

The class of congruence meet semidistributive varieties is not strong Maltsev

PALS Seminar

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October 7, 2025

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The interpretability lattice and Maltsev conditions

Definition

Let $\mathcal{V}_1 = \text{Mod}(\Sigma_1)$ and $\mathcal{V}_2 = \text{Mod}(\Sigma_2)$ be varieties over the signatures τ_1, τ_2 , respectively. An *interpretation* of $\mathcal{V}_1$ in $\mathcal{V}_2$ is a mapping $I : \tau_1 \rightarrow \text{Terms}(\tau_2)$ which preserves the satisfaction of Σ_1 identities, i.e.

$$f(x_1, \dots, x_n) \approx g(x_1, \dots, x_n) \in \Sigma_1 \implies \mathcal{V}_2 \models (If)(x_1, \dots, x_n) \approx (Ig)(x_1, \dots, x_n).$$

This induces a preorder $\preceq$ on the class of varieties and we write $\mathcal{V}_1 \preceq \mathcal{V}_2$ to indicate that $\mathcal{V}_1$ interprets in $\mathcal{V}_2$. If both $\mathcal{V}_1 \preceq \mathcal{V}_2$ and $\mathcal{V}_2 \preceq \mathcal{V}_1$, we say $\mathcal{V}_1$ and $\mathcal{V}_2$ are *equi-interpretable*.

Example

- The variety of groups interprets in the variety of abelian groups.
- The varieties of Boolean rings and Boolean algebras are equi-interpretable.
- The variety of sets and the variety of semigroups are equi-interpretable.
- The variety $\mathcal{V}$ in the signature with a single ternary operation $m(xyz)$ axiomatized by the Maltsev identities $m(yxx) \approx m(yxx) \approx y$ interprets in the variety of groups, by mapping $m(xyz)$ to the term $xy^{-1}z$.

Theorem

The class of varieties modulo the equi-interpretability relation forms a bounded lattice.

- Roughly, the interpretability lattice of varieties orders equivalence classes of varieties by 'strength of identities satisfied'.
- Upwards closed subclasses of the interpretability poset are therefore natural objects to study.

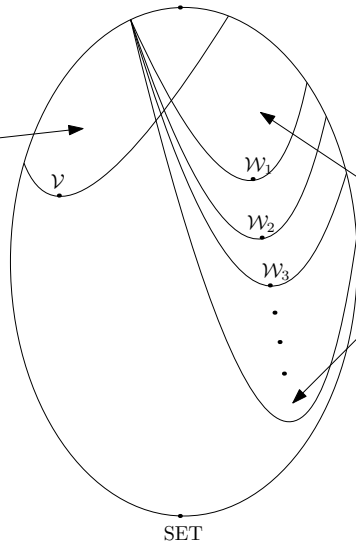
Definition

- A class of varieties $\mathcal{C}$ is said to be characterized by a *strong Maltsev condition* if there exists a finitely presented (finite signature and finitely based) variety $\mathcal{V}$ such that $\mathcal{C} = \{\mathcal{W} : \mathcal{V} \preceq \mathcal{W}\}$.
- A class of varieties $\mathcal{C}$ is said to be characterized by a *Maltsev condition* if there exists a countable sequence of finitely presented varieties $\mathcal{V}_1 \succeq \mathcal{V}_2 \succeq \cdots \succeq \mathcal{V}_i \succeq \cdots$ such that $\mathcal{C} = \bigcup_{1 \leq i} \{\mathcal{W} : \mathcal{V}_i \preceq \mathcal{W}\}$.

Interpretability lattice of varieties

TRIVIAL 1-ELEMENT ALGEBRAS

This class is characterized by the strong Maltsev condition given by $\mathcal{V}$ (which is assumed to be finitely presented).



This class is characterized by a proper Maltsev condition and is the union of an increasing sequence of strong Maltsev classes.

Maltsev conditions have been extensively studied and used in Universal algebra. The following are a few classical examples.

- Maltsev showed that a variety has permuting congruences if and only if it has a Maltsev term (hence the class of permutable varieties is characterized by a strong Maltsev condition).
- Pixley showed that the class of varieties which are congruence distributive and congruence permutable is a strong Maltsev class (Pixley term).
- The class of congruence distributive varieties is definable with a Maltsev condition (Jónsson).
- The class of congruence modular varieties is definable with a Maltsev condition (Day).
- Independently, Fichtner (1972) and Kelly (1973) proved that both congruence distributivity and congruence modularity are not definable with strong Maltsev conditions. Each of their arguments relies on some syntactic analysis.

Definition

A variety $\mathcal{V}$ of algebras is *congruence meet semidistributive* if each congruence lattice of its members satisfies the implication

$$\gamma \wedge \alpha = \gamma \wedge \beta \implies \gamma \wedge (\alpha \vee \beta) = \gamma \wedge \alpha.$$

- Kearnes and Szendrei, and independently Lipparini showed that the class of congruence meet semidistributive varieties is characterized by a Maltsev condition.
- Question: Is congruence meet semidistributivity definable with a strong Maltsev condition?

