Finite Simple Groups in the Primitive Positive Constructability Poset and Minor Conditions Associated to Permutation Groups

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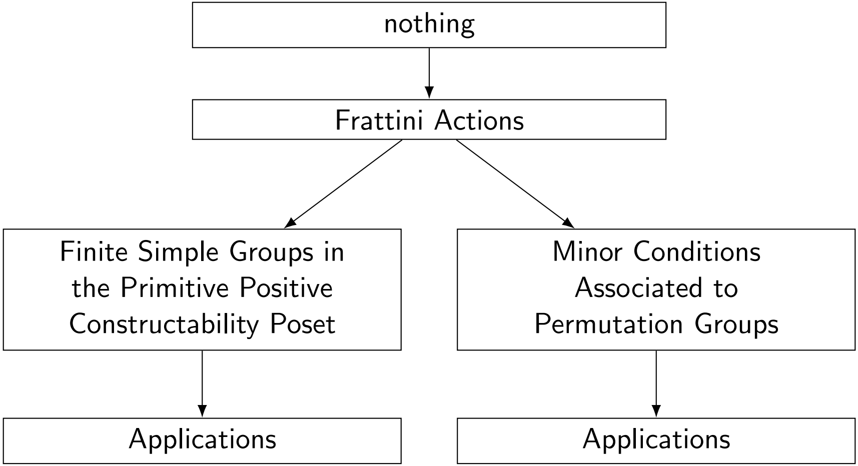
Two Theorems with one Proof.

The flowchart of understanding.



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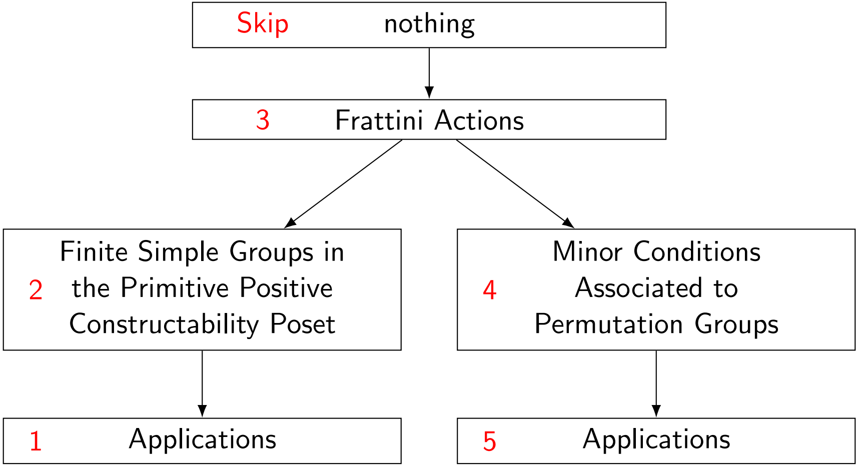

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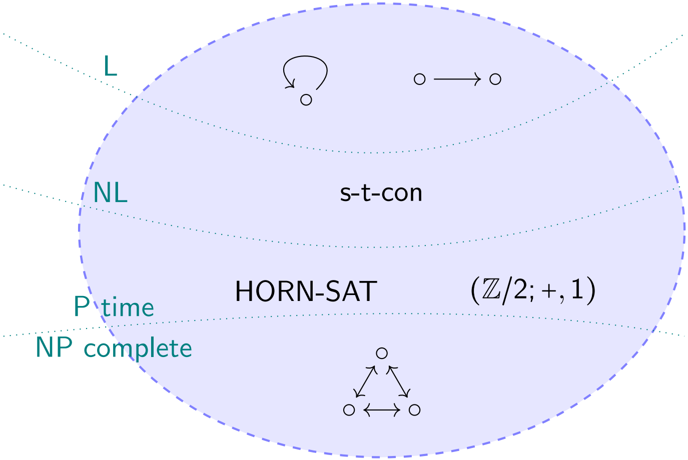
Finite Domain CSPs



Primitive positive Constructions

Theorem (Barto, Opršal, Pinsker 2018)

For two finite structures $\underline{A}$ and $\underline{B}$, the following is equivalent:

1. $\underline{A}$ pp-constructs $\underline{B}$.
2. There is a minion-homomorphism $\mathrm{Pol}(\underline{A}) \to \mathrm{Pol}(\underline{B})$.
3. Every minor condition valid in $\mathrm{Pol}(\underline{A})$ is valid in $\mathrm{Pol}(\underline{B})$.

