When invariance implies exchangeability

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Joint work with

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Exchangeable graphs

$\mathrm{Graph}(\mathbb{N}) :=$ set of graphs with domain $\mathbb{N}$;

Definition (Exchangeable graph)

An **exchangeable graph** is a Borel probability measure on $\mathrm{Graph}(\mathbb{N})$ invariant under all permutations of $\mathbb{N}$.

Example: The standard construction of the random graph yields an exchangeable graph.

Exchangeable structures

Natural to generalise from graphs to arbitrary relational structures!

$\mathcal{C}'$:= a **hereditary class**[1] of finite relational structures;

$\mathrm{Struc}(\mathbb{N}, \mathcal{C}')$:= set of structures with

- domain $\mathbb{N}$;
- **age** (i.e., class of finite substructures) contained in $\mathcal{C}'$.

Definition (Exchangeable structure)

An **exchangeable structure** is a Borel probability measure on $\mathrm{Struc}(\mathbb{N}, \mathcal{C}')$ invariant under all permutations of $\mathbb{N}$.

[1] closed under isomorphisms and substructures,
e.g. graphs, $\triangle$-free graphs, linear orders, partitions in $\leq k$ many classes ...

Exchangeable structures

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Definition (Exchangeable structure)

An **exchangeable structure** is a Borel probability measure on $\mathrm{Struc}(\mathbb{N}, \mathcal{C}')$ invariant under all permutations of $\mathbb{N}$.

Exchangeable structures are (relatively) well-understood:

- De Finetti 1929 characterises exchangeable colourings;
- Aldous 1981 and Hoover 1979 give a "representation theorem" for exchangeable graphs and hypergraphs;
- This generalises to exchangeable structures.

(cf. Ackerman, Freer, Kruckman, and Patel 2017; Crane and Towsner 2018)

Invariant random expansions

We understand invariance with respect to ALL symmetries.
What about invariance with respect to SOME symmetries?

$\mathcal{M}$:= a relational structure with domain $\mathbb{N}$;

Definition (Invariant random expansion[1] $\mathrm{IRE}(\mathcal{M}, \mathcal{C}')$)

An **invariant random expansion** of $\mathcal{M}$ to $\mathcal{C}'$, $\mathrm{IRE}(\mathcal{M}, \mathcal{C}')$, is a Borel probability measure on $\mathrm{Struc}(\mathbb{N}, \mathcal{C}')$ invariant under $\mathrm{Aut}(\mathcal{M}) \curvearrowright \mathbb{N}$.

[1] Related notions are defined in Aldous 1985; Angel, Kechris, and Lyons 2014; Crane and Towsner 2018.

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Take $\mathcal{M}$ **homogeneous:** any isomorphism between finite substructures extends to an automorphism.

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Take $\mathcal{M}$ **homogeneous:** any isomorphism between finite substructures extends to an automorphism.
The ages of homogeneous structures correspond to well-behaved hereditary classes known as **Fraïssé classes**.

Homogeneous structure	Fraïssé class
$(\mathbb{N}, =)$	finite sets with $=$
$(\mathbb{Q}, <)$	finite linear orders
Random graph	finite graphs
Generic tetrahedron-free 3-hypergraph	finite tetrahedron-free 3-hypergraphs

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IREs of homogeneous structures occur naturally in statistical networks,[1] spin glass models,[2] probabilistic programming[3]...

[1] Holland, Laskey, and Leinhardt 1983; Crane 2018.

[2] Austin and Panchenko 2014

[3] Jung, Lee, Staton, and Yang 2021.

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Our main interest comes from **model theory**:
We show: invariant Keisler measures are a special case of IREs;[4]
We describe the former in previously not understood contexts.

(cf. Albert 1994; Ensley 2001; Chernikov, Hrushovski, Kruckman, Krupiński, Moconja, Pillay, and Ramsey 2023)

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[1] Holland, Laskey, and Leinhardt 1983; Crane 2018.

