

What happens when you enumerate a structure?

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Funded by the European Union (project POCOCOP, ERC Synergy grant No. 101071674). Views and opinions expressed are however those of the author only and do not necessarily reflect those of the European Union or the European Research Council Executive Agency. Neither the European Union nor the granting authority can be held responsible for them.



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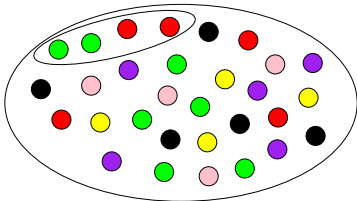
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Now fix $c: \text{Emb}(2, \omega) \rightarrow k$. Apply the infinite Ramsey theorem twice, first to stabilize c on the increasing embeddings and then to stabilize it also on the decreasing embeddings. \square

Definition

Let \mathbf{M} be a structure and \mathbf{A} its finite substructure. The **big Ramsey degree of \mathbf{A} in \mathbf{M}** , is the least $\text{brd}(\mathbf{A}) \in \omega \cup \{\omega\}$ such that for every $k \in \omega$:

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- ▶ In $(\mathbb{Q}, <)$, the big Ramsey degree of a vertex is 1.

Useful colourings

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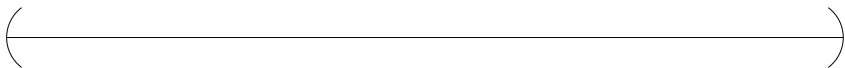
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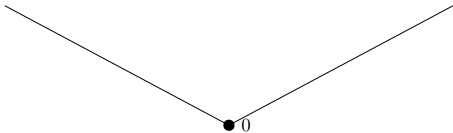
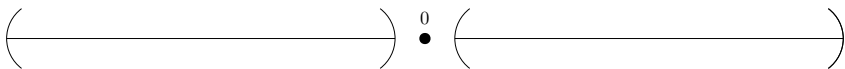
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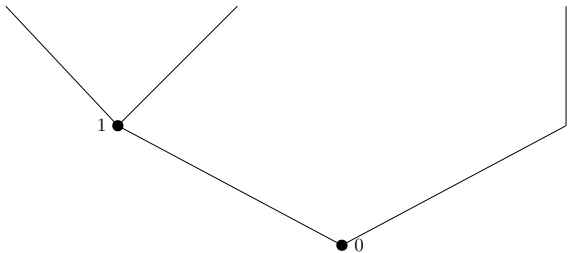
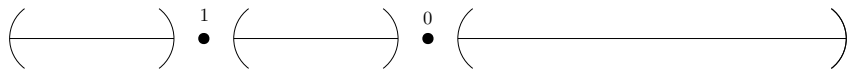
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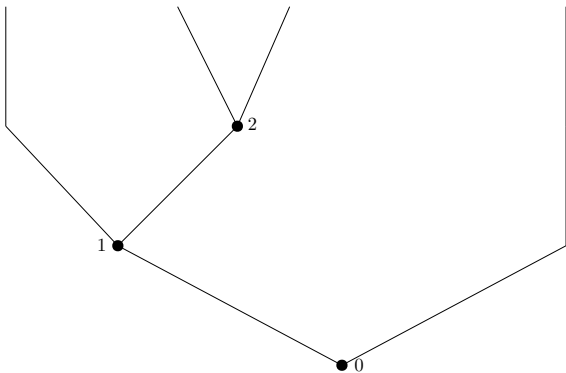
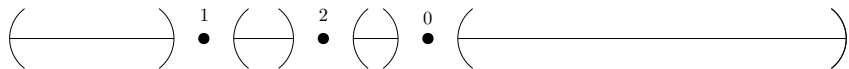
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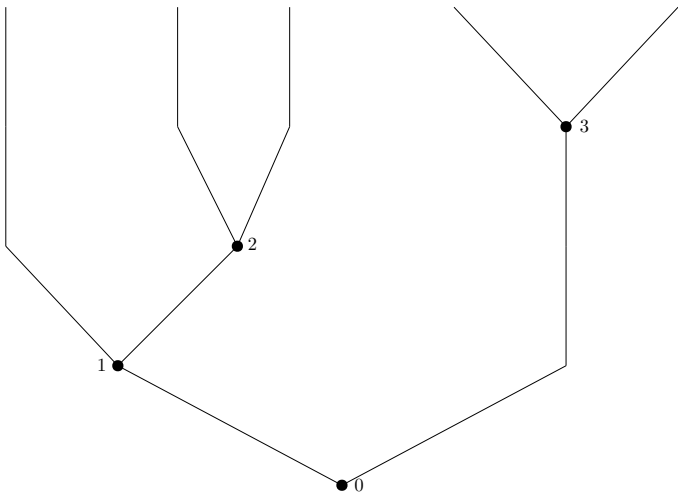
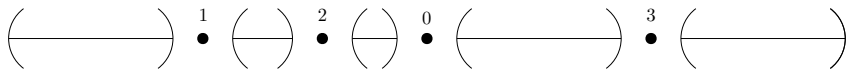
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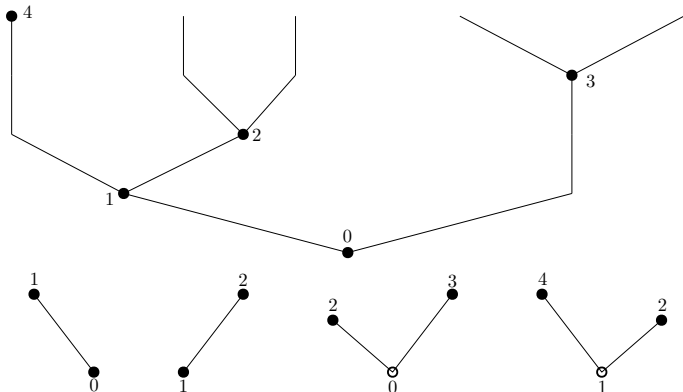
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See Natasha's and Honza's talks