In this case, $\mathrm{CSP}(\underline{B})$ reduces to $\mathrm{CSP}(\underline{A})$ in logspace (L).

Important examples of Minor Conditions

A *height-1-condition* or *minor condition* of $\underline{A}$ is a condition of the form

$$\exists f \in \mathrm{Pol}(\underline{A}) : \bigwedge f_\alpha = f_\beta$$

Examples

$f(x) = f(y)$	constant
$f(x,x,x) = f(x,y,y) = f(y,y,x)$	quasi Maltsev
$f(x,y,z) = f(y,z,x) = f(z,x,y)$	cyclic of arity 3
$f(x,y,z) = f(y,z,x) = f(y,x,z)$	(fully) symmetric of arity 3

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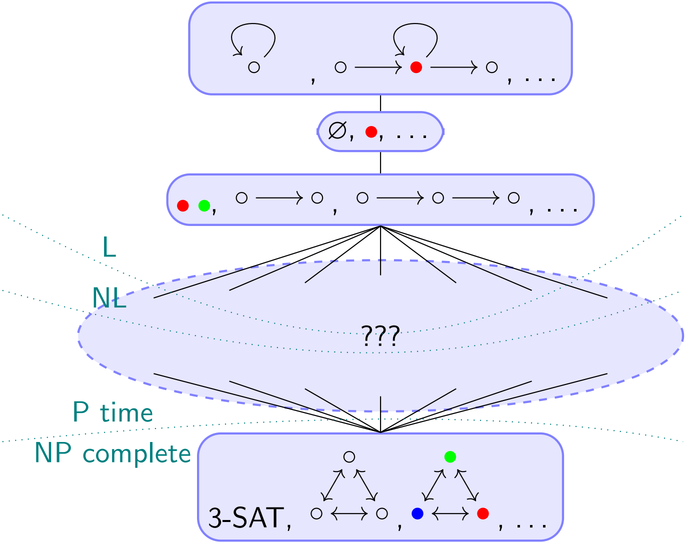
More general, for any group G acting on a set X, consider $\Sigma(G \curvearrowright X)$ as

$$\forall g \in G : f((y_x)_{x \in X}) = f((y_{g.x})_{x \in X})$$

The Primitive Positive Constructability Poset on Finite Structures



The PP-Constructability Poset on Finite Structures

Classified sub-posets

1. The poset of all smooth digraphs (Bodirsky, Starke, Vucaj 2021)
2. The poset of all 2-element structures (Bodirsky, Vucaj 2020)

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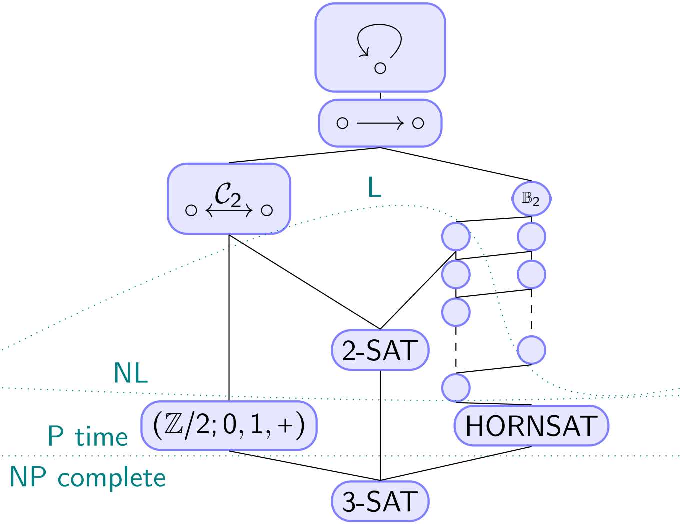
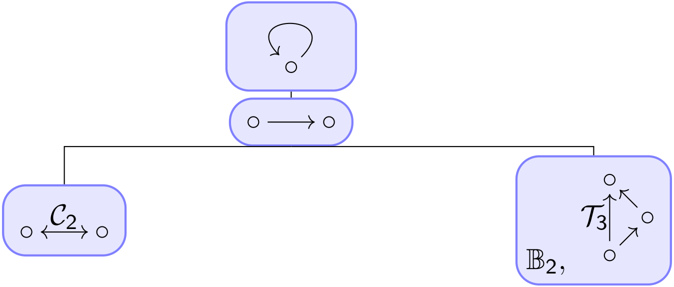
3. The poset of all 3-element structures with a Maltsev operation (Fioravanti, Kompatscher, Rossi, Vucaj. preprint 2025)