- The interpretability order can be restricted to the class of locally finite varieties. Siggers showed in 2015 that the class of locally finite Taylor varieties is characterized by a strong Maltsev condition. Olšák later proved that this is actually true for the class of all Taylor varieties.

TABLE 1. The six conditions

Type Omitting Class	Equivalent Property	Strong for l.f. varieties?	Strong in general?
$\mathcal{M}_{\{1\}}$	satisfies a nontrivial idempotent Maltsev condition	YES (Siggers)	YES (Olšák)
$\mathcal{M}_{\{1,5\}}$	satisfies a nontrivial congruence identity (see [17])	NO (KKVW)	NO
$\mathcal{M}_{\{1,4,5\}}$	congruence n -permutable, for some $n > 1$	NO (KKVW)	NO
$\mathcal{M}_{\{1,2\}}$	congruence meet semidistributive	YES (KKVW)	??
$\mathcal{M}_{\{1,2,5\}}$	congruence join semidistributive (see [17])	NO (KKVW)	NO
$\mathcal{M}_{\{1,2,4,5\}}$	congruence n -permutable for some n and congruence join semidistributive	NO (KKVW)	NO

Table taken from ‘Characterizations of several Maltsev conditions’ by Kozik, Krokhin, Valeriote, Willard.

The connection to the commutator and 2-congruences

Definition

Let $\mathbb{A}$ be an algebra and let θ_1, θ_2 be congruences of $\mathbb{A}$. We define the *algebra of (θ_1, θ_2) -matrices* as follows.

$$M(\theta_1, \theta_2) = \text{Sg}_{A^{2^2}} \left(\left\{ \begin{array}{c|c} x & y \\ \hline x & y \end{array} : (x, y) \in \theta_1 \right\} \cup \left\{ \begin{array}{c|c} y & y \\ \hline x & x \end{array} : (x, y) \in \theta_2 \right\} \right).$$

We then say that θ_1 *term condition centralizes* θ_2 if no matrix of $M(\theta_1, \theta_2)$ has one column which determines a pair of equal elements, while the opposite column determines a pair of unequal elements. The *term condition commutator* is the least congruence δ that one can factor $\mathbb{A}$ by so that θ_1/δ centralizes θ_2/δ . We denote this δ by $[\theta_1, \theta_2]_{TC}$. An *abelian* congruence of $\mathbb{A}$ is an α such that $[\alpha, \alpha]_{TC} = 0$

Theorem (Kearnes + Szendrei, and Lipparini)

The following are equivalent for a variety $\mathcal{V}$:

- 1. $\mathcal{V}$ is congruence meet semidistributive.*
- 2. $[\gamma, \gamma]_{TC} = \gamma$ for all congruences γ of algebras in $\mathcal{V}$ (such $\mathcal{V}$ are often called congruence neutral varieties).*
- 3. $[\gamma, \gamma]_{TC} = \gamma$, where γ is the principle congruence of $\mathbb{F}_{\mathcal{V}}(x, y)$ generated by (x, y) .*

Informally, the above theorem is stating that the class of congruence meet semidistributive varieties is exactly the class of varieties which have no nontrivial abelian congruences, and the latter condition (2) holds for a variety $\mathcal{V}$ if and only if there are no abelian principle congruences, which is true if and only if the ‘free’ principle congruence for $\mathcal{V}$ is neutral.

Definition

Let A be a set and let $R \subseteq A^{2^2}$. We say that R is

$$1. \text{ (2)-reflexive if } \begin{array}{|c|c|} \hline b & d \\ \hline a & c \\ \hline \end{array} \in R \text{ implies } \begin{array}{|c|c|} \hline a & c \\ \hline a & c \\ \hline \end{array}, \begin{array}{|c|c|} \hline b & d \\ \hline b & d \\ \hline \end{array}, \begin{array}{|c|c|} \hline d & d \\ \hline c & c \\ \hline \end{array}, \begin{array}{|c|c|} \hline b & b \\ \hline a & a \\ \hline \end{array} \in R,$$

$$2. \text{ (2)-symmetric if } \begin{array}{|c|c|} \hline b & d \\ \hline a & c \\ \hline \end{array} \in R \text{ implies } \begin{array}{|c|c|} \hline a & c \\ \hline b & d \\ \hline \end{array}, \begin{array}{|c|c|} \hline b & d \\ \hline c & a \\ \hline \end{array} \in R,$$

3. (2)-transitive if

$$\bullet \begin{array}{|c|c|} \hline b & d \\ \hline a & c \\ \hline \end{array}, \begin{array}{|c|c|} \hline d & f \\ \hline c & e \\ \hline \end{array} \in R \text{ implies } \begin{array}{|c|c|} \hline b & f \\ \hline a & e \\ \hline \end{array} \in R$$

$$\bullet \begin{array}{|c|c|} \hline b & d \\ \hline a & c \\ \hline \end{array}, \begin{array}{|c|c|} \hline e & f \\ \hline b & d \\ \hline \end{array} \in R \text{ implies } \begin{array}{|c|c|} \hline e & f \\ \hline a & c \\ \hline \end{array} \in R$$

Definition

We say that R is a (2)-equivalence relation on A if it is (2)-reflexive, (2)-symmetric, and (2)-transitive. If A is the universe of an algebra $\mathbb{A}$, we say that R is

1. A (2)-tolerance of $\mathbb{A}$ if it is $\mathbb{A}$ -invariant, (2)-reflexive, and (2)-symmetric.
2. A (2)-congruence of $\mathbb{A}$ if it is $\mathbb{A}$ -invariant, (2)-reflexive, (2)-symmetric, and (2)-transitive.