[2] Austin and Panchenko 2014

[3] Jung, Lee, Staton, and Yang 2021.

[4] In their generalisation to arbitrary domains. (Braunfeld, Jahel, and Marimon 2024).

The main question

Problem (Aldous 1985)

What conditions *prima facie* weaker than exchangeability imply exchangeability?
I.e., when can we say that all IREs of $\mathcal{M}$ by $\mathcal{C}'$ are exchangeable?

Note: if all IREs of $\mathcal{M}$ by $\mathcal{C}'$ are exchangeable, we have a description of them from Aldous 1981 and Hoover 1979.

Previous results had strong restrictions on either $\mathcal{C}'$ or $\mathcal{M}$!

- $\mathcal{C}' = \{\text{linear orders}\}$; (Angel, Kechris, and Lyons 2014; Jahel and Tsankov 2022)
- $\mathcal{C}'$ is unary; (Jahel and Tsankov 2022)
- $\mathcal{M}$ is the random k-hypergraph. (Crane and Towsner 2018)

We are especially interested in IREs of homogeneous hypergraphs with interesting omitted configurations by graphs.

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When invariance implies exchangeability

Main Theorem (Braunfeld, Jahel, and Marimon 2024)

*Let $k \geq 1$ and $\mathcal{M}$ be homogeneous with **k-overlap closed** age.*
Let $\mathcal{C}'$ have labelled growth rate $O(e^{n^{k+\delta}})$ for every $\delta > 0$.
Then every IRE of $\mathcal{M}$ by $\mathcal{C}'$ is exchangeable.

k-overlap closed: the age of $\mathcal{M}$ is closed under a "random placement" construction that works for $(k+1)$-hypergraphs and allows for interesting omitted configurations. ▸ See precise definition

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Examples of k-overlap closed Fraïssé classes

- **1-overlap closed:** free amalgamation classes in arity > 1 (e.g. graphs, $\triangle$-free graphs), tournaments;
- **2-overlap closed:** 3-hypergraphs, tetrahedron-free 3-hypergraphs;
- ***k*-overlap closed:** $\mathrm{Forb}(\mathcal{F})$ of arity $> k$ and all $A \in \mathcal{F}$ are
  - $(k+1)$-irreducible;[5]
  - of bounded size and k-irreducible (for $k \geq 2$).

Non-example: linear orders are not 1-overlap closed.

[5] ***k*-irreducible**: every k-many elements are related.

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Examples of $\mathcal{C}'$ with labelled growth rate $O(e^{n^{k+\delta}})$ for all $\delta > 0$

- $\boldsymbol{O(e^{n^{1+\delta}})}$: unary structures, linear orders, the age of any NIP homogeneous structure;
- $\boldsymbol{O(e^{n^{2+\delta}})}$: graphs;
- $\boldsymbol{O(e^{n^{k+\delta}})}$: structures with finitely many k-ary relations.

When invariance implies exchangeability

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Corollary (Braunfeld, Jahel, and Marimon 2024)

IREs of the generic tetrahedron-free 3-hypergraph by graphs are exchangeable.

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Corollary (Braunfeld, Jahel, and Marimon 2024)

IREs of the generic tetrahedron-free 3-hypergraph by graphs are exchangeable.

Moral of the story: If $\mathcal{M}$ is k-transitive and "looks random enough", IREs by "essentially k-ary" classes are exchangeable.

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Corollary (Braunfeld, Jahel, and Marimon 2024)

IREs of the generic tetrahedron-free 3-hypergraph by graphs are exchangeable.

We also recover previous results:

- IREs of $\mathcal{M}$ transitive homogeneous with free amalgamation by linear orders or colourings are exchangeable;
  (Angel, Kechris, and Lyons 2014; Jahel and Tsankov 2022)
- IREs of the random k-hypergraph by l-hypergraphs are exchangeable for $k > l \geq 1$.(Crane and Towsner 2018)

Thank you!

