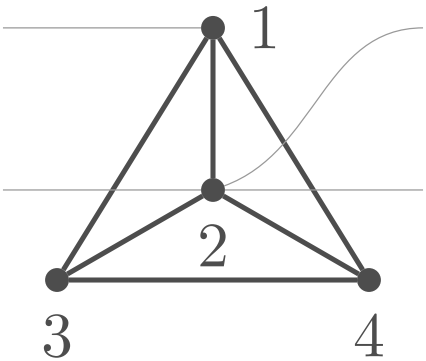
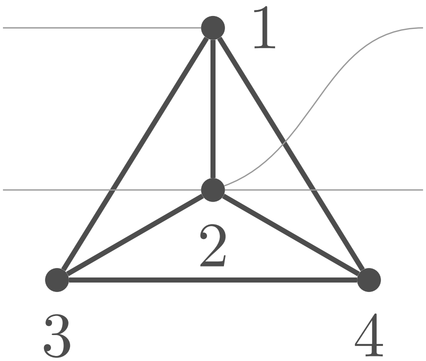


How to draw bilabelled graphs

$\big(\mathrm{K}_4, (1,2), (2,2)\big)$



A bilabelled graph is **planar** if it can be drawn with no crossings.

Homomorphism matrices

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So $\boldsymbol{A}_G$ is the **adjacency matrix** of G.

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Theorem. For a graph G and bilabelled graphs $\mathbf{F}, \mathbf{F'}$,

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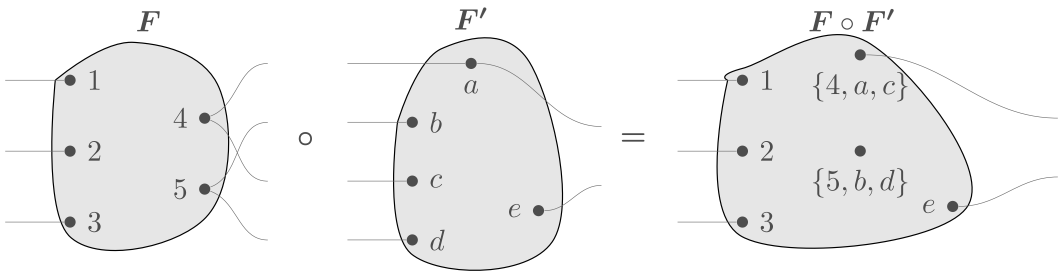

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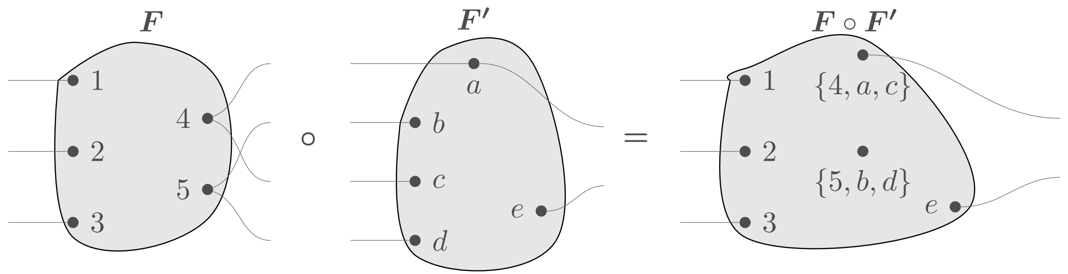

