Binary symmetries of tractable non-rigid structures

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LICS 2025, 24th of June

POCOCOP ERC Synergy Grant No. 101071674. Views and opinions expressed are those of the authors only and do not necessarily reflect those of the European Union or the European Research Council Executive Agency. Neither the European Union nor the granting authority can be held responsible for them.

Outline

Constraint Satisfaction Problems

$\mathbb{B}$:= a relational structure with signature τ.

Definition ($\mathrm{CSP}(\mathbb{B})$)

$\mathrm{CSP}(\mathbb{B})$ is the following computational problem:

- **INPUT:** a primitive positive τ-sentence

$$\phi(x_1, \dots, x_n) := \exists x_1 \dots \exists x_n (R_1(\dots) \wedge \dots \wedge R_m(\dots))$$

- **OUTPUT:** does $\mathbb{B} \models \phi$?

We focus on $\mathbb{B}$

- finite; OR
- countably infinite and ω-**categorical**:[1]
  $\mathrm{Aut}(\mathbb{B}) \curvearrowright \mathbb{B}^n$ has finitely many orbits for each $n \in \mathbb{N}$.

[1] Examples: $(\mathbb{N}, =)$, $(\mathbb{Q}, <)$, the (countable) Rado graph, ...

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Examples

- k-colourability of graphs;
- 3SAT;
- solving linear equations over a finite field;

} $\mathbb{B}$ is finite

- digraph acyclicity;
- graph colourability omitting monochromatic triangles.

} $\mathbb{B}$ is ω-categorical

The algebraic approach to CSPs

Algebraic approach to CSPs:

Higher arity symmetries of $\mathbb{B}$ (**polymorphisms**)[1] capture the computational complexity of $\mathrm{CSP}(\mathbb{B})$.

Polymorphisms=higher arity homomorphisms.

$\mathrm{Pol}(\mathbb{B})$:= **polymorphism clone** of $\mathbb{B}$, the set of polymorphisms of $\mathbb{B}$.

---

[1] $f : \mathbb{B}^n \to \mathbb{B}$ is a **polymorphism** if it **preserves all relations** of $\mathbb{B}$:

$$\begin{pmatrix} a_1^1 \\ \vdots \\ a_k^1 \end{pmatrix}, \dots, \begin{pmatrix} a_1^n \\ \vdots \\ a_k^n \end{pmatrix} \in R^{\mathbb{B}} \Rightarrow \begin{pmatrix} f(a_1^1, \dots, a_1^n) \\ \vdots \\ f(a_k^1, \dots, a_k^n) \end{pmatrix} \in R^{\mathbb{B}}.$$

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Higher arity symmetries of $\mathbb{B}$ (**polymorphisms**) capture the computational complexity of $\mathrm{CSP}(\mathbb{B})$.

Highly successful in the finite setting:

Theorem (Bulatov 2017; Zhuk 2017)

Let $\mathbb{B}$ be finite. Then:

- *EITHER $\mathbb{B}$ has a Siggers polymorphism.*[1]
  In this case, $\mathrm{CSP}(\mathbb{B})$ is in P;
- *OR $\mathbb{B}$ "pp-constructs" EVERYTHING (i.e., all finite structures)*
  In this case, $\mathrm{CSP}(\mathbb{B})$ is NP-complete.

[1] A polymorphism $s : \mathbb{B}^6 \to \mathbb{B}$ such that

$$\forall x, y, z \; s(x, y, x, z, y, z) = s(y, x, z, x, z, y).$$

The algebraic approach to CSPs

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Higher arity symmetries of $\mathbb{B}$ (**polymorphisms**) capture the computational complexity of $\mathrm{CSP}(\mathbb{B})$.

Often successful for $\mathbb{B}$ ω-categorical: complexity dichotomies for CSPs of structures first-order definable in:

- $(\mathbb{Q}, <)$ (Bodirsky and Kára 2010);
- homogeneous graphs (Bodirsky, Martin, Pinsker, and Pongrácz 2019);
- countable unary structures (Bodirsky and Mottet 2018);

  ⋮

Bodirsky-Pinsker conjecture: CSPs of a large class of ω-categorical structures[1] satisfy a complexity dichotomy analogous to the finite-domain one.

---

[1] First-order reducts of finitely bounded homogeneous structures.

Understanding low arity polymorphisms

Question

What is the minimal amount of structure in $\mathrm{Pol}(\mathbb{B})$ when $\mathrm{CSP}(\mathbb{B})$ is not NP-hard (due to pp-constructing EVERYTHING)?

