

VALUED CONSTRAINTS ON INFINITE DOMAINS

Why should you care?

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Outline

- 1) Examples and definitions
- 2) Complexity dichotomies
- 3) A few algebraic tools
- 4) Valued cores
- 5) Why should you care?

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Example: hike clustering

Variables: participants of CWC

Constraints: who wants to (not) go on the same hike
e.g. $\text{Tomas} \neq \text{Kristina} \wedge \text{Jakub} = \text{Michal} \wedge \dots$

Question: What is the minimal number of constraints to violate so that people can be partitioned for hikes?

Example: talk scheduling

Variables: talks at CWC

Constraints: who wants to have a talk before/after someone
e.g. $\text{Libor} < \text{Standa} \wedge \check{\text{Žaneta}} < \text{Libor} \wedge \dots$

Question: What is the minimal number of conditions to violate so that the talks can be scheduled?

Example: shower pairs

Variables: participants of CWC

Constraints: $A \rightarrow B$ for each time A has seen B in the shower
e.g. not available due to GDPR

Question: How many shower events would have to unhappen so that there is no pair (A, B) who have seen each other in the shower?

Valued CSPs

valued structure Γ consists of:

- (countable) domain C
- (finite, relational) signature τ
- $\forall R \in \tau, R^\Gamma : C^{\text{ar}(R)} \rightarrow \mathbb{Q} \cup \{\infty\}$

classical relations are
 $C^{\text{ar}(R)} \rightarrow \{0, \infty\}$



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DEF.: $\text{VCSP}(\Gamma)$

Input: $u \in \mathbb{Q}$, an expression

$$\phi(x_1, \dots, x_n) = \sum_i \psi_i$$

atomic τ -formulas

Output: Is $\inf_{t \in C^n} \phi(t) \leq u$ in Γ ?

Examples revisited

$$R_a^b(x) = \begin{cases} a & x \in R \\ b & x \notin R \end{cases}$$

Hike clustering = $(\mathbb{N}; (=)_0^1, (\neq)_0^1)$

Talks scheduling = $(\mathbb{Q}; (<)_0^1)$

Shower pairs = $(V, (E)_0^1)$,

where (V, E) is the random tournament

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NP-complete

in P

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↳ all valued structures above are examples

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From now on we always assume that $\text{Aut}(\Gamma)$ is oligomorphic.

Resilience problems

bag database - a finite structure whose relations are multisets

μ - conjunctive query = pp-formula over signature τ

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DEF. (resilience of μ):

Input: A finite bag database A over τ , $u \in \mathbb{Q}$

Output: Can we remove $\leq u$ tuples from A so that $A \not\models \mu$?

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bag database - a finite structure whose relations are **multisets**

m - **conjunctive query** = **pp-formula** over signature τ

DEF. (resilience of m):

Input: A finite **bag database** A over τ , $u \in \mathbb{Q}$

Output: Can we **remove $\leq u$ tuples** from A so that $A \not\models m$?

Example:

$$m = \exists x, y, z \ R(x, y) \wedge R(y, z)$$

$(A, 2)$ is a YES-instance



Resilience problems are VCSPs

THEOREM (essentially Cherlin, Shelah, Shi '99)

For every m , there exists a relational structure B_m with $\text{Aut}(B_m)$ oligomorphic s.t. for every finite A ,

$A \rightarrow B_m$ iff $|A| \leq m$

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B_μ = the random tournament $\Rightarrow$ needs to be infinite!

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PROPOSITION (Bodirsky, S., Lutz '24)

The resilience of μ is equal to $\text{VCSP}(B_\mu, (R^{B_\mu}_i)^1, R \in \mathcal{T})$.

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$\hookrightarrow$ interesting class of VCSPs with applications in DB theory

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Complexity dichotomies

Follows from [Kozik, Ochremiak '15], [Kolmogorov, Krokhin, Rodneev '15]
[Bulatov '17], [Zhuk '17]:

THEOREM:

VCSP (Γ) for a finite-domain Γ is in P or NP-complete.

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- For VCSPs of Γ with $\text{Aut}(\Gamma)$ oligomorphic, there is no hope for dichotomy (CSP reasons, [Bodirsky, Grohe '08])

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$\hookrightarrow$ implies dichotomy for resilience problems for incidence-acyclic queries

- For VCSPs of Γ with $\text{Aut}(\Gamma)$ oligomorphic, there is no hope for dichotomy (CSP reasons, [Bodirsky, Grohe '08])
- If $\text{Aut}(\Gamma) \cong \text{Aut}(\mathbb{B})$ for a finitely bounded homogeneous structure $\mathbb{B}$, then $\text{VCSP}(\Gamma) \in NP$

Complexity dichotomies

CONJECTURE (Bodirsky, Pinsker '11)

If $\mathbb{A}$ is a relational structure with $\text{Aut}(\mathbb{A}) \geq \text{Aut}(\mathbb{B})$
for a finitely bounded homogeneous structure $\mathbb{B}$, then
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Subgoals: Dichotomies for subclasses where the conjecture holds for CSPs.

