# Numeric CSPs

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Lukas Pieper

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# Constraint Satisfaction Problems

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### Example

3-COLORING (CSP($\{r, g, b\}, \neq$))

# Numeric CSPs

CSP$(A; (R_i)_{i \in I})$ is *numeric* if

- $A \in \{\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}, \dots\}$ and
- all $R_i$ are fo-definable over $(A; +, \cdot, 0, 1, <)$.

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## Example

CSP$(\mathbb{R}; x^2 + y^2 \leq 1, x + 1 = y)$

# Tractable and NP-complete CSPs

### Example (LP-feasibility)

CSP($\mathbb{R}$; $a_1x_1 + \cdots + a_nx_n \geq b, \ldots$) with relations for all $n \in \mathbb{N}$ and $a_1, \ldots, a_n, b \in \mathbb{Q}$ is solved in polynomial time by Khachiyan's ellipsoid algorithm.

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### Example (ILP-feasibility)

CSP($\mathbb{Z}; a_1x_1 + \cdots + a_nx_n \geq b, \ldots$) with relations for all $n \in \mathbb{N}$ and $a_1, \ldots, a_n, b \in \mathbb{Z}$ is NP-complete.

# Open Complexity

### Definition (SDP-feasibility)

INPUT symmetric matrizes $A_1, \dots, A_n, B \in \mathbb{Q}^{m \times m}$

OUTPUT YES if there exists $x \in \mathbb{R}^n$ s.t. $x_1 A_1 + \cdots + x_n A_n - B \succeq 0$, NO otherwise

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**Theorem ([Ram97])**

*SDP-feasibility* $\in$ NP $\cap$ coNP *or SDP-feasibility* $\notin$ NP $\cup$ coNP.

# Semilinear Constraints

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**Theorem ([JT16])**

$\mathrm{CSP}(\mathbb{R}; +, R_1, \ldots, R_n)$ *where* $R_1, \ldots, R_n$ *are semilinear is in* P *or* NP*-complete.*

# P-NP-Dichotomy

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**Theorem ([Bul17]; [Zhu20])**

*If* $\mathbb{A}$ *is finite, then* CSP($\mathbb{A}$) *is in* P *or* NP*-complete.*

# Dichotomies for FO-Reducts

## Definition

A relational structure $(A; R_1, \ldots, R_n)$ is a (finite signature) *first-order reduct* of $\mathbb{A}$ if each $R_i$ is first-order definable in $\mathbb{A}$.

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## Theorem ([BK10], [BMM18], [Bod+18])

*Every first-order reduct of*

1. $(\mathbb{Q}; <)$,
2. $(\mathbb{Z}; <)$,
3. $(\mathbb{Z}; +, 1)$ *containing* $+$

*has a CSP which is in* P *or is* NP*-complete.*

# Arbitrarily Complex CSPs

**Theorem ([Mat93])**

*$S \subseteq \mathbb{Z}$ is recursively enumerable iff it is pp-definable in $(\mathbb{Z}; \cdot, +, 1)$.*

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### Corollary

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**Corollary**

$\mathrm{CSP}(\mathbb{Z}; \cdot, +, 1)$ *is undecidable.*

**Theorem ([BM17])**

*Every recursively enumerable problem is polynomial-time Turing equivalent to the CSP of a first-order reduct of $(\mathbb{Z}; \cdot, +, 1)$.*

# Complicated Subproblems

## Definition (Sum-of-Square-Roots)

INPUT a list of natural numbers $a_1, \ldots, a_n, b$

OUTPUT YES if $b \leq \sqrt{a_1} + \cdots + \sqrt{a_n}$, NO otherwise

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In PSPACE but membership in PH $\supseteq$ NP unknown.

$$\text{SDP-feasibility}$$
$$\uparrow$$
$$\text{CSP}(\mathbb{R}, +, 1, x^2 \leq y)$$
$$\uparrow$$
$$\text{CSP}(\mathbb{R}, +, 1, x^2 + y^2 \leq 1)$$
$$\uparrow$$
$$\text{Sum-of-Square-Roots}$$

# Failure of the Algebraic Approach

$f : A^n \to A$ is a *polymorphism* of $(A, (R_i)_{i \in I})$ means

$$
\begin{array}{cccccc}
f( & a_1^1, & \ldots, & a_1^n & ) & = & b_1 \\
f( & a_2^1, & \ldots, & a_2^n & ) & = & b_2 \\
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f( & a_m^1, & \ldots, & a_m^n & ) & = & b_m \\
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For finite structures the collection of all polymorphisms determines the CSP's complexity.

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**Theorem ([JC95])**

*If* $\mathbb{A}$ *is finite and has binary* max *as a polymorphism, then* $\mathrm{CSP}(\mathbb{A}) \in \mathrm{P}$.

# Failure of the Algebraic Approach

### Theorem ([JC95])

*If* $\mathbb{A}$ *is finite and has binary* max *as a polymorphism, then* CSP($\mathbb{A}$) $\in$ P.

This does not hold for infinite CSPs.

### Example

$(\mathbb{Z}; 1, -1, 2x = y, x \leq y + z)$ has max as a polymorphism but its CSP is NP-complete.

# What can we do? Sampling

## Definition

$\mathbb{A}$ has a *sampling procedure* if for every pp-sentence $\phi$ we can construct in polynomial time a finite substructure $\mathbb{A}_\phi \subseteq \mathbb{A}$ such that

$$\mathbb{A} \models \phi \iff \mathbb{A}_\phi \models \phi.$$

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$(\mathbb{Z}, \text{succ}, 0) \models \phi \iff (\{-n, \ldots, n\}, \text{succ}, 0) \models \phi$ where $n$ is the number of variables in $\phi$.

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## Observation

*If* $\mathbb{A}$ *has a sampling procedure and* max *as a polymorphism, then* $\mathrm{CSP}(\mathbb{A}) \in \mathrm{P}$*.*

# What can we do? Saturation

### Theorem ([BMM18])

*Let $\mathbb{B}$ be a first-order reduct of a countable, saturated structure $\mathbb{A}$. If $R \subseteq A^n$ is first-order definable in $\mathbb{A}$ and consists of $k$ orbits of $n$-tuples in $\mathbb{B}$, then*

$$R \text{ pp-definable in } \mathbb{B} \iff R \text{ preserved by all } k\text{-ary polymorphisms of } \mathbb{B}.$$

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Main ingredient in classification of fo-reducts of $(\mathbb{Z}, <)$.

# Open Questions

1. Can we pp-define $x^6 \leq y$ over $\mathbb{R}$ using linear relations and $x^2 \leq y$?
2. Is $\mathrm{CSP}(\mathbb{R}; x+1=y, x^2 \leq y)$ in P?
3. $\mathrm{CSP}(\mathbb{Q}; x+1=y, x^2 \leq y) = \mathrm{CSP}(\mathbb{R}; x+1=y, x^2 \leq y)$?
4. Classify the CSPs of first-order reducts of
   4.1 $(\mathbb{Z}; \mathrm{succ}, 0)$,
   4.2 $(\mathbb{Q}; +)$,
   4.3 $(\mathbb{Z}; +, <)$.

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