The PP-Constructability Poset on 2-Element Structures



The First Nontrivial Layer of the PP-Constructability Poset



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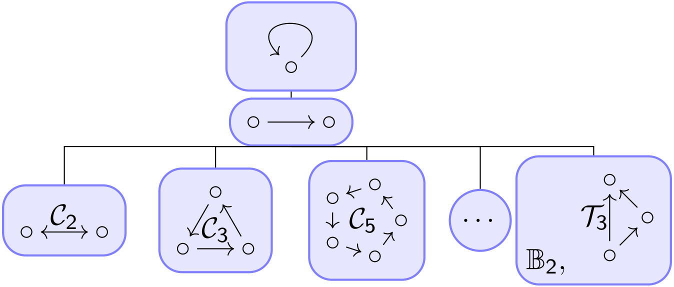

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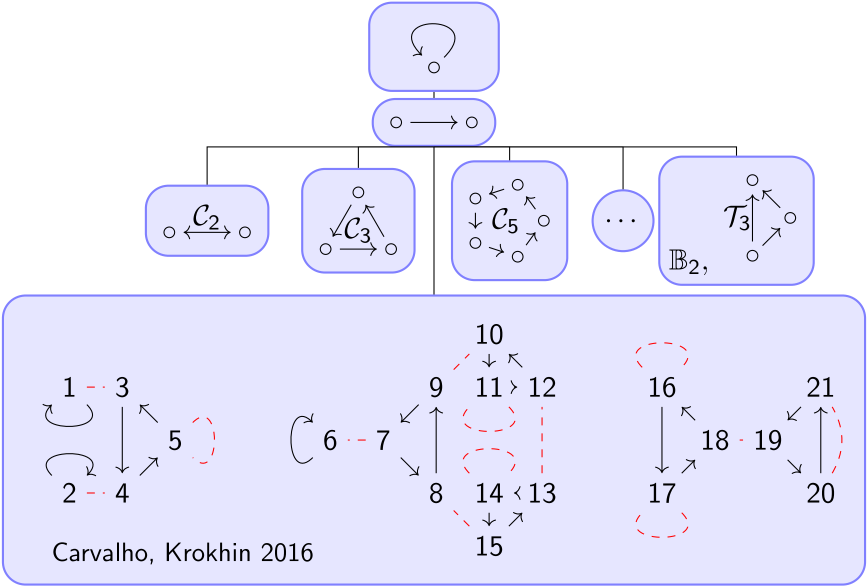

The Frattini Action $\mathbb{P}(H)$

Take ...

1. A group H.

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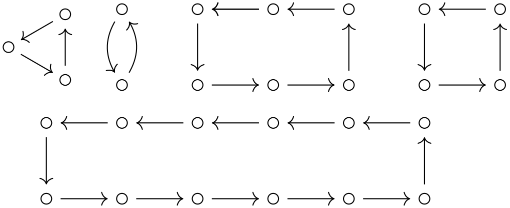

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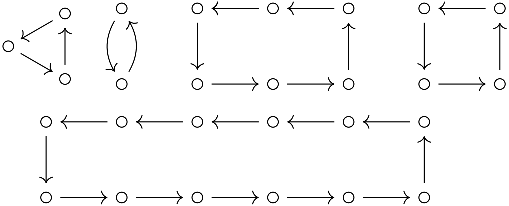

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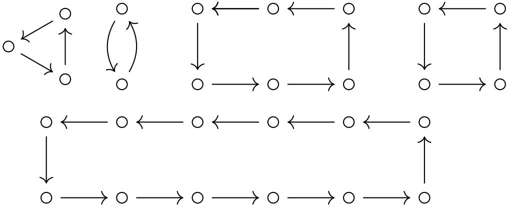

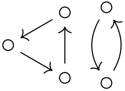

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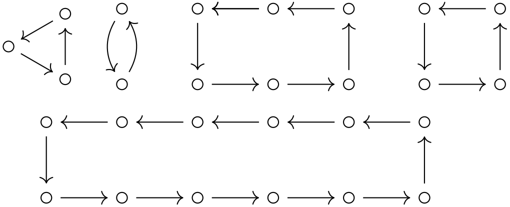

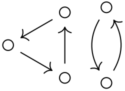

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3. The core structure $\mathbb{P}(H)$.