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Done: $\text{Aut}(\Gamma) \geq \text{Aut}(\mathbb{Q}; <)$ [Bodirsky, Bonnet, S. '25]

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Done: $\text{Aut}(\Gamma) \geq \text{Aut}(\mathbb{Q}, <)$ [Bodirsky, Bonnet, S. '25]

Work in progress: $\text{Aut}(\Gamma) \geq \text{Aut}(\text{random graph})$ (with Bodirsky, Jonsson, Osipov)

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Fractional homomorphisms

Γ, Δ valued structures, signature τ

DEF.:

A probability distribution ω on $\mathcal{C}^{\mathcal{D}}$ is a fractional homomorphism from Δ to Γ if for every $R \in \tau$, $t \in \mathcal{D}^{\text{ar}(R)}$ we have

$$E_{\omega} [f \mapsto R^{\Gamma}(f(t))] \leq R^{\Delta}(t).$$

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↳ homomorphisms between relational structures are a special case

Fractional polymorphisms

Γ^l : domain $\mathbb{C}^l$

$$R^{\Gamma^l}(x_1^1, \dots, x_{ar(R)}^1) := \frac{1}{l} \sum_{i=1}^l R^{\Gamma}(x_i^1, \dots, x_i^{ar(R)})$$

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A fractional polymorphism of Γ of arity l is
a fractional homomorphism from Γ^l to Γ .

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DEF.:

A fractional polymorphism of Γ of arity l is a fractional homomorphism from Γ^l to Γ .

Slogan: "the expected value is at most the arithmetic mean"

$$E_w[f \mapsto R^{\Gamma}(f(t_1^1, \dots, t_l^1))] \leq \frac{1}{l} \sum_{j=1}^l R^{\Gamma}(t_j^1)$$

We say that w improves R .

Expressibility

$\langle \Gamma \rangle$ is the set of valued relations on C containing relations of Γ , $(=)_0^\infty$ and $\perp$ closed under

- *shifting* by $a \in \mathbb{Q}$
- *scaling* by $a \in \mathbb{Q}_{\geq 0}$

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- operator **Feas** : $\text{Feas}(R) = \{t \in C^{\text{ar}(R)} \mid R(t) < \infty\}$

- operator **Opt** : $\text{Opt}(R) = \{t \in \text{Feas}(R) \mid R(t) \leq R(s) \forall s\}$

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PROPOSITION: We can add $R \in \langle \Gamma \rangle$ to the signature without changing the complexity of the VCSP.

Galois connection

THEOREM (Cohen, Cooper, Creed, Jeavons, Živný '13;
Fulla, Živný '16)

If Γ has a finite domain, $\text{Imp}(\text{fPol}(\Gamma)) = \langle \Gamma \rangle$.

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If Γ has a finite domain, $\text{Imp}(\text{fPol}(\Gamma)) = \langle \Gamma \rangle$.

Proof idea (as I understand it):

- $\supseteq$ is straightforward
- For $\subseteq$, we take $R \notin \langle \Gamma \rangle$ and want to construct $w \in \text{fPol}(\Gamma)$ that does not improve R .
- To get the probabilities for w , we use Farkas's lemma.

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Problem with generalization: We need to apply Farkas's lemma ∞ -many times and use compactness. $>$ may become $\geq$.

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Natural question: Are our definitions correct to get

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 $\Rightarrow$ don't restrict to discrete probability distributions.

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Since $\langle \Gamma \rangle \subseteq \text{Imp}(\text{fPol}(\Gamma))$, it is tactical to keep $\text{fPol}(\Gamma)$ as large as possible
 $\Rightarrow$ don't restrict to discrete probability distributions.

Do we have all relevant operators in $\langle \Gamma \rangle$?

PP-constructions

Δ is a **pp-power** of Γ if $D = C^d$ for some d and all relations $R \subseteq D^{\text{ar}(R)} = C^{d \cdot \text{ar}(R)}$ of Δ lie in $\langle \Gamma \rangle$

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DEF :

We say that Γ **pp-constructs** Δ if Δ is **fractionally homomorphically equivalent** to a **pp-power** of Γ .

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PROPOSITION:

If Γ **pp-constructs** Δ , then $\text{VCSP}(\Delta)$ **reduces** to $\text{VCSP}(\Gamma)$ in PTIME.