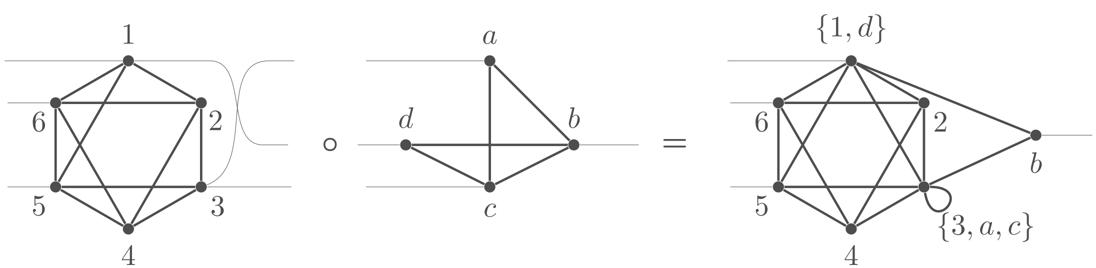

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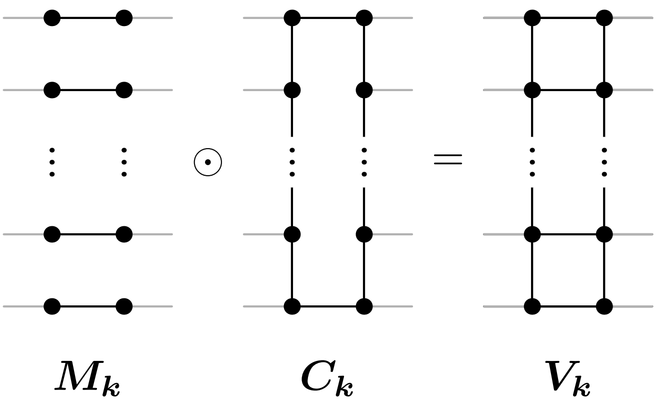

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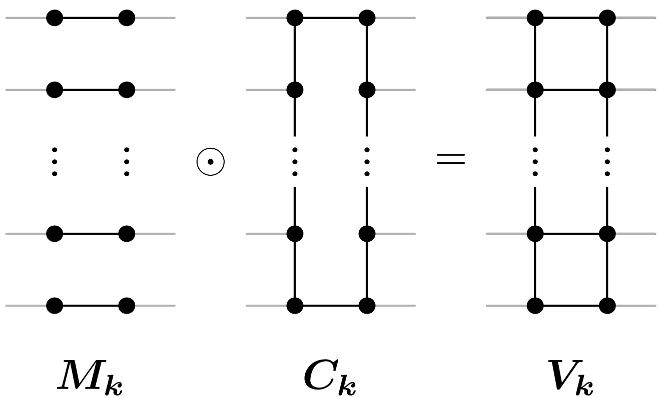

Other operations: transposition and cyclic permutations.

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Definition. $\mathcal{P}_k = \{F \ : \ \exists(F, \boldsymbol{u}, \boldsymbol{v}) \in \boldsymbol{\mathcal{P}}_k\}$

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- The other direction works too.

NPA hierarchy (intuition)

Let $k \in \mathbb{N}$. For all $\ell \leqslant k$, define

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This gives rise to a semidefinite program $\rightarrow$ NPA hierarchy.

NPA hierarchy for the isomorphism game

Let $\Sigma = V(G) \times V(H)$.

A matrix $\mathcal{R} \in \mathbb{M}_{\Sigma^{\leqslant k}}(\mathbb{C})$ is a **certificate for the k^{th} level of the NPA hierarchy for the (G, H)-isomorphism game** if

1. $\mathcal{R} \succeq 0$,
2. $\mathcal{R}_{\varepsilon,\varepsilon} = 1$,
3. $\mathcal{R}_{s,t}$ depends only on the equivalence class of $s^R t$,
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Thm. (G, H)-isomorphism game has a perfect quantum strategy iff there is a certificate for the k^{th} level of the NPA hierarchy, $k \in \mathbb{N}$.

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Thm (Choi, 1975). Φ is completely positive $\Leftrightarrow$ C_Φ is positive.

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4. The constraints on $\mathcal{R}$ translate into constraints on the map Φ.

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Definition. A linear map $\Phi : \mathbb{M}_{V(G)^k}(\mathbb{C}) \to \mathbb{M}_{V(H)^k}(\mathbb{C})$ is a **level-k quantum isomorphism map** if

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Converse requires some previous results and a bit of combinatorics.

Recap

The k^{th} level of the NPA hierarchy for the (G, H)-isomorphism game is feasible

$\Updownarrow$

There exists a level-k quantum isomorphism map from G to H

$\Updownarrow$

G and H are homomorphism indistinguishable over $\mathcal{P}_k$

Corollary

Corollary. $G \cong_{qc} H$ if and only if $G \cong_{\mathcal{P}} H$, where $\mathcal{P} = \cup_{k=1}^{\infty} \mathcal{P}_k$.