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A tree?

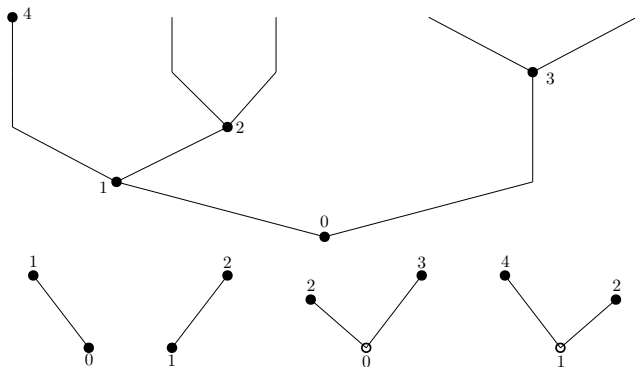
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Example

A dense order and a well-order together define a tree with a distinguished vertex on every level:



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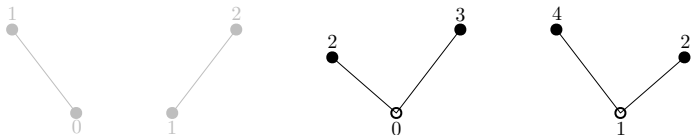
There is an embedding $\delta: (\mathbb{Q}, <) \rightarrow (\mathbb{Q}, <)$ such that for every $n \in \omega$ and every $g: (n, <) \rightarrow (\mathbb{Q}, <)$ it holds that the elements of $g[n]$ are pairwise incomparable in the tree $\xi_n(\delta \circ g)$.

