A Topological Proof of the Hell–Nešetřil Dichotomy

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Hardness Theorems

Theorem

The problem 3-SAT is NP-hard.

Theorem

Graph 3-coloring is NP hard.

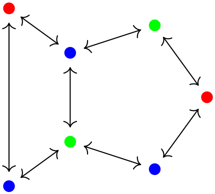
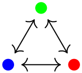


Figure: A graph colored in  .

Graph H-Coloring

Problem (Graph H-Coloring)

input	*an (undirected) graph G*
output yes	*if there is a map from the vertices of G to the vertices of H mapping edges to edges.*
output no	*else*

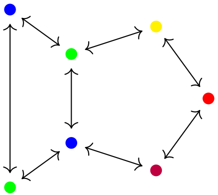
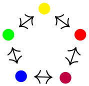


Figure: A graph colored in  .

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Theorem (Hell, Nešetřil 1990)

If H is bipartite or has a loop, then H-coloring is in P.
Else, the H-coloring problem is NP-complete.

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If H is bipartite or has a loop, then H-coloring is in P.
Else, the H-coloring problem is NP-complete.

We give a new, short, topological proof.

Simplicial Complex

Examples

- Geometrical

Figure: A contractible and a non-contractible simplicial complex.

Simplicial Complex

Definition

An abstract simplicial complex is a set C of vertices together with a set $F \subseteq \wp(C)$ called faces such that

$$\begin{aligned} \forall c \in C : &\qquad \{c\} \in F \\ \forall A \in F, A' \subseteq A : &\qquad A' \in F \end{aligned}$$

holds.

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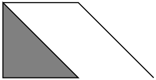
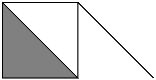
- Geometrical





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holds.

Examples

- C attendees of this conference,
  F groups that joint the same session at some time
- Geometrical

Figure: A contractible and a non-contractible simplicial complex.

Simplicial Complex Coloring

Problem (Coloring problem of a simplicial complex (C, F))

input *a set of variables V,*
for some variables v a vertex c_v in C
some subsets of V called connected variables

output yes *if there is a map f from V to C such that*

- *$f(v) = c_v$ for respective variables and*
- *connected variables are mapped to faces*

output no *else*

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Theorem (new)

If (C, F) is not contractible, then the (C, F)-coloring problem is NP-hard.

When and Why do Efficient Algorithms Exists for Constraint Satisfaction?

Theorem (Bulatov, Jeavons, Krokhin 2005)(Bulatov 2017; Zhuk 2017)

- *The constraint satisfaction problem associated to a structure A is NP-hard, if A has no Taylor polymorphism.*
- *Else, it is in P.*

When and Why do Efficient Algorithms Exists for Constraint Satisfaction?

A polymorphism of A is a homomorphism $A^n \to A$.

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A polymorphism of A is a homomorphism $A^n \to A$.
A polymorphism f is Taylor, it it is prevented from being a projection by specific identities.

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Taylor Polymorphism

A polymorphism f is Taylor, if it satisfies identities of the form

$$
\begin{aligned}
\exists s_1 \forall x, y: &\quad s_1(x,y) = f(x, \bullet, \dots, \bullet) = f(y, \bullet, \dots, \bullet) \\
\exists s_2 \forall x, y: &\quad s_2(x,y) = f(\bullet, x, \dots, \bullet) = f(\bullet, y, \dots, \bullet) \\
&\quad \vdots \\
\exists s_n \forall x, y: &\quad s_n(x,y) = f(\bullet, \bullet, \dots, x) = f(\bullet, \bullet, \dots, y)
\end{aligned}
$$

Examples:

$$
\begin{aligned}
f(x,y) &= f(y,x) \\
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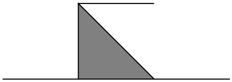
\end{aligned}
$$

Left to Show

What is left to show the Topological Hell–Nešetřil dichotomy?

- Show that non-bipartite Taylor graphs have a loop. (Bulatov 2005)
- Show that Taylor simplicial complexes are contractible.



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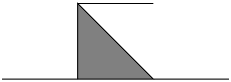

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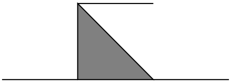

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How? Topology . . .

Theorem (Corollary of Lefschetz fixed-point theorem)

Every automorphism of a finite contractible simplicial complex has a fixed face.