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All that is left is to show that $\mathcal{P}$ is the class of all planar graphs.

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All that is left is to show that $\mathcal{P}$ is the class of all planar graphs.

Proof that $\mathcal{P} \subseteq$ planar graphs: All generators of $\boldsymbol{\mathcal{P}}_k$ are planar, all operations preserve planarity.

Proof that $\mathcal{P} \supseteq$ planar graphs

Lemma. Each $\boldsymbol{\mathcal{P}}_k$ is minor-closed, and thus so is each $\mathcal{P}_k$ and $\mathcal{P}$.

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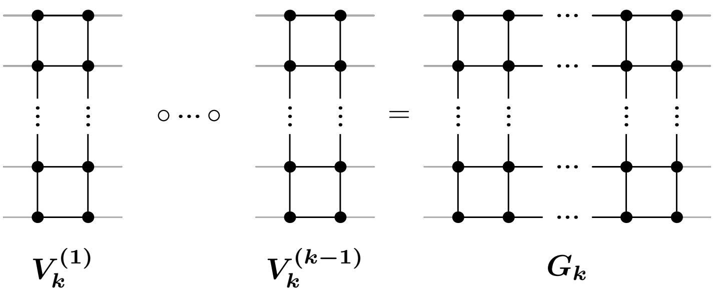
Proof.



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Lemma. Each $\boldsymbol{\mathcal{P}}_{\mathrm{k}}$ is minor-closed, and thus so is each $\mathcal{P}_{\mathrm{k}}$ and $\mathcal{P}$.

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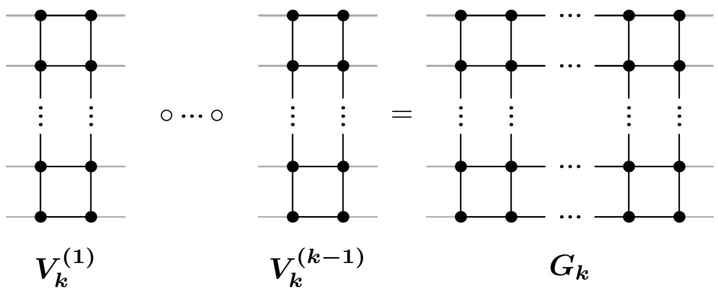

Well-known: Every planar graph is the minor of some $\mathrm{k} \times \mathrm{k}$ grid.

Some remarks/questions

- $\mathcal{P}_k$ has treewidth bounded by $3k - 1$. This implies there is a randomized poly time algorithm for determining if $G \cong_{\mathcal{P}_k} H$, and thus whether the k^{th} level of the NPA hierarchy for the (G, H)-isomorphism game is feasible.

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- There are graphs G and H of size $\leqslant 72k^2$ that are not quantum isomorphic, but the k^{th} level of the NPA hierarchy is feasible for the (G, H)-isomorphism game.
- Can we obtain a better description of the classes $\mathcal{P}_k$?

Thank you!

The hierarchy of Navascués, Pironio, and Acín (NPA)

CHSH game, $X = Y = A = B = \{0, 1\}$. Deterministic strategies M:

$$
\begin{pmatrix} 1 & 1 & 1 & 1 \\ & & & \\ & & & \\ & & & \end{pmatrix}
\begin{pmatrix} 1 & & 1 & \\ & 1 & & 1 \\ & & & \\ & & & \end{pmatrix}
\begin{pmatrix} & 1 & & 1 \\ 1 & & 1 & \\ & & & \\ & & & \end{pmatrix}
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$$

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The classical value is $\sup_M \langle K, M \rangle = \sup_M \mathrm{Tr}[K^* M] = 3/4$.

Nonlocal games

Def. A nonlocal game is a 6-tuple (X, Y, A, B, π, V), where

1. X, Y, A, and B are finite and nonempty sets,
2. $\pi \in P(X \times Y)$ is a probability vector, and
3. $V\colon A \times B \times X \times Y \to \{0, 1\}$ is a predicate.