A brief recap:

- We study **invariant random expansions**: probability measures on spaces of countable structures (with age $\subseteq \mathcal{C}'$) invariant under automorphisms of a fixed structure $\mathcal{M}$;
- We show: $\mathrm{Aut}(\mathcal{M})$-invariance implies exchangeability when:
  - $\mathcal{M}$ looks "random enough for arity $k+1$";
  - $\mathcal{C}'$ has "essentially arity k".

QR code to preprint:

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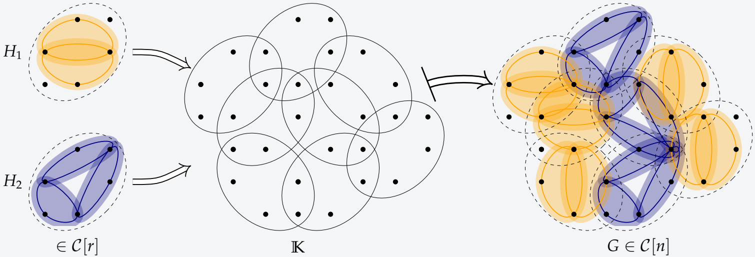
k-overlap closed classes

Definition (k-overlap closedness ▸ Back to main presentation)

$\mathcal{L}$ of arity $> k$. $\mathcal{C}$ is **k-overlap closed** if for every $r > k$ and arbitrarily large n, there exists an r-uniform hypergraph $\mathbb{K}$ on n vertices s.t.

1. $\mathbb{K}$ has at least $C(r)n^{k+\alpha(r)}$ many hyperedges for some $\alpha(r) > 0$;
2. No two $\mathbb{K}$-hyperedges intersect in more than k points;
3. For every $H_1, H_2 \in \mathcal{C}[r]$, pasting them into the $\mathbb{K}$-hyperedges yields $G \in \mathcal{C}[n]$ (possibly after adding extra relations).



The key lemma for exchangeability

$\mathcal{C}$:= age of $\mathcal{M}$;
$\mathcal{C}[k]$:=structures in $\mathcal{C}$ of size k.

Lemma (Braunfeld, Jahel, and Marimon 2024)

Suppose that for all $\mathbf{H}_1$, $\mathbf{H}_2 \in \mathcal{C}[k]$, and $\epsilon > 0$, there is some n, $\mathbf{G} \in \mathcal{C}[n]$ and non-empty families Θ_i of embeddings of $\mathbf{H}_i$ in $\mathbf{G}$ such that for all $\mathbf{H}' \in \mathcal{C}'[k]$ and $\mathbf{G}' \in \mathcal{C}'[n]$ we have

$$\left| \frac{N_{\Theta_1}(\mathbf{H}_1^\star, \mathbf{G}^*)}{|\Theta_1|} - \frac{N_{\Theta_2}(\mathbf{H}_2^\star, \mathbf{G}^*)}{|\Theta_2|} \right| < \varepsilon,$$

where $\mathbf{G}^\star := \mathbf{G} \star \mathbf{G}', \mathbf{H}_i^\star := \mathbf{H}_i \star \mathbf{H}'$ and $N_{\Theta_i}(\mathbf{H}_i^\star, \mathbf{G}^\star)$ is the number of embeddings in Θ_i that are also embeddings of $\mathbf{H}_i^\star$ in $\mathbf{G}^\star$.
Then every IRE of $\mathcal{M}$ by $\mathcal{C}'$ is exchangeable.

Applications to invariant Keisler measures

Definition (Invariant Keisler measure)

An **invariant Keisler measure** is a finitely additive probability measure on $\mathrm{Def}_x(M)$, invariant under $\mathrm{Aut}(\mathcal{M}) \curvearrowright \mathrm{Def}_x(M)$.[5]

We show: invariant Keisler measures are a special case of IREs.

[5] Outside the homogeneous context: $\mathcal{M}$ is sufficiently saturated and symmetric (i.e., strongly ω-homogeneous).