Alternatively, $\mathbb{P}(H)$ is the disjoint union of all primitive group actions of H.

The First Nontrivial Layer of the PP-Constructability Poset

Theorem

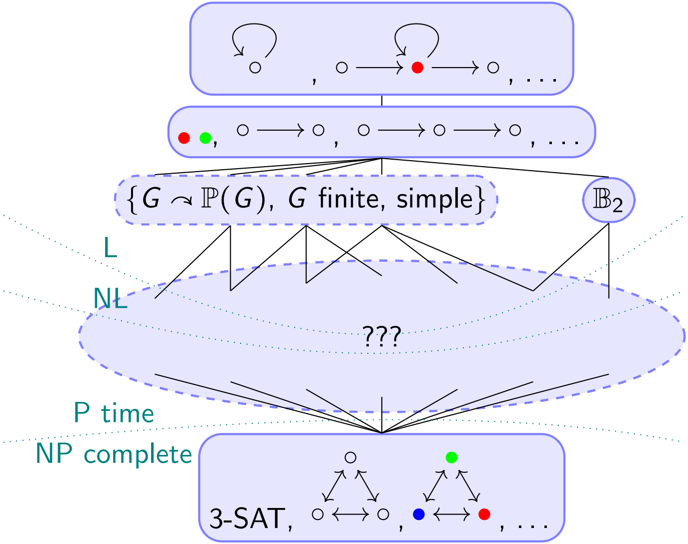
The pp-constructabillity poset has a first nontrivial layer consisting of the equivalence classes of

1. $\mathbb{B}_2$ and
2. for all finite simple groups G, the structure $\mathbb{S}(G \curvearrowright \mathbb{P}(G))$, where $\mathbb{P}(G)$ is the disjoint union of all primitive group actions.

Moreover,

$$\mathbb{P}(G) = \begin{cases} G & \text{(with multiplication)} \\ & \text{if } G \text{ is abelian simple} \\ \{M \leq G \text{maximal subgroup}\} & \text{(with conjugation)} \\ & \text{if } G \text{ is nonabelian simple} \end{cases}$$

The PP-Constructability Poset on Finite Structures



Proof overview

Let $\underline{A}$ be a structure.

1. If $\underline{A}$ has a quasi Maltsev polymorphism and fully symmetric polymorphisms of all arities, then $\circ \longrightarrow \circ$ pp-constructs $\underline{A}$.
2. If $\underline{A}$ has no quasi Maltsev polymorphism, then $\underline{A}$ pp-constructs $\mathbb{B}_2$.
3. If $\underline{A}$ has not fully symmetric polymorphism of an arity n, then $\underline{A}$ pp-constructs $\mathbb{S}(G \curvearrowright \mathbb{P}(G))$ for G finite simple group.
4. $\mathbb{S}(G \curvearrowright \mathbb{P}(G))$ does not pp-construct $\mathbb{S}(G' \curvearrowright \mathbb{P}(G'))$ for $G \neq G'$ different, finite simple

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Proof. Omitted. Uses Vucaj, Zhuk 2023.

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Proof. See Opršal 2018.

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- Note $\mathrm{Pol}(\underline{A}) \models \Sigma(S_n \curvearrowright [n])$.