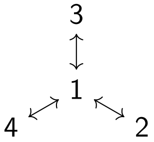
The Box Complex

Definition

The box complex of a graph is the set of complete bipartite subgraphs.

- It is a finite poset with inclusion.
- It is a simplicial complex where a face is a totally ordered subset.

Example



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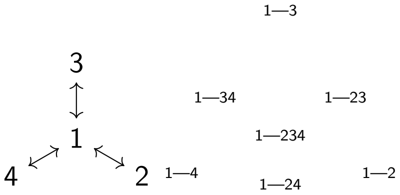

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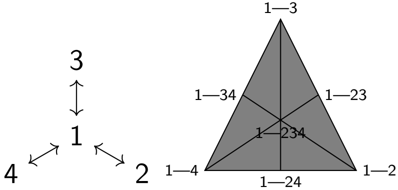

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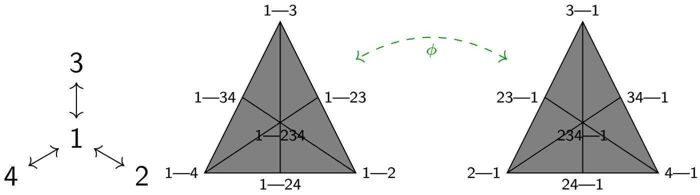

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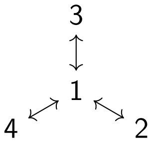

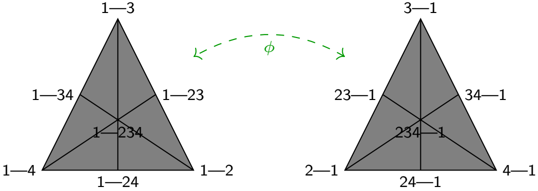

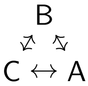

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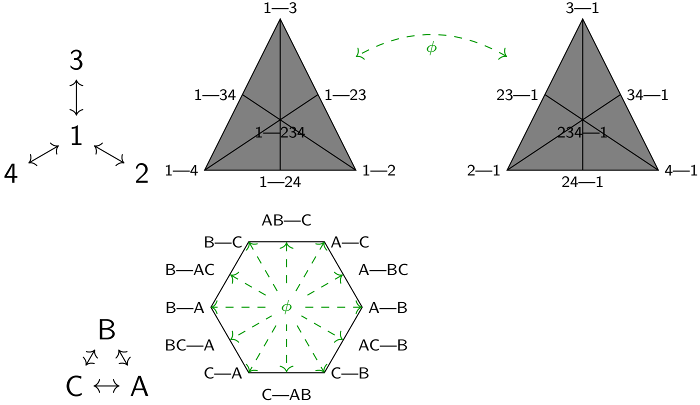

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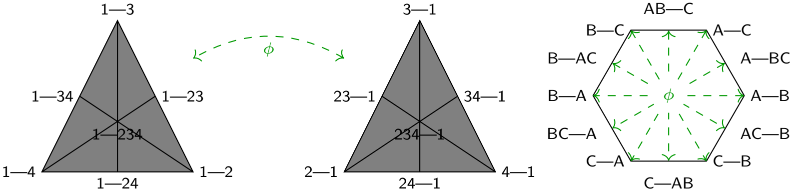

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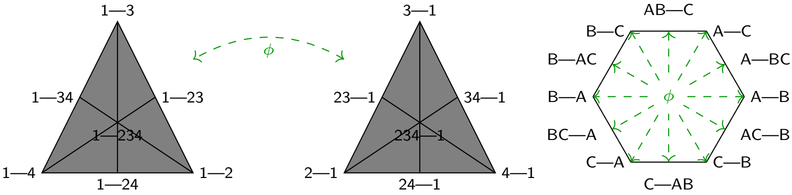

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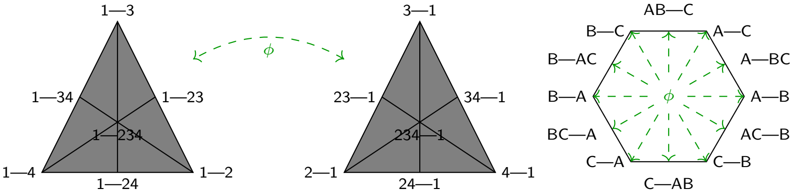

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- It is connected (if the graph is connected, non-bipartite).

