Valued Constraint Satisfaction Problem and Resilience in Database Theory

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Resilience of queries

database – a relational structure $\mathfrak{A}$

conjunctive query – a formula q of the form $\exists y_1, \ldots, y_l \, (\psi_1 \wedge \cdots \wedge \psi_m)$, where ψ_i are atomic

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Fixed conjunctive query q.
Input: a finite database $\mathfrak{A}$, $u \in \mathbb{N}$
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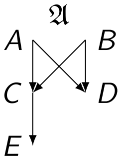
Example: The resilience of

$$q = \exists x, y, z (R(x, y) \wedge R(y, z))$$

with respect to $\mathfrak{A}$ is 1 – remove (C, E).



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Goal: Classify complexity of resilience for all q.

Constraint satisfaction

Fixed τ-structure $\mathfrak{A}$ (τ – finite relational signature)
Input: list of atomic τ-formulas (constraints)
Output:

- **CSP**: Decide whether there is a solution that satisfies all constraints.
- **MinCSP**: Find the minimal number of constraints to violate so that the remaining constraints are satisfiable simultaneously.
- **VCSP**: Find the minimal cost with which the constraints can be satisfied (each constraint comes with a cost depending on the chosen values).

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Observation: VCSP generalizes CSP and MinCSP.
Proof: Model the tuples in relations with cost 0 and outside with cost 1 (for MinCSP) or ∞ (for CSP).

Focus on VCSP

A valued structure Γ consists of:

- (countable) domain C
- (finite, relational) signature τ
- for each $R \in \tau$ of arity k, a function $R^\Gamma\colon C^k \to \mathbb{Q} \cup \{\infty\}$

the concrete choice of the ordered semiring is not crucial

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Definition (VCSP(Γ))

Input: $u \in \mathbb{Q}$, an expression

$$\phi(x_1, \ldots, x_n) = \sum_i \psi_i,$$

where each ψ_i is an atomic τ-formula

Question: Is

$$\inf_{t \in C^n} \phi(t) \leq u \text{ in } \Gamma?$$

Example: Max-Cut as a VCSP

Input: $G = (V, E)$ – finite directed (multi)graph
Goal: Find a partition $A \cup B$ of V such that $E \cap (A \times B)$ is maximal.
Equivalently: $E \cap (A^2 \cup B^2 \cup B \times A)$ is minimal.

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Let Γ_{MC} be a valued structure where:

- $C = \{0, 1\}$
- $\tau = \{R\}$, R binary

$$R(x, y) = \begin{cases} 0 \text{ if } x = 0 \text{ and } y = 1 \\ 1 \text{ otherwise} \end{cases}$$

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Take vertices of G as variables. The size of a maximal cut of G is

$\min_{x \in C^n} \sum_{(x_i, x_j) \in E} R(x_i, x_j)$. The partition of V is given by the values 0 and 1.

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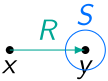
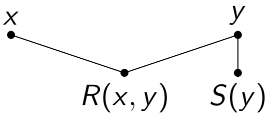
every instance of VCSP(Γ_{MC}) corresponds to a directed multigraph
$\rightsquigarrow$ VCSP(Γ_{MC}) is the Max-Cut problem (NP-hard)

Translation to a dual problem

Example:
$q := \exists x, y(R(x,y) \wedge S(y))$



canonical structure



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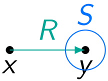

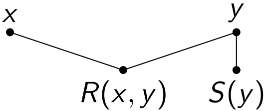

Theorem (Cherlin, Shelah, Shi '99)

Let q be a query and $\mathfrak{Q}$ its canonical structure. If $I(q)$ is connected, then there exists a structure $\mathfrak{B}_q$, such that for every finite $\mathfrak{A}$:

$$\mathfrak{A} \not\models q \;\Leftrightarrow\; \mathfrak{Q} \not\to \mathfrak{A} \;\Leftrightarrow\; \mathfrak{A} \to \mathfrak{B}_q$$

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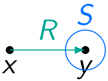

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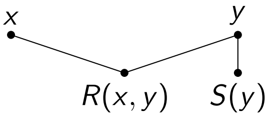

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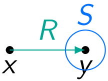

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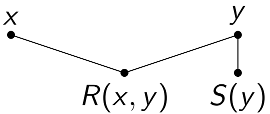

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- *$\mathfrak{B}_q$ can be chosen finite iff q is homomorphically equivalent to q' such that $I(q')$ is a tree. (Nešetřil, Tardiff '00; Larose, Loten, Tardiff '07)*

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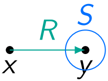

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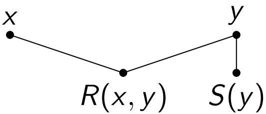

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oligomorphic – countable domain B_q and the action of $\mathrm{Aut}(\mathfrak{B}_q)$ on B_q^n has finitely many orbits for every $n \geq 1$

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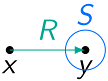

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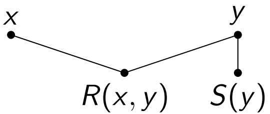

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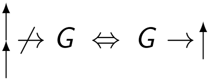
Example: For every finite directed graph G we have:



