Temporal Valued Constraint Satisfaction Problems

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Observation: VCSP generalizes CSP and MinCSP.
Proof: Model the tuples in relations with cost 0 and outside with cost 1 (for MinCSP) or ∞ (for CSP).

Valued Constraint Satisfaction Problem

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- (finite, relational) signature τ
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Definition (VCSP(Γ))

Input: $u \in \mathbb{Q}$, an expression

$$\phi(x_1, \dots, x_n) = \sum_i \psi_i,$$

where each ψ_i is an atomic τ-formula
Output: Is

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Notation: For $S \subseteq C^k$ and $a, b \in \mathbb{Q} \cup \{\infty\}$, S_a^b denotes the valued relation such that $S_a^b(t) = a$ if $t \in S$ and $S_a^b(t) = b$ otherwise.

Optimization problems

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P = class of efficiently solvable problems
NP = class of problems with efficiently verifiable solution
NP-complete problems = hardest problems in NP

Complexity of VCSPs

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Let Γ *be a valued structure with a finite domain. Then* $\mathrm{VCSP}(\Gamma)$ *is in P or NP-complete.*

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Γ – valued structure on a countable domain C over a signature τ

- automorphism of Γ – permutation α of C such that for $R \in \tau$ of arity k and every $t \in C^k$, $R(\alpha(t)) = R(t)$

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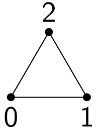
- a temporal structure if $\mathrm{Aut}(\mathbb{Q}; <) \subseteq \mathrm{Aut}(\Gamma)$ (e.g., $(\mathbb{Q}; (<)^1_0)$)

Pp-constructability

K_3 is the valued structure on $\{0, 1, 2\}$ with single binary relation E defined:

$$E(x, y) = \begin{cases} 0 \text{ if } x \neq y \\ \infty \text{ if } x = y \end{cases}$$



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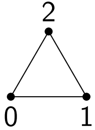

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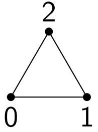

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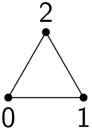

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Proposition (Bodirsky, S., Lutz '24)

If $\mathrm{Aut}(\mathbb{Q}; <) \subseteq \mathrm{Aut}(\Gamma)$ *and* Γ *pp-constructs* K_3*, then* $\mathrm{VCSP}(\Gamma)$ *is NP-complete.*

Fractional polymorphisms

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Definition

A map $f : C^n \to C$ is called

- a polymorphism of $\mathfrak{C}$ if for every k-ary $R \in \tau$ and $t^1, \dots, t^n \in C^k$

$$R(t^1) \wedge \dots \wedge R(t^n) \Rightarrow R(f(t^1, \dots, t^n));$$

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- a fractional polymorphism if for every k-ary $R \in \tau$ and $t^1, \dots, t^n \in C^k$

$$\frac{1}{n}(R(t^1) + \cdots + R(t^n)) \geq R(f(t^1, \dots, t^n)).$$

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Pol($\mathfrak{C}$) – set of all polymorphisms of $\mathfrak{C}$
fPol(Γ) – set of all fractional polymorphisms of Γ
$\hookrightarrow$ contains more than covered in the definition above

Classification of equality VCSPs

Known for CSPs:

Theorem (Bodirsky, Kára '08)

If $\mathfrak{A}$ is an equality relational structure, then exactly one of the following holds:

- $\mathrm{Pol}(\mathfrak{A})$ *contains a unary constant operation or a binary injection and* $\mathrm{CSP}(\mathfrak{A})$ *is in P.*
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Theorem (Bodirsky, Bonnet, S. '24)

If Γ is an equality valued structure, then exactly one of the following holds:

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Theorem (Bodirsky, Kára '10)

Let $\mathfrak{A}$ *be a temporal relational structure. Then exactly one of the following holds:*

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Remark:

- $\mathrm{ll} \in \mathrm{Pol}(\mathfrak{A}) \Rightarrow \mathrm{lex} \in \mathrm{Pol}(\mathfrak{A})$
- $\mathrm{lex} \in \mathrm{Pol}(\mathfrak{A})$ does not imply tractability of $\mathrm{CSP}(\mathfrak{A})$!

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Corollary (of the proof): Given a temporal valued structure Γ, it is decidable whether VCSP(Γ) is in P or NP-complete.

Open questions

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- Classify the complexity of VCSPs of valued structures Γ such that $\mathrm{Aut}(\Gamma)$ contains the automorphism group of the countable random graph. Is $\mathrm{VCSP}(\Gamma)$ in P whenever Γ does not pp-construct K_3?

Thank you for your attention

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