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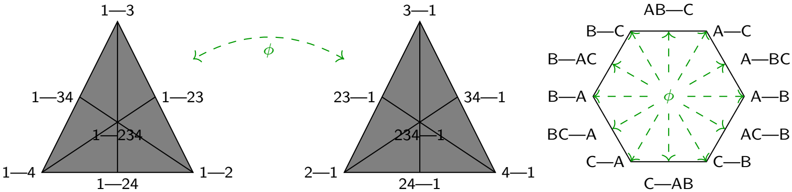

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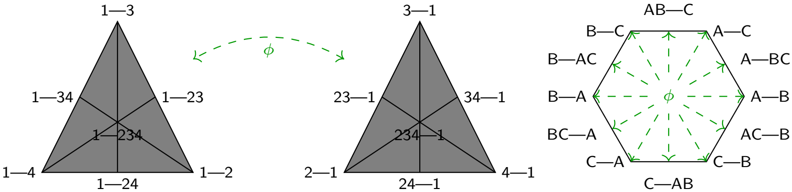

- It is a poset and topological space.
- It has a automorphism ϕ.
- It is connected (if the graph is connected, non-bipartite).
- It has a sub-Taylor polymorphism (if the graph has a Taylor polymorphism). That is no projection.

Sub-Taylor Polymorphism

A monotone map f is sub-Taylor, if it satisfies identities of the form

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\exists s_1 \forall x, y: &\quad f(x, \bullet, \dots, \bullet) \geq s_1(x, y) \leq f(y, \bullet, \dots, \bullet) \\
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\exists s_n \forall x, y: &\quad f(\bullet, \bullet, \dots, x) \geq s_n(x, y) \leq f(\bullet, \bullet, \dots, y) \text{ and} \\
\forall x: &\quad f(x, x, \dots, x) \geq x
\end{aligned}$$

Polymorphisms of Posets

Theorem (Larose, Zadori 2005)

The complex associated to a connected ramified poset with a polymorphism, which is not a projection, is contractible.

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- Larose, Zadori: Finite posets and topological spaces in locally finite varieties (2005)
- Larose: Taylor operations on finite reflexive structures (2006)
- The ArXiv-version of this paper.

Recap

graph $\longrightarrow$ box poset $\longrightarrow$ box complex

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connected non-bipartite Taylor graph

Recap

graph ⟶ box poset ⟶ box complex

graph	box poset	box complex
connected non-bipartite Taylor graph	sub-Taylor connected poset	

Recap

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Recap

graph	$\longrightarrow$	box poset	$\longrightarrow$	box complex
connected non-bipartite Taylor graph		sub-Taylor connected poset automorphism ϕ		contractible simplicial complex

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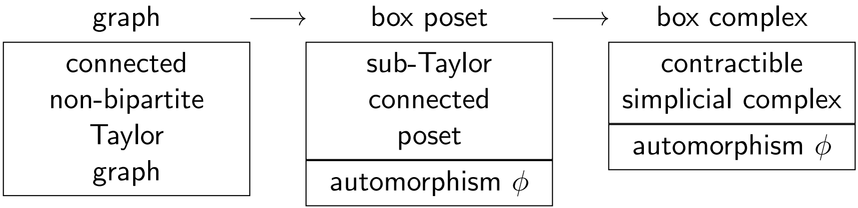

- So ϕ has a fixed point.
- So the graph has a loop.

Recap

graph	$\longrightarrow$	box poset	$\longrightarrow$	box complex
connected non-bipartite Taylor graph		sub-Taylor connected poset		contractible simplicial complex
		automorphism ϕ		automorphism ϕ

- So ϕ has a fixed point.
- So the graph has a loop. $\square$

Main Results

A (connected) non-bipartite Taylor graph has a loop.

Theorem (Hell, Nešetřil 1990)

If graph H is bipartite or has a loop, then H-coloring is in P.
Else, the H-coloring problem is NP-complete.

Every simplicial complex with a (simplicial idempotent) Taylor polymorphism is contractible.

Theorem (new)

If simplicial complex (C, F) is not contractible, then the (C, F)-coloring problem is NP-complete.

Thank you for your attention

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