We refer to the (2)-congruence *generated* by $\text{Cg}_2(X)$. We now define the relation

$$\Delta(\theta, \theta) = \text{Cg}_2 \left(\left\{ \begin{array}{cc} x & \text{---} & y \\ | & & | \\ x & \text{---} & y \end{array} : (x, y) \in \theta_1 \right\} \cup \left\{ \begin{array}{cc} y & \text{---} & y \\ | & & | \\ x & \text{---} & x \end{array} : (x, y) \in \theta_2 \right\} \right),$$

for congruences θ_1 and θ_2 of an algebra $\mathbb{A}$. We say θ_1 *hypercentralizes* θ_2 if no matrix of $\Delta(\theta_1, \theta_2)$ has one column which determines a pair of equal elements, while the opposite column determines a pair of unequal elements. We denote by $[\theta_1, \theta_2]_H$ the corresponding commutator.

Let $S \leq A^{2^2}$. We define

$$\begin{aligned} \bullet \quad H(S) &:= \left\{ \begin{array}{ccc} b & \text{---} & d \\ | & & | \\ a & \text{---} & c \end{array} : \exists e, f \left(\begin{array}{ccc} b & \text{---} & f \\ | & & | \\ a & \text{---} & e \end{array}, \begin{array}{ccc} f & \text{---} & d \\ | & & | \\ e & \text{---} & c \end{array} \in S \right) \right\} \text{ and} \\ \bullet \quad V(S) &:= \left\{ \begin{array}{ccc} b & \text{---} & d \\ | & & | \\ a & \text{---} & c \end{array} : \exists e, f \left(\begin{array}{ccc} b & \text{---} & d \\ | & & | \\ e & \text{---} & f \end{array}, \begin{array}{ccc} e & \text{---} & f \\ | & & | \\ a & \text{---} & c \end{array} \in S \right) \right\} \end{aligned}$$

It is not hard to see that

$$\Delta(\theta, \theta) = \bigcup_{n \geq 0} (V \circ H)^n(M(\theta, \theta)).$$

In particular, $M(\theta, \theta) \subseteq \Delta(\theta, \theta)$, so it follows that $[\theta_1, \theta_2]_{TC} \leq [\theta_1, \theta_2]_H$.

The following two theorems are important for our characterization of congruence meet semidistributivity.

Theorem

Let $\mathbb{A}$ be an algebra and let θ be a congruence of $\mathbb{A}$. The following are equivalent.

1. $(a, b) \in [\theta, \theta]_H$, and

2. $\begin{array}{ccc} x & \text{---} & y \\ | & & | \\ x & \text{---} & x \end{array} \in \Delta(\theta, \theta)$

Theorem (Follows from a result of Kearnes and Szendrei)

Let $\mathbb{A}$ be a Taylor algebra and let α be a congruence of $\mathbb{A}$. Then

$$[\alpha, \alpha]_{TC} = [\alpha, \alpha]_H.$$

We define for a variety $\mathcal{V}$ and the (x, y) -elementary matrices:

$$E_{\mathcal{V}}(x, y) := \text{Sg}_{\mathbb{F}_{\mathcal{V}}(x, y)^{2^2}} \left(\left\{ \begin{array}{cc} x & \text{---} & x \\ | & & | \\ x & \text{---} & x \end{array}, \begin{array}{cc} y & \text{---} & y \\ | & & | \\ y & \text{---} & y \end{array}, \begin{array}{cc} y & \text{---} & y \\ | & & | \\ x & \text{---} & x \end{array}, \begin{array}{cc} x & \text{---} & x \\ | & & | \\ y & \text{---} & y \end{array}, \begin{array}{cc} x & \text{---} & y \\ | & & | \\ x & \text{---} & y \end{array}, \begin{array}{cc} y & \text{---} & x \\ | & & | \\ y & \text{---} & x \end{array} \right\} \right).$$

Theorem

Let $\mathcal{V}$ be a variety. The following are equivalent.