↳ needs oligomorphic automorphism groups or similar

Dichotomy theorem revisited

THEOREM:

Γ on a finite domain. Then

- 1) either Γ has a cyclic fractional polymorphism and $\text{VCSP}(\Gamma)$ is in P , or ([KKR '15])
- 2) Γ pp-constructs K_3 and $\text{VCSP}(\Gamma)$ is NP-hard.
(essentially [KO '15])

Tractability via the reduction to finite

Reduction to finite in style of [Bodirsky, Mottet '18]:

THEOREM (Bodirsky, S., Lutz '24):

If $\text{Aut}(\Gamma) \supseteq \text{Aut}(\mathbb{B})$ for a finitely bounded homogeneous $\mathbb{B}$ and $\exists w \in \text{Pol}(\Gamma)$, w pseudo cyclic and canonical w.r.t. $\text{Aut}(\mathbb{B})$, then $\text{VCSP}(\Gamma)$ is in P.

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Question: When could it be enough to capture tractability?

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Question: When could it be enough to capture tractability?

- classes of valued structures where the tractability of CSPs is captured by canonical polymorphisms
- resilience problems (conjecture from [Bodirsky, S., Lutz '24])

Cheating via CSP results

$$\langle \Gamma \rangle_0^\infty := \{ R \in \langle \Gamma \rangle \mid R \text{ is } 0\text{-}\infty \text{ valued} \}$$

$$\Gamma^* := (C, \langle \Gamma \rangle_0^\infty) \dots \text{underlying crisp structure}$$

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$\Gamma^* := (C; \langle \Gamma \rangle_0^\infty)$... underlying crisp structure

PROPOSITION (Bodirsky, Bonnet, S. '25)

Γ pp-constructs $K_3 \iff \Gamma^*$ pp-constructs K_3 .

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$$\Gamma \text{ pp-constructs } K_3 \iff \Gamma^* \text{ pp-constructs } K_3.$$

In particular, the complexity of VCSP(Γ) could be encoded in the CSP of (finite-signature reducts of) Γ^* .

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PROPOSITION (Bodirsky, Bonnet, S. '25)

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In particular, the complexity of $\text{VCSP}(\Gamma)$ could be encoded in the CSP of (finite-signature reducts of) Γ^* .

However, we don't understand the relationship between $\text{fPol}(\Gamma)$ and $\text{Pol}(\Gamma^*)$.

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Valued cores (ongoing work with Bodirsky, Jahnke & Mottet)

Motivation:

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- hope for Olšák-style exclusivity result:

"if Γ is a valued core and Γ has a pseudo cyclic fractional polymorphism, then Γ does not pp-construct K_3 "

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Motivation:

- cores can help to get **idempotence** [Kozik, Ochremiak '15]
- hope for **Olšák-style exclusivity result**:

"if Γ is a valued core and Γ has a pseudo cyclic fractional polymorphism, then Γ does not pp-construct K_3 "

DEF.:

Γ is a **valued core** if $w(\overline{\text{Aut}(\Gamma)}) = 1$ for
 $\forall w \in \text{End}(\Gamma)$.

in the topology
of pointwise
convergence

Existence and uniqueness of cores

Conjecture: For every Γ , there exists a **valued core** Δ which is **fractionally homomorphically equivalent** to Γ .

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- Γ with **≤ 2 finite values** (the fractional homomorphisms are of the form $\omega(f) = 1$)

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- Γ with **≤ 2 finite values** (the fractional homomorphisms are of the form $\omega(f) = 1$)

THEOREM:

If two **valued cores** are **fractionally homomorphically equivalent**, then they are **isomorphic**.

Underlying crisp structures of valued cores

PROPOSITION

If Γ^* is a model-complete core, then Γ is a valued core.

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Conjecture: The reverse implication also holds.

$\hookrightarrow$ is equivalent to a very special case of the Imp-flof preservation theorem

$\hookrightarrow$ would imply that if Γ is a valued core with a pseudo cyclic fractional polymorphism, then Γ does not pp-construct K_3

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Why should you care?

Because it's pretty cool !

Why should you care?

VCSPs of Γ with oligomorphic $\text{Aut}(\Gamma)$:

- model natural problems
- their complexity relates to the complexity of the underlying CSPs

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VCSs of Γ with oligomorphic $\text{Aut}(\Gamma)$:

- model natural problems
- their complexity relates to the complexity of the underlying CSPs
- unexplored algebraic notions:
 - ↳ fractional homomorphisms
 - ↳ fractional polymorphisms
 - ↳ expressibility
 - ↳ valued cores

For your attention,
I am ever so grateful,
I hope you have no regrets,
for not being now in the whirlpool!

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