Connection of resilience and VCSPs

query q with $I(q)$ connected (WLOG) $\rightsquigarrow$ obtain the dual structure $\mathfrak{B}_q$ $\rightsquigarrow$ turn it into a valued structure Γ_q with cost functions taking values 0 and 1

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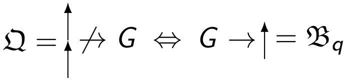

$\mathfrak{B}_q \rightsquigarrow \Gamma_{\mathrm{MC}} = (\{0, 1\}; R)$
Resilience of $q = \mathrm{VCSP}(\Gamma_{\mathrm{MC}}) =$ Max-Cut is NP-hard

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Remark: We have to consider bag databases – a database $\mathfrak{A}$ might contain a tuple with multiplicity > 1 (differs from the original setting).

Example: Input $R(x, y) + R(x, y)$ for VCSP(Γ) corresponds to a database with multiplicity 2 for $R(x, y)$.

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For a finite-domain valued structure Γ, $\mathrm{VCSP}(\Gamma)$ *is in P or NP-complete.*

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Question: For general queries, choose $\mathfrak{B}_q$ such that $\mathrm{Aut}(\mathfrak{B}_q)$ is oligomorphic $\Rightarrow$ finitely many orbits of n-tuples for every n.
Can we use some results for finite domains?

Hard resilience problems

pp-construction – a notion of 'expressing' one valued structure in another (generalizes pp-constructions for classical structures)

Fact: If Aut(Γ) is oligomorphic and Γ pp-constructs Δ, then VCSP(Δ) reduces to VCSP(Γ) in poly-time.

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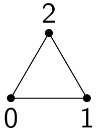

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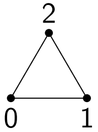

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Fractional polymorphisms

polymorphism of a relational structure $\mathfrak{A}$ – $f : A^n \to A$ such that for all relations R of $\mathfrak{A}$ and $t^1, \dots, t^n \in R$, $f(t^1, \dots, t^n) \in R$ (applied row-wise)

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Example: The operation min is a polymorphism of $(\mathbb{Q}; <)$.

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Definition (fractional polymorphism)

A fractional polymorphism of Γ of arity n is a probability distribution ω on the maps $f : C^n \to C$ such that for every k-ary $R \in \tau$ and $t^1, \ldots, t^n \in C^k$

$$\underbrace{E_\omega[f \mapsto R(f(t^1, \ldots, t^n))]}_{\text{expected value}} \leq \underbrace{\frac{1}{n} \sum_{j=1}^{n} R(t^j)}_{\text{arithmetic mean}} \quad (\omega \text{ improves } R).$$

Tractability conjecture

Known for finite-domain VCSPs:

Theorem

Γ – *a finite-domain valued structure*

- *If* Γ *does not pp-construct* K_3, *then* Γ *has cyclic fractional polymorphism (essentially [Kozik, Ochremiak '15]).*
- *If* Γ *has a cyclic fractional polymorphism, then* $\mathrm{VCSP}(\Gamma)$ *is in P [Kolmogorov, Krokhin, Rolínek '15].*

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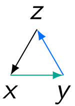
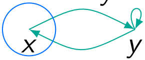
Conjecture: If Γ_q does not pp-construct K_3, then the tractability theorem applies and VCSP(Γ_q) and hence resilience of q is in P.

Examples

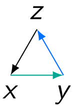
- $q_{\mathrm{MC}} := \exists x, y, z\big(R(x, y) \wedge R(y, z)\big)$
- $q_{\mathrm{path}} := \exists x, y, z\big(R(x, y) \wedge S(y, z)\big)$
- $q_{\triangle} := \exists x, y, z\big(R(x, y) \wedge S(y, z) \wedge T(z, x)\big)$
- $q_{\mathrm{new}} := \exists x, y\big(S(x) \wedge R(x, y) \wedge R(y, x) \wedge R(y, y)\big)$

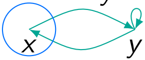




Examples

- $q_{\mathrm{MC}} := \exists x, y, z\big(R(x,y) \wedge R(y,z)\big)$
- $q_{\mathrm{path}} := \exists x, y, z\big(R(x,y) \wedge S(y,z)\big)$
- $q_{\triangle} := \exists x, y, z\big(R(x,y) \wedge S(y,z) \wedge T(z,x)\big)$
- $q_{\mathrm{new}} := \exists x, y\big(S(x) \wedge R(x,y) \wedge R(y,x) \wedge R(y,y)\big)$





query	**complexity**	**VCSP proof**
q_{MC}	NP-hard (FGIM '20)	Max-Cut/pp-constructs K_3
q_{path}	P (MG)	binary cyclic frac. polymorphism
$q_{\triangle}$	NP-hard (FGIM '15)	pp-constructs K_3
q_{new}	P (BLS)	binary pseudo cyclic frac. polymorphism

References:

- Freire, Gatterbauer, Immerman, Meliou '15
- Freire, Gatterbauer, Immerman, Meliou '20
- Makhija, Gatterbauer '22
- Bodirsky, S., Lutz '24

Thank you for your attention

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