1. $\mathcal{V}$ is congruence meet semidistributive,

2. $\begin{array}{cc} x & \text{---} & x \\ | & & | \\ x & \text{---} & y \end{array} \in \Delta(\gamma, \gamma)$, where γ is the congruence of the two generated free algebra $\mathbb{F}_{\mathcal{V}}(x, y)$ in $\mathcal{V}$ generated by the pair (x, y) , and

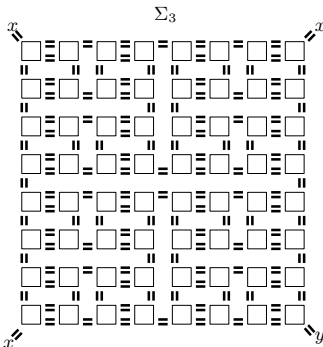
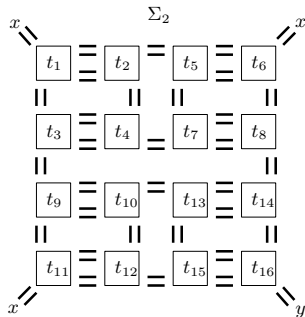
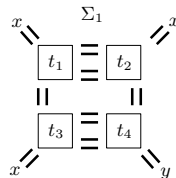
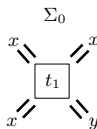
3. $\begin{array}{cc} x & \text{---} & x \\ | & & | \\ x & \text{---} & y \end{array} \in \text{Cg}_2(E_{\mathcal{V}}(x, y)) = \bigcup_{n \geq 0} (V \circ H)^n(E_{\mathcal{V}}(x, y)).$

$$\boxed{t} \equiv \equiv \boxed{s}$$

stands for

$$\begin{array}{ccccccc} t(xxyyyy) & \text{---} & t(xyxyyx) & = & s(xxyyyy) & \text{---} & s(xyxyyx) \\ | & & | & & | & & | \\ t(xyxyxy) & \text{---} & t(xyxyyx) & = & s(xyxyxy) & \text{---} & s(xyxyyx) \end{array}$$

The equational conditions $\Sigma_1, \Sigma_2, \dots, \Sigma_i, \dots$ which determine a Maltsev condition for congruence meet semidistributivity:



...

- We denote by $\mathcal{V}_n$ the variety of algebras in the signature $\{t_1, \dots, t_{4^n}\}$ which satisfy the package of identities Σ_n .
- To show that there is no strong Maltsev condition that characterizes congruence meet semidistributivity, it suffices to produce for each $n \geq 0$ some congruence meet semidistributive variety $\mathcal{W}$ that does not interpret $\mathcal{V}_n$.
- We define the following sequence of conditions $\Lambda_1, \dots, \Lambda_l, \dots$ and let $\mathcal{W}_l$ be the variety of algebras satisfying Λ_l .

$$\begin{array}{ccc} xyxx - xxyx & & s(xyxx) - s(xxyx) \\ | \quad s \quad | & \text{stands for} & | \quad | \\ yyxx - yxyx & & s(yyxx) - s(yxyx) \end{array}$$

$$\begin{array}{ccccccc} x & & & & & & x \\ // & & & & & & // \\ xxxx - xyxx = xyxx - xxyx = xxyx - xyxx = & \dots & = xxyx - xyxx = xyxx - xyyy & & & & \\ | \quad s_0 \quad | & & | \quad s_{2l} \quad | & & | \quad s_{2l+1} \quad | & & \\ yxxx - yyxx = yyxx - yxyx = yxyx - yyxx = & \dots & = yxyx - yyxx = yyxx - yyyy & & & & \\ x & & & & & & y \\ // & & & & & & // \end{array}$$

The condition Λ_l

Term analysis

Recall that

$$E_{\mathcal{W}_l}(x, y) := \text{Sg}_{(\mathbb{F}_l)^{2^2}} \left(\left\{ \begin{array}{ccc|ccc} x & \text{---} & x & y & \text{---} & y \\ | & & | & | & & | \\ x & \text{---} & x & y & \text{---} & y \end{array} , \begin{array}{ccc|ccc} y & \text{---} & y & y & \text{---} & y \\ | & & | & | & & | \\ y & \text{---} & y & x & \text{---} & x \end{array} , \begin{array}{ccc|ccc} y & \text{---} & y & x & \text{---} & x \\ | & & | & | & & | \\ x & \text{---} & x & y & \text{---} & y \end{array} , \begin{array}{ccc|ccc} x & \text{---} & x & x & \text{---} & y \\ | & & | & | & & | \\ y & \text{---} & y & x & \text{---} & y \end{array} , \begin{array}{ccc|ccc} x & \text{---} & y & y & \text{---} & x \\ | & & | & | & & | \\ x & \text{---} & y & y & \text{---} & x \end{array} , \begin{array}{ccc|ccc} y & \text{---} & x & y & \text{---} & x \\ | & & | & | & & | \\ y & \text{---} & x & y & \text{---} & x \end{array} \right\} \right).$$

We want to show that for any n , there exists l so that it is impossible to glue together such squares to obtain a diagram witnessing the condition Σ_n . Let $\tau_l = \{s_0, \dots, s_{2l+1}\}$ be the signature corresponding to Λ_l . Consider the sets $E_0 \subseteq E_1 \subseteq \dots \subseteq E_k \subseteq \dots$ whose union is $E_{\mathcal{W}_l}(x, y)$ defined by