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We show: invariant Keisler measures are a special case of IREs.

We describe the spaces of invariant Keisler measures of several homogeneous hypergraphs.
This answers questions of Albert 1994 and Ensley 2001.

[5]Outside the homogeneous context: $\mathcal{M}$ is sufficiently saturated and symmetric (i.e., strongly ω-homogeneous).

Two notions of smallness in simple theories

We are interested in invariant Keisler measures in simple theories.[6]

[6] Theories endowed with a good notion of independence:
vector spaces with forms over finite fields, pseudofinite fields, random graph, etc.

Two notions of smallness in simple theories

We are interested in invariant Keisler measures in simple theories.

Recent work[6] shows the following notions of smallness for a definable set X disagree for some simple theories:

- X **forks**: there are $(\sigma_i)_{i\in\omega} \in \mathrm{Aut}(\mathcal{M})$ such that $\{\sigma_i X | i \in \omega\}$ is k-inconsistent;
- X is **universally measure zero**: for any invariant Keisler measure $\mu(X) = 0$.

[6] Chernikov, Hrushovski, Kruckman, Krupiński, Moconja, Pillay, and Ramsey 2023 and Marimon 2024.

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- X is **universally measure zero**: for any invariant Keisler measure $\mu(X) = 0$.

Previous examples are somewhat ad-hoc! We show:

- There are $2^{\aleph_0}$ simple ternary homogeneous structures with non-forking sets which are universally measure zero; (cf. Koponen 2018)
- More generally, the above is ubiquitous amongst simple homogeneous structures; ▸ More on this

[6] Chernikov, Hrushovski, Kruckman, Krupiński, Moconja, Pillay, and Ramsey 2023 and Marimon 2024.

Non-forking universally measure zero formulas everywhere

$\mathcal{C}$ has **n-DAP** for all n: given $A_I \in \mathcal{C}[I]$ for each $I \in [n]^{n-1}$, such that for all $I, J \in [n]^{n-1}$ $A_I \restriction_{I\cap J} = A_J \restriction_{I \cap J}$, there is $A \in \mathcal{C}[n]$ such that for all $I \in [n]^{n-1}$, $A \restriction_I = A_I$.

Corollary (Braunfeld, Jahel, and Marimon 2024)

Let $\mathcal{M}$ be simple, k-transitive, homogeneous in a finite $(k+1)$-ary language, k-overlap closed and with free amalgamation. Then, any IKM of $\mathcal{M}$ in the variable x is exchangeable. Moreover,

1. *EITHER: $\mathrm{Age}(M)$ has n-DAP for all n. In this case there is an IKM assigning positive measure to every non-forking formula;*
2. *OR: $\mathrm{Age}(M)$ fails n-DAP for some n. In this case $\mathcal{M}$ has non-forking formulas which are universally measure zero.*

For $k > 1$, there are $2^{\aleph_0}$-many structures in 2 (Koponen 2018). Meanwhile, only countably many structures in 1. Back

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The Aldous-Hoover theorem

Theorem (Aldous 1981 and Hoover 1979)

Let μ be an exchangeable graph.
Then, there is a Borel function[7] $f : [0,1]^4 \to \{0,1\}$ *and* $\mathrm{Uniform}[0,1]$ *independent identically distributed random variables*

$$U_\emptyset, (U_a | a \in \mathbb{N}), (U_{\{a,b\}} | \{a,b\} \in [\mathbb{N}]^2)$$

such that the random graph built by setting

$$E(a,b) \text{ if and only if } f(U_\emptyset, U_a, U_b, U_{\{a,b\}}) = 1 \qquad (\diamondsuit)$$

has the same distribution as μ.

EASY TO SEE: $(\diamondsuit)$ gives an exchangeable graph.
HARD TO PROVE: any exchangeable graph is of the form $(\diamondsuit)$.

[7] symmetric in the second and third argument.