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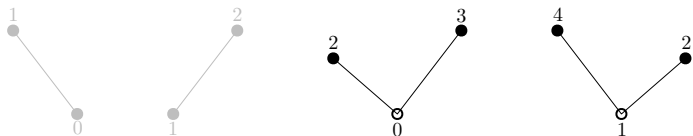


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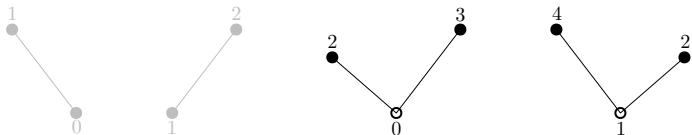
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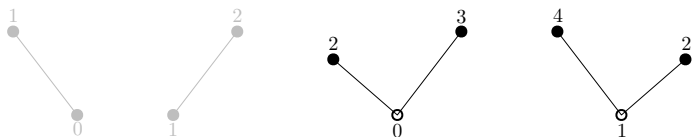
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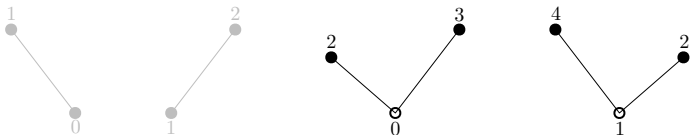
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Devlin types

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Put χ_n to be the colouring induced by ξ_n on $\delta[\mathbb{Q}]$. Then χ_n is both universal and unavoidable. Consequently, See Andy's talk.

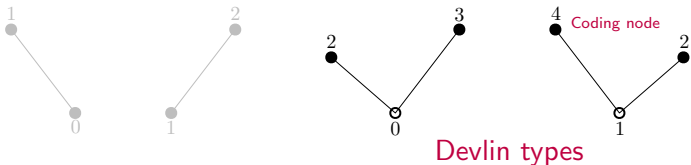
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Is ξ_n unavoidable?

Recall: $\xi_n(g)$ is the isomorphism-type of $(\overline{g[n]}, <, \triangleleft, g[n])$.

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There is an embedding $\delta: (\mathbb{Q}, <) \rightarrow (\mathbb{Q}, <)$ such that for every $n \in \omega$ and every $g: (n, <) \rightarrow (\mathbb{Q}, <)$ it holds that the elements of $g[n]$ are pairwise incomparable in the tree $\xi_n(\delta \circ g)$.



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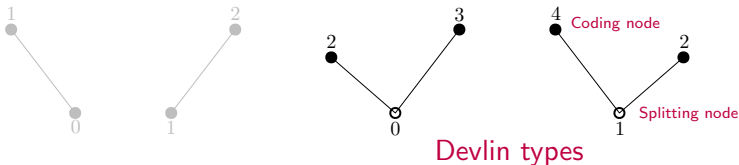
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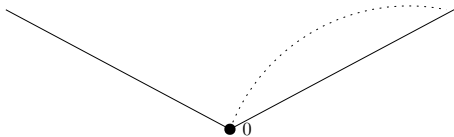
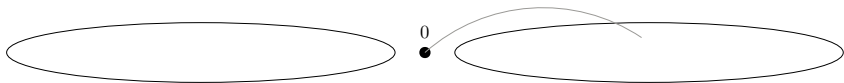
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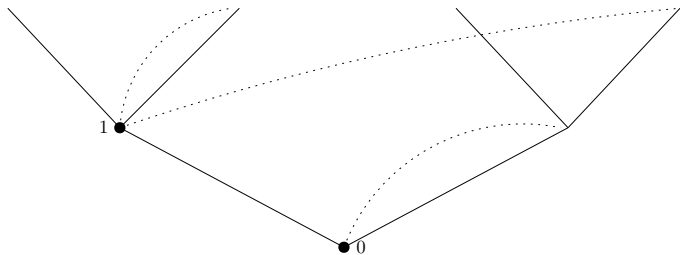
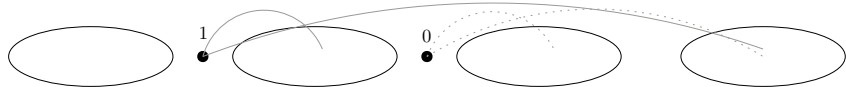
The random graph \mathbf{R}