$$E_0 = \left\{ \begin{array}{ccc|ccc} x & \text{---} & x & y & \text{---} & y \\ | & & | & | & & | \\ x & \text{---} & x & y & \text{---} & y \end{array} , \begin{array}{ccc|ccc} y & \text{---} & y & y & \text{---} & y \\ | & & | & | & & | \\ y & \text{---} & y & x & \text{---} & x \end{array} , \begin{array}{ccc|ccc} y & \text{---} & y & x & \text{---} & x \\ | & & | & | & & | \\ x & \text{---} & x & y & \text{---} & y \end{array} , \begin{array}{ccc|ccc} x & \text{---} & x & x & \text{---} & y \\ | & & | & | & & | \\ y & \text{---} & y & x & \text{---} & y \end{array} , \begin{array}{ccc|ccc} x & \text{---} & y & y & \text{---} & x \\ | & & | & | & & | \\ x & \text{---} & y & y & \text{---} & x \end{array} , \begin{array}{ccc|ccc} y & \text{---} & x & y & \text{---} & x \\ | & & | & | & & | \\ y & \text{---} & x & y & \text{---} & x \end{array} \right\} \text{ and}$$

$$E_{k+1} = \{r^{(\mathbb{F}_l)^{2^2}}(\alpha, \beta, \gamma, \delta) : r \in \tau_l \text{ and } \alpha, \beta, \gamma, \delta \in E_k\} \text{ for } k \geq 0.$$

Our goal is to show that, for large enough l , there is no k where

$$\begin{array}{ccc} x & \text{---} & x \\ | & & | \\ x & \text{---} & y \end{array} \in (V \circ H)^n(E_k).$$

- An idea: What if we could treat subterms like free generators?
- If we define $\Lambda_{l,i}$ to be the condition produced by deleting the operation s_i from the signature τ_l and all identities mentioning it from Λ_l , then the corresponding variety $\mathcal{W}_{l,i}$ is equi-interpretable with SET.
- Then it would be possible to reduce the complexity of terms of any diagram witnessing Σ_n by interpreting $\tau_{l,i}$ operation symbols as projections, which would lead to a contradiction, since squares belonging E_0 cannot be arranged to witness Σ_n .

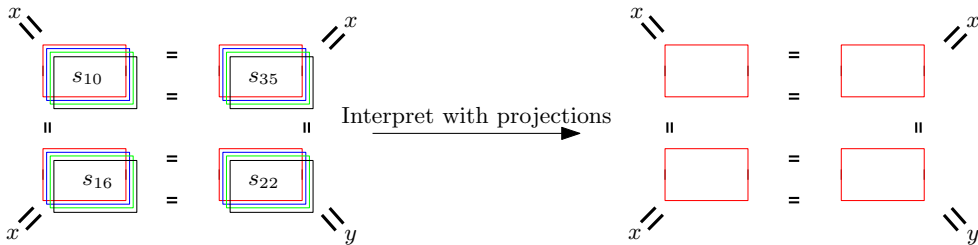
The condition Λ_l

$$\begin{array}{ccccccc}
 x & & & & & & x \\
 \parallel & & & & & & \parallel \\
 xxx - yxx & = & yxx - xxy & = & xxy - yxx & = & \dots = xxy - yxx = yxx - xyy \\
 | \quad s_0 & | & | \quad s_1 & | & | \quad s_2 & | & | \quad s_{2l} & | & | \quad s_{2l+1} \\
 yxx - yyx & = & yyx - yxy & = & yxy - yyx & = & \dots = yxy - yyx = yyx - yyy \\
 x & & & & & & & & y \\
 \parallel & & & & & & & & \parallel
 \end{array}$$

The condition $\Lambda_{l,1}$ is modeled by projections.

$$\begin{array}{ccc}
 x & & x \\
 \parallel & & \parallel \\
 xxx - yxx & & xxy - yxx = \dots = xxy - yxx = yxx - xyy \\
 | \quad s_0 & & | \quad s_2 & | & | \quad s_{2l} & | & | \quad s_{2l+1} \\
 yxx - yyx & & yxy - yyx = \dots = yxy - yyx = yyx - yyy \\
 x & & & & & & y \\
 \parallel & & & & & & \parallel \\
 \pi_4 & & \pi_1 & & \pi_1 & & \pi_1
 \end{array}$$

delete s_1



- To prove that the strategy actually works, we prefer to construct the free algebra recursively so that its underlying set is a collection of minimal term complexity normal forms. We do this to avoid worrying that we overlooked any equalities between terms which follow from equational logic.
- We define a sequence of sets $\{x, y\} = F_l^0 \subseteq F_l^1 \subseteq \dots \subseteq F_l^{k-1} \subseteq F_l^k \dots$, where each F_l^k is the domain of a partial τ_l -algebra $\mathbb{F}_l^k$ with all τ_l operations defined on $(F_l^{k-1})^4$, for every $k \geq 1$.
- Given the partial τ_l -algebra $\mathbb{F}_l^k$, the partial τ_l -algebra $\mathbb{F}_l^{k+1}$ is defined by extending the operation $r^{\mathbb{F}_l^k}$ to $(F_l^k)^l$ either by applying Λ_l -identities or choosing a new term and adding it to F_l^{k+1} .
- Then define $F_l = \bigcup_{1 \leq k} F_l^k$ and interpret the operations in the obvious way.