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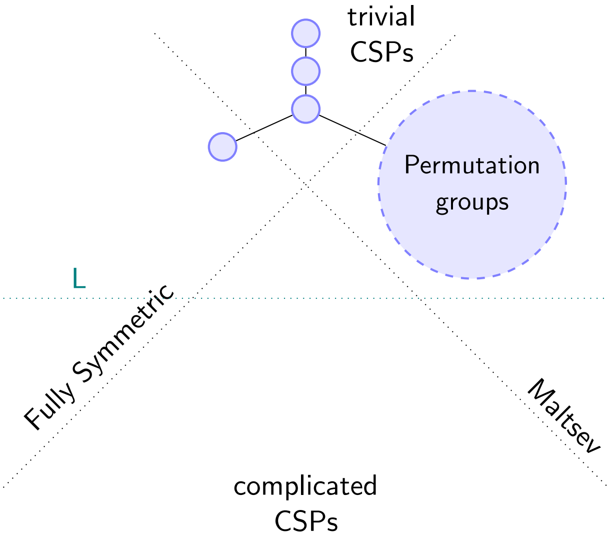
$\underline{A}$ pp-constructs a nonempty group action without fixed point, namely $\mathbb{S}(S_n \curvearrowright \mathrm{Pol}_n(\underline{A}))$.


trivial
CSPs
Permutation
groups
L
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Part 3 + 4

We are left with the problems:

Given $\mathbb{S}(S \curvearrowright X)$ (nonempty, without fixed-point), show that there exists a finite simple group G such that $\mathbb{S}(S \curvearrowright X)$ pp-constructs $\mathbb{S}(G \curvearrowright \mathbb{P}(G))$.

and

$\mathbb{S}(G \curvearrowright \mathbb{P}(G))$ does not pp-construct $\mathbb{S}(G' \curvearrowright \mathbb{P}(G'))$ for $G \neq G'$ different, finite simple.

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Idea: Classify all structures of the type $\mathbb{S}(G \curvearrowright X)$.

Classification of Group actions

The structure $\mathbb{S}(G \curvearrowright \mathbb{P}(G))$ is nice.

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Theorem

Given a group G, a structure $\underline{A}$.
Either

1. $\underline{A}$ pp-constructs $\mathbb{S}(G \curvearrowright \mathbb{P}(G))$ or
2. $\mathrm{Pol}(\underline{A}) \vDash \Sigma(G \curvearrowright \mathbb{P}(G))$

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Which elements of G act trivial in $\mathbb{P}(G)$?
Elements of the Frattini Subgroup $\Phi(G)$.
$\mathbb{S}(G \curvearrowright \mathbb{P}(G))$ and $\mathbb{S}(G/\Phi(G) \curvearrowright \mathbb{P}(G/\Phi(G)))$ are equal up to renaming.

Classification of Group actions

Theorem

Given groups G and H with trivial Frattini subgroup.
The following are equivalent:

1. $\mathbb{S}(G \curvearrowright \mathbb{P}(G))$ pp-constructs $\mathbb{S}(H \curvearrowright \mathbb{P}(H))$,
2. There is a group epimorphism $G \to H$,
3. $\Sigma(H \curvearrowright \mathbb{P}(H))$ implies $\Sigma(G \curvearrowright \mathbb{P}(G))$,
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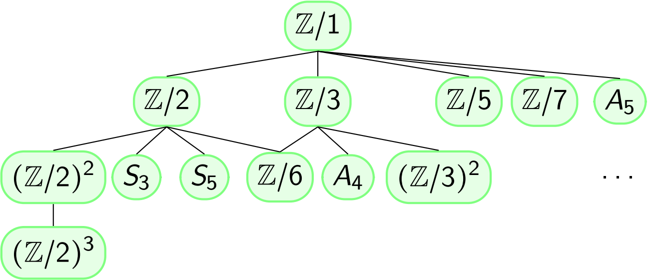
So simple groups cannot pp-construct each other.

Classification of Group actions



Classification of Group actions

Theorem

The structure $\mathbb{S}(G \curvearrowright X)$ is pp-interconstructable with the product

$$\prod_{H} \mathbb{S}(H \curvearrowright \mathbb{P}(H))$$

where H runs over all subgroups of G such that $H \curvearrowright X$ is *minimal*(!) fixed-point free.

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So any structure $\mathbb{S}(S \curvearrowright X)$ (nonempty, without fixed-point) pp-constructs $\mathbb{S}(G \curvearrowright \mathbb{P}(G))$ for some G simple. □

Classification of Group actions

Slogan

In general,

- $\mathbb{S}(G \curvearrowright X)$ is the meet of $\mathbb{S}(H \curvearrowright \mathbb{P}(H))$ for some subgroups H,
- then, delete Frattini subgroups,
- then, these actions are compared by group-epimorphisms.