- Sufficient to consider case of a **core**;
- We can assume $\mathrm{Pol}(\mathbb{B})$ is **essential**: it has an **essential polymorphism** (depending on more than one variable): if $\mathrm{Pol}(\mathbb{B})$ is NOT essential, then $\mathbb{B}$ pp-interprets EVERYTHING;
- **Bottom-up approach to CSPs:** several complexity classifications identify the behaviours of low arity essential polymorphisms.

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(Bodirsky and Kára 2008):

Lemma 5. *Every essentially at least binary operation together with all permutations locally generates a binary operation that depends on both arguments.*

(Bodirsky and Kára 2010)

Lemma 10. *Let Γ be a relational structure and let R be a k-ary relation that is a union of l orbits of k-tuples of $Aut(\Gamma)$. If R is violated by a polymorphism g of Γ of arity $m \geq l$, then R is also violated by an l-ary polymorphism of Γ.*

(Bodirsky and Pinsker 2014):

LEMMA 40: *Let $f\colon V^k \to V$ be an essential operation. Then f generates a binary essential operation.*

(Bodirsky 2021; Mottet and Pinsker 2024):

LEMMA 6.1.29. *Let $\mathscr{C}$ be a clone with an essential operation that contains a permutation group $\mathscr{G}$ with the orbital extension property. Then $\mathscr{C}$ must also contain a binary essential operation.*

PROPOSITION 23. *Let $\mathbb{A}$ be a first-order reduct of a homogeneous structure $\mathbb{B}$ such that $\mathbb{B}$ has a free orbit. If $\mathrm{Pol}(\mathbb{A})$ contains an essential function, then it contains a binary essential operation.*

(Mottet, Nagy, and Pinsker 2024)

Lemma 27. *Let $\mathbb{A}$ be a first-order reduct of $\mathbb{H}$ that is a model-complete core. If $\mathrm{Pol}(\mathbb{A})$ does not have a uniformly continuous clone homomorphism to $\mathscr{P}$, then it contains a binary essential operation.*

Also done in:

- Bodirsky, Jonsson, and Van Pham 2017;
- Bodirsky and Mottet 2018;
- Kompatscher and Van Pham 2018;
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- Bodirsky and Greiner 2020.

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ISSUE: These techniques are ad-hoc!

Answering a question of Bodirsky

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Answer (Marimon and Pinsker 2025a): No.

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Does every ω-categorical core with an essential polymorphism also have a binary essential polymorphism?

"Moral" answer (for the purposes of CSPs): Yes!

Theorem (Marimon and Pinsker 2025a)

Let $\mathbb{B}$ be a core which is

- *EITHER ω-categorical;*
- *OR finite where $\mathrm{Aut}(\mathbb{B})$ is not a Boolean group acting freely.*[2]

Suppose that $\mathbb{B}$ does not pp-interpret EVERYTHING.
Then, $\mathrm{Pol}(\mathbb{B})$ contains a binary essential polymorphism.

[2] **Boolean group**: every non-identity element has order 2.
$\mathrm{Aut}(\mathbb{B})$ acts **freely**: any automorphism fixing a point is the identity.

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This is optimal.

Rosenberg's Theorem

Our strategy: generalise a classic theorem of Rosenberg.

Theorem (Rosenberg 1986)

Let $\mathbb{B}$ be a finite core and $\mathrm{Aut}(\mathbb{B}) = \{1\}$. Suppose $\mathrm{Pol}(\mathbb{B})$ is essential. Then, $\mathrm{Pol}(\mathbb{B})$ contains one of the following:

1. *a binary essential operation;*
2. *a ternary majority operation;*
3. *a minority of the form $x + y + z$ in some Boolean group $(B, +)$;*
4. *a k-ary essential semiprojection for some $k \geq 3$.*

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Ternary majority: an operation $m : \mathbb{B}^3 \to \mathbb{B}$ such that

$$\forall x, y \; m(x, x, y) = m(x, y, x) = m(y, x, x) = m(x, x, x) = x;$$

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Ternary minority: an operation $\mathfrak{m} : \mathbb{B}^3 \to \mathbb{B}$ such that

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Semiprojection: $f : \mathbb{B}^k \to \mathbb{B}$ such that there is an $i \in \{1, \ldots, k\}$ such that whenever $(a_1, \ldots, a_k)$ is a non-injective tuple from $\mathbb{B}$,

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Rosenberg's Theorem classifies "minimal" operations in $\mathrm{Pol}(\mathbb{B})$.