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$$M \in L(\mathbb{R}^X \otimes \mathbb{R}^Y, \mathbb{R}^A \otimes \mathbb{R}^B).$$

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The probability that M wins a game is

$$\sum_{(x,y)\in X\times Y} \pi(x, y) \sum_{(a,b)\in A\times B} V(a, b|x, y)M(a, b|x, y) = \langle K, M\rangle,$$

where $K(a, b|x, y) = \pi(x, y)V(a, b|x, y)$.

Commuting measurement strategies

An operator M represents a **commuting measurement strategy** if there exists a Hilbert space $\mathcal{H}$, a unit vector $u \in \mathcal{H}$, and projection operators

$$\{P_a^x : x \in X, a \in A\} \quad \text{and} \quad \{Q_b^y : y \in Y, b \in B\}$$

acting on $\mathcal{H}$ such that the follwoing are satisfied:

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2. $[P_a^x, Q_b^y] = 0$, $x \in X, y \in Y, a \in A, b \in B$,
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A commuting measurement value of game G is

$$\omega^c(G) = \sup_{M \in \mathcal{C}} \langle K, M \rangle,$$

where K is defined from G as before and $\mathcal{C}$ is the class of commuting measurement strategies.

1st level of the NPA hierarchy (intuition)

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3. $R((x,a),(x,c)) = 0$, $x \in X, a, c \in A$, $a \neq c$,
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3. $R((x, a), (x, c)) = 0$, $x \in X$, $a, c \in A$, $a \neq c$,
$R((y, b), (y, d)) = 0$, $y \in Y$, $b, d \in B$, $b \neq d$,
(P_a^x and P_c^x, Q_b^y and Q_d^y are orthogonal)
4. $R((z, c), (z, c)) = R(\varepsilon, (z, c)) = R((z, c), \varepsilon)$, ($P^2 = P$)

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Suppose that $R \in L(\mathbb{C}^{\Sigma^{\leqslant 1}}, \mathbb{C}^{\Sigma^{\leqslant 1}})$. We observe the following:

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2. $\sum_{a \in A} R((x, a), s) = R(\varepsilon, s)$ and $\sum_{a \in A} R(s, (x, a)) = R(s, \varepsilon)$,
   $\sum_{b \in B} R((y, b), s) = R(\varepsilon, s)$ and $\sum_{b \in B} R(s, (y, b)) = R(s, \varepsilon)$,
   (summing over operators in measurements is identity)
3. $R((x, a), (x, c)) = 0$, $x \in X, a, c \in A$, $a \neq c$,
   $R((y, b), (y, d)) = 0$, $y \in Y, b, d \in B$, $b \neq d$,
   (P_a^x and P_c^x, Q_b^y and Q_d^y are orthogonal)
4. $R((z, c), (z, c)) = R(\varepsilon, (z, c)) = R((z, c), \varepsilon)$, ($P^2 = P$)
5. $R((x, a), (y, b)) = R((y, b), (x, a))$. (commutativity)

1st level of the NPA hierarchy (intuition)

Let $\mathcal{C}_1$ be the class containing all strategies M for which there is a positive semidefinite R such that $M(a,b|x,y) = R((x,a),(y,b))$. We have

$$\omega^c(G) = \sup_{M \in \mathcal{C}} \langle K, M \rangle \leqslant \sup_{M \in \mathcal{C}_1} \langle K, M \rangle.$$

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$$H((x,a),(y,b)) = H((y,b),(x,a)) = \frac{1}{2}\pi(x,y)V(a,b|x,y),$$

we get

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which gives us a semidefinite program, where we optimize $\langle H, R \rangle$ over positive semidefinite R satisfying (affine) linear constraints given in items 1–5 above.

k^{th} level of the NPA hierarchy (intuition)

In the k^{th} level of the NPA hierarchy we consider operators R indexed by $\Sigma^{\leqslant k}$ satisfying conditions similar to 1–5.

Then the class $\mathcal{C}_k$ contains all strategies M for which there exists such admissible operator R. We have:

$$\mathcal{C}_1 \supseteq \mathcal{C}_2 \supseteq \mathcal{C}_3 \supseteq \cdots \supseteq \mathcal{C}.$$

Thm. The following are equivalent:

- M is a commuting measurement strategy.
- $M \in \mathcal{C}_k$ for every k.

Equivalently:

$$\mathcal{C} = \bigcap_{k=1}^{\infty} \mathcal{C}_k.$$