	$s_0^{\mathbb{F}_l^1}$	$s_1^{\mathbb{F}_l^1}$	$s_2^{\mathbb{F}_l^1}$	$\dots$	$s_{2l}^{\mathbb{F}_l^1}$	$s_{2l+1}^{\mathbb{F}_l^1}$
$xxxx$	x	x	x	$\dots$	x	x
$xxxy$	$s_0(xxxy)$	$s_1(xxxy)$	$s_2(xxxy)$	$\dots$	$s_{2l}(xxxy)$	$s_{2l+1}(xxxy)$
$xyyx$	$s_0(xyyx)$	$s_1(xyyx)$	$\leftarrow \mathbf{s}_1(\mathbf{xyyx})$	$\dots$	$s_{2l}(xyyx)$	$s_{2l+1}(xyyx)$
$xyyy$	$s_0(xyyy)$	$\leftarrow \mathbf{s}_0(\mathbf{xyyy})$	$s_2(xyyy)$	$\dots$	$s_{2l}(xyyy)$	$\leftarrow \mathbf{s}_{2l}(\mathbf{xyyy})$
$xyxx$	$s_0(xyxx)$	$\leftarrow \mathbf{s}_0(\mathbf{xyxx})$	$s_2(xyxx)$	$\dots$	$s_{2l}(xyxx)$	$\leftarrow \mathbf{s}_{2l}(\mathbf{xyxx})$
$xyxy$	$s_0(xyxy)$	$s_1(xyxy)$	$\leftarrow \mathbf{s}_1(\mathbf{xyxy})$	$\dots$	$s_{2l}(xyxy)$	$s_{2l+1}(xyxy)$
$xyyx$	$s_0(xyyx)$	$s_1(xyyx)$	$s_2(xyyx)$	$\dots$	$s_{2l}(xyyx)$	$s_{2l+1}(xyyx)$
$xyyy$	y	$s_1(xyyy)$	$s_2(xyyy)$	$\dots$	$s_{2l}(xyyy)$	x
$yxxx$	x	$s_1(yxxx)$	$s_2(yxxx)$	$\dots$	$s_{2l}(yxxx)$	y
$yxxxy$	$s_0(yxxxy)$	$s_1(yxxxy)$	$s_2(yxxxy)$	$\dots$	$s_{2l}(yxxxy)$	$s_{2l+1}(yxxxy)$
$yxyyx$	$s_0(yxyyx)$	$s_1(yxyyx)$	$\leftarrow \mathbf{s}_1(\mathbf{yxyyx})$	$\dots$	$s_{2l}(yxyyx)$	$s_{2l+1}(yxyyx)$
$yxyyy$	$s_0(yxyyy)$	$\leftarrow \mathbf{s}_0(\mathbf{yxyyy})$	$s_2(yxyyy)$	$\dots$	$s_{2l}(yxyyy)$	$\leftarrow \mathbf{s}_{2l}(\mathbf{yxyyy})$
$yyxx$	$s_0(yyxx)$	$\leftarrow \mathbf{s}_0(\mathbf{yyxx})$	$s_2(yyxx)$	$\dots$	$s_{2l}(yyxx)$	$\leftarrow \mathbf{s}_{2l}(\mathbf{yyxx})$
$yyxy$	$s_0(yyxy)$	$s_1(yyxy)$	$\leftarrow \mathbf{s}_1(\mathbf{yyxy})$	$\dots$	$s_{2l}(yyxy)$	$s_{2l+1}(yyxy)$
$yyyx$	$s_0(yyyx)$	$s_1(yyyx)$	$s_2(yyyx)$	$\dots$	$s_{2l}(yyyx)$	$s_{2l+1}(yyyx)$
$yyyy$	y	y	y	$\dots$	y	y

Input tuples $(a, b, c, d) \in F_l^k$ satisfy $\{a, b, c, d\} \cap (F_l^k \setminus F_l^{k-1}) \neq \emptyset$

	$\mathbb{F}_l^{k+1}$ s_0	$\mathbb{F}_l^{k+1}$ s_1	$\mathbb{F}_l^{k+1}$ s_2	$\dots$	$\mathbb{F}_l^{k+1}$ s_{2l}	$\mathbb{F}_l^{k+1}$ s_{2l+1}
$pppp$	p	p	p	$\dots$	p	p
$pppq$	$s_0(pppq)$	$s_1(pppq)$	$s_2(pppq)$	$\dots$	$s_{2l}(pppq)$	$s_{2l+1}(pppq)$
$ppqp$	$s_0(ppqp)$	$s_1(ppqp)$	$\leftarrow \mathbf{s_1(ppqp)}$	$\dots$	$s_{2l}(ppqp)$	$s_{2l+1}(ppqp)$
$ppqq$	$s_0(ppqq)$	$\leftarrow \mathbf{s_0(ppqq)}$	$s_2(ppqq)$	$\dots$	$s_{2l}(ppqq)$	$\leftarrow \mathbf{s_{2l}(ppqq)}$
$pqpp$	$s_0(pqpp)$	$\leftarrow \mathbf{s_0(pqpp)}$	$s_2(pqpp)$	$\dots$	$s_{2l}(pqpp)$	$\leftarrow \mathbf{s_{2l}(pqpp)}$
$pqpq$	$s_0(pqpq)$	$s_1(pqpq)$	$\leftarrow \mathbf{s_1(pqpq)}$	$\dots$	$s_{2l}(pqpq)$	$s_{2l+1}(pqpq)$
$pqqp$	$s_0(pqqp)$	$s_1(pqqp)$	$s_2(pqqp)$	$\dots$	$s_{2l}(pqqp)$	$s_{2l+1}(pqqp)$
$pqqq$	q	$s_1(pqqq)$	$s_2(pqqq)$	$\dots$	$s_{2l}(pqqq)$	p