Classification of Group actions

Can we also classify conditions $\Sigma(G \curvearrowright X)$?

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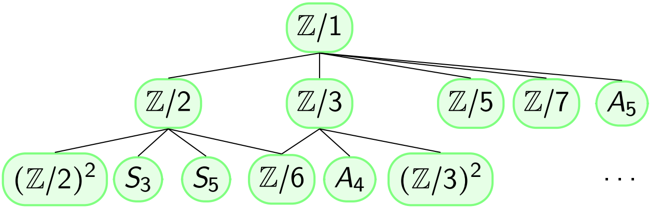
Theorem

The condition $\Sigma(G \curvearrowright X)$ is equivalent to satisfying all of the conditions

$$\{\mathbb{S}(H \curvearrowright \mathbb{P}(H)) \mid H\}$$

where H runs over all subgroups of G such that $H \curvearrowright X$ is fixed-point free.

Classification of Group actions



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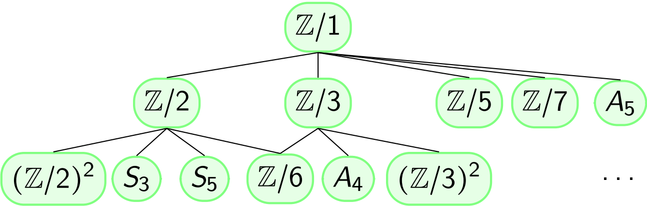
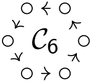


Consider the cycle 

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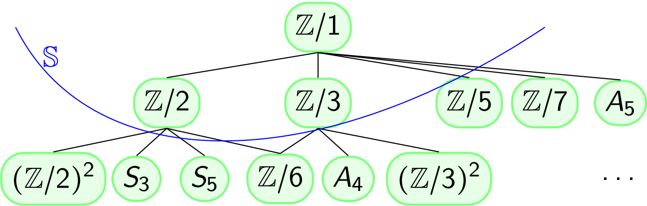
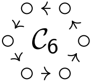


Consider the cycle 

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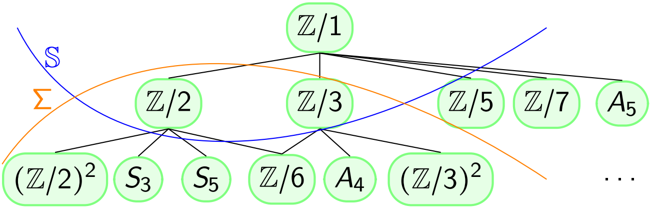
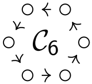


Consider the cycle 

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  - Fully symmetric arity 5 $\iff$ 7 specific Frattini actions

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  - Fully symmetric arity 5 $\iff$ 7 specific Frattini actions
    $\Longleftarrow$ Frattini actions of $Z/2, Z/3, Z/5, A_5$.

Applications

- We can classify fully symmetric conditions:
  - Fully symmetric arity 2 $\iff$ Frattini actions of $\mathbb{Z}/2$,
  - Fully symmetric arity 3 $\iff$ Frattini actions of $\mathbb{Z}/3$ and S_3,
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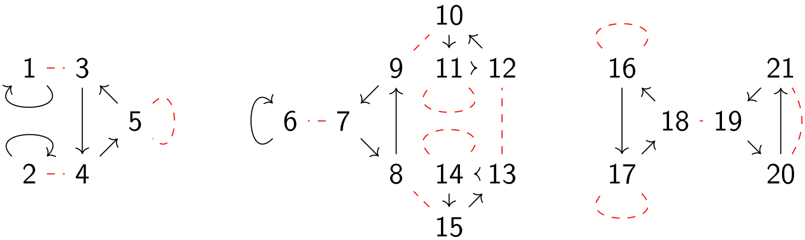
Thank you for your attention



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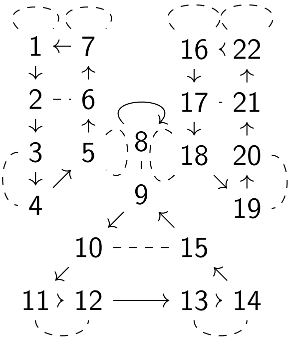

$$\mathrm{PSL}(2,7) \rightsquigarrow [22]$$