Rosenberg's theorem for non-rigid structures

Theorem (Marimon and Pinsker 2025a)

Let $\mathbb{B}$ be a (possibly infinite) core.
Suppose $\mathrm{Aut}(\mathbb{B})$ is not a Boolean group acting freely.
Suppose $\mathrm{Pol}(\mathbb{B})$ is essential. Then, it contains one of the following:

1. *a binary essential operation;*
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Orbit-semiprojection: $f : \mathbb{B}^k \to \mathbb{B}$ such that there is an $i \in \{1, \ldots, k\}$ and some $\alpha \in \mathrm{End}(\mathbb{B})$ such that whenever $(a_1, \ldots, a_k)$ contains at least two elements in the same orbit,

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For $\mathbb{B}$ ω-categorical:

- $\mathrm{Aut}(\mathbb{B}) \curvearrowright \mathbb{B}$ is not free, so theorem always applies;
- We strictly improve a previous result of Bodirsky and Chen 2007, which included a “majority” case, and a much weaker 2.

▸ More on this

Rosenberg's theorem for non-rigid structures

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When $\mathrm{Aut}(\mathbb{B})$ is the free action of a (non-trivial) Boolean group, if $s = 2^n$ for some $n \in \mathbb{N}$ or is infinite, $\mathrm{Pol}(\mathbb{B})$ may also contain:

3. $\mathfrak{q} = \alpha\mathfrak{m}$, where
   - $\alpha \in \mathrm{Aut}(\mathbb{B})$;
   - $\mathfrak{m}$ is a minority of the form $x + y + z$ in a Boolean group;
   - for all $\alpha, \beta, \gamma \in \mathrm{Aut}(\mathbb{B})$, $\forall x, y, z$ $\mathfrak{m}(\alpha x, \beta y, \gamma z) = \alpha\beta\gamma\mathfrak{m}(x, y, z)$.

Thank you!

A brief recap:

- We study polymorphisms of cores when $\mathrm{Aut}(\mathbb{B}) \neq \{1\}$;
- When $\mathrm{CSP}(\mathbb{B})$ is not NP-hard we can in general find binary essential polymorphisms;
- We classify polymorphisms that have to appear if $\mathrm{Pol}(\mathbb{B})$ is essential and $\mathrm{Aut}(\mathbb{B}) \neq \{1\}$;
- Surprisingly, we get fewer cases than if $\mathrm{Aut}(\mathbb{B}) = \{1\}$.

QR code to preprint:

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Bodirsky and Chen's Theorem

Theorem (Bodirsky and Chen 2007)

Let $\mathbb{B}$ be an ω-categorical core. Suppose $\mathrm{Pol}(\mathbb{B})$ is essential. Then, $\mathrm{Pol}(\mathbb{B})$ contains one of the following:

1. *a binary essential operation;*
2. *a ternary quasi-majority operation;*
3. *an essential k-ary semiprojection for $3 \leq k \leq 2r - s$, where*
   - *s is the number of $\mathrm{Aut}(\mathbb{B})$-orbits on $\mathbb{B}$;*
   - *r is the number of $\mathrm{Aut}(\mathbb{B})$-orbits on $\mathbb{B}^2$.*

Back to main presentation

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Ternary quasi-majority: an operation $m : \mathbb{B}^3 \to \mathbb{B}$ such that

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Quasi-semiprojection: $f : \mathbb{B}^k \to \mathbb{B}$ such that there is an $i \in \{1, \ldots, k\}$ and $g \in \mathrm{End}(\mathbb{B})$ such that whenever $(a_1, \ldots, a_k)$ is a non-injective tuple,

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Our improvements in the ω-categorical context:

Theorem (Marimon and Pinsker 2025a)

Let $\mathbb{B}$ be an ω-categorical core. Suppose $\mathrm{Pol}(\mathbb{B})$ is essential. Then, $\mathrm{Pol}(\mathbb{B})$ contains one of the following:

1. *a binary essential operation;*
2. ~~*a ternary quasi-majority operation;*~~
3. *an essential k-ary orbit-semiprojection for $3 \leq k \leq s$*

Back to main presentation

pp-interpretations and pp-constructions

A ***pp*-formula** is a first-order formula consisting only of existential quantifiers, conjunctions, and atomic formulas.

Definition (pp-interpretation, pp-construction)

$\mathbb{B}$ ***pp*-interprets** $\mathbb{A}$ if there is partial surjective $h : \mathbb{B}^d \to \mathbb{A}$ such that for every $R \subseteq \mathbb{A}^n$ that is a relation of $\mathbb{A}$ (or $\mathbb{A}$, or equality on $\mathbb{A}$), $h^{-1}(R)$ is defined by a pp-formula in $\mathbb{B}^{nd}$.

$\mathbb{B}$ ***pp*-constructs** $\mathbb{A}$ if it is homomorphically equivalent to a structure that pp-interprets $\mathbb{A}$.