Input tuples $(a, b, c, d) \in F_l^k$ satisfy $\{a, b, c, d\} \cap (F_l^k \setminus F_l^{k-1}) \neq \emptyset$ and $|\{a, b, c, d\}| \geq 3$

	$\mathbb{F}_l^{k+1}$ s_0	$\mathbb{F}_l^{k+1}$ s_1	$\mathbb{F}_l^{k+1}$ s_2	$\dots$	$\mathbb{F}_l^{k+1}$ s_{2l}	$\mathbb{F}_l^{k+1}$ s_{2l+1}
$abcd$	$s_0(abcd)$	$s_1(abcd)$	$s_2(abcd)$	$\dots$	$s_{2l}(abcd)$	$s_{2l+1}(abcd)$

Lemma

Let $l \geq 1$, and $m \geq 1$. Consider a set of τ_l -terms of the form

$$T = \{r_j(a_j, b_j, c_j, d_j) : r_j \in \tau_l \text{ and } a_j, b_j, c_j, d_j \in F_l, \text{ for } 1 \leq j \leq m\}$$

and let

$$Z = \{z : \text{there exists } r_j(a_j, b_j, c_j, d_j) \in T \text{ with } r_j = s_z\}$$

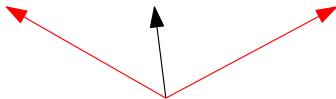
be the set of indices of basic τ_l -operation symbols which appear as the outer symbol for a term in T . If there exists $0 \leq i \leq 2l + 1$ such that $|z - i| \geq 2$ for all $z \in Z$, then

$$\begin{aligned} r_{j_1}^{\mathbb{F}_l}(a_{j_1}, b_{j_1}, c_{j_1}, d_{j_1}) &= r_{j_2}^{\mathbb{F}_l}(a_{j_2}, b_{j_2}, c_{j_2}, d_{j_2}) \iff \\ r_{j_1}^{\mathbb{G}}(a_{j_1}, b_{j_1}, c_{j_1}, d_{j_1}) &= r_{j_2}^{\mathbb{G}}(a_{j_2}, b_{j_2}, c_{j_2}, d_{j_2}), \end{aligned}$$

where $\mathbb{G} = \mathbb{F}_{\mathcal{W}_{l,i}}(\bigcup_{1 \leq j \leq m} \{a_j, b_j, c_j, d_j\})$ is the algebra for $\mathcal{W}_{l,i}$ freely generated by

$$\bigcup_{1 \leq j \leq m} \{a_j, b_j, c_j, d_j\}.$$

	$\mathbb{F}_l^{k+1}$ s_0	$\mathbb{F}_l^{k+1}$ s_1	$\mathbb{F}_l^{k+1}$ s_2	$\dots$	$\mathbb{F}_l^{k+1}$ s_{2l}	$\mathbb{F}_l^{k+1}$ s_{2l+1}
$pppp$	p	p	p	$\dots$	p	p
$pppq$	$s_0(pppq)$	$s_1(pppq)$	$s_2(pppq)$	$\dots$	$s_{2l}(pppq)$	$s_{2l+1}(pppq)$
$ppqp$	$s_0(ppqp)$	$s_1(ppqp)$	$\leftarrow \mathbf{s_1(ppqp)}$	$\dots$	$s_{2l}(ppqp)$	$s_{2l+1}(ppqp)$
$ppqq$	$s_0(ppqq)$	$\leftarrow \mathbf{s_0(ppqq)}$	$s_2(ppqq)$	$\dots$	$s_{2l}(ppqq)$	$\leftarrow \mathbf{s_{2l}(ppqq)}$
$pqpp$	$s_0(pqpp)$	$\leftarrow \mathbf{s_0(pqpp)}$	$s_2(pqpp)$	$\dots$	$s_{2l}(pqpp)$	$\leftarrow \mathbf{s_{2l}(pqpp)}$
$pqpq$	$s_0(pqpq)$	$s_1(pqpq)$	$\leftarrow \mathbf{s_1(pqpq)}$	$\dots$	$s_{2l}(pqpq)$	$s_{2l+1}(pqpq)$
$pqqp$	$s_0(pqqp)$	$s_1(pqqp)$	$s_2(pqqp)$	$\dots$	$s_{2l}(pqqp)$	$s_{2l+1}(pqqp)$
$pqqq$	q	$s_1(pqqq)$	$s_2(pqqq)$	$\dots$	$s_{2l}(pqqq)$	p



Intution: if $i \in Z$, where Z is the set of indices of basic τ_l -operation symbols being used, then we *at most* need to use identities involving s_{i-1} , s_i , and s_{i+1} to find the normal form for some $s_i(a, b, c, d)$.

Hence, for fixed n , there is obviously l large enough so that the lemma applies when $|Z| = 4^n$.

Summarizing:

- Fix $n \geq 0$ and choose l so that the lemma applies ($l > 2 \cdot 4^n$ for example).
- We want to show that $\mathcal{W}_l$ has no Σ_n -terms. Assume to the contrary, then

$$\begin{array}{ccc} x & \text{---} & x \\ | & & | \\ x & \text{---} & y \end{array} \in \bigcup_{n \geq 0} (V \circ H)^n (E_{\mathcal{V}}(x, y)).$$

We defined $E_0 = \left\{ \begin{array}{ccc} x & \text{---} & x \\ | & & | \\ x & \text{---} & x \end{array}, \begin{array}{ccc} y & \text{---} & y \\ | & & | \\ y & \text{---} & y \end{array}, \begin{array}{ccc} y & \text{---} & y \\ | & & | \\ x & \text{---} & x \end{array}, \begin{array}{ccc} x & \text{---} & x \\ | & & | \\ y & \text{---} & y \end{array}, \begin{array}{ccc} x & \text{---} & y \\ | & & | \\ x & \text{---} & y \end{array}, \begin{array}{ccc} y & \text{---} & x \\ | & & | \\ y & \text{---} & x \end{array} \right\}$ and

$$E_{k+1} = \{r^{(\mathbb{F}_l)^{2^2}}(\alpha, \beta, \gamma, \delta) : r \in \tau_l \text{ and } \alpha, \beta, \delta, \gamma \in E_k\} \text{ for } k \geq 0.$$

- Choose k minimal so that $\begin{array}{ccc} x & \text{---} & x \\ | & & | \\ x & \text{---} & y \end{array} \in \bigcup_{n \geq 0} (V \circ H)^n (E_k(x, y))$. Obviously, $k \neq 0$.

We apply the lemma and find that $k - 1$ works also, contradiction.

$$\begin{array}{ccccccc}
r_1^{\mathbb{F}_l}(xxxx) & & & & & & r_6^{\mathbb{F}_l}(xxxx) \\
\parallel & & & & & & \parallel \\
\boxed{r_1(\alpha_1, \beta_1, \gamma_1, \delta_1)} & \equiv & \boxed{r_2(\alpha_2, \beta_2, \gamma_2, \delta_2)} & = & \boxed{r_5(\alpha_5, \beta_5, \gamma_5, \delta_5)} & \equiv & \boxed{r_6(\alpha_6, \beta_6, \gamma_6, \delta_6)} \\
\parallel & & & \parallel & \parallel & & \parallel \\
\boxed{r_3(\alpha_3, \beta_3, \gamma_3, \delta_3)} & \equiv & \boxed{r_4(\alpha_4, \beta_4, \gamma_4, \delta_4)} & = & \boxed{r_7(\alpha_7, \beta_7, \gamma_7, \delta_7)} & \equiv & \boxed{r_8(\alpha_8, \beta_8, \gamma_8, \delta_8)} \\
\parallel & & & & & & \parallel \\
\boxed{r_9(\alpha_9, \beta_9, \gamma_9, \delta_9)} & \equiv & \boxed{r_{10}(\alpha_{10}, \beta_{10}, \gamma_{10}, \delta_{10})} & = & \boxed{r_{13}(\alpha_{13}, \beta_{13}, \gamma_{13}, \delta_{13})} & \equiv & \boxed{r_{14}(\alpha_{14}, \beta_{14}, \gamma_{14}, \delta_{14})} \\
\parallel & & & \parallel & \parallel & & \parallel \\
\boxed{r_{11}(\alpha_{11}, \beta_{11}, \gamma_{11}, \delta_{11})} & \equiv & \boxed{r_{12}(\alpha_{12}, \beta_{12}, \gamma_{12}, \delta_{12})} & = & \boxed{r_{15}(\alpha_{15}, \beta_{15}, \gamma_{15}, \delta_{15})} & \equiv & \boxed{r_{16}(\alpha_{16}, \beta_{16}, \gamma_{16}, \delta_{16})} \\
\parallel & & & & & & \parallel \\
r_{11}^{\mathbb{F}_l}(xxxx) & & & & & & r_{16}^{\mathbb{F}_l}(yyyy)
\end{array}$$

$$\boxed{r(\alpha, \beta, \gamma, \delta)} \quad \text{stands for} \quad r^{\mathbb{F}_l^{2^2}} \left(\boxed{\alpha} \quad \boxed{\beta} \quad \boxed{\gamma} \quad \boxed{\delta} \right)$$

Remarks

- Working on a more direct argument to prove the equivalence of congruence meet semidistributivity, the neutrality of the hypercommutator, and the condition involving $E_{\mathcal{V}}(x, y)$.
- I believe that a similar style of argument shows that the class of varieties having a weak difference term is also not definable with a strong Maltsev condition.

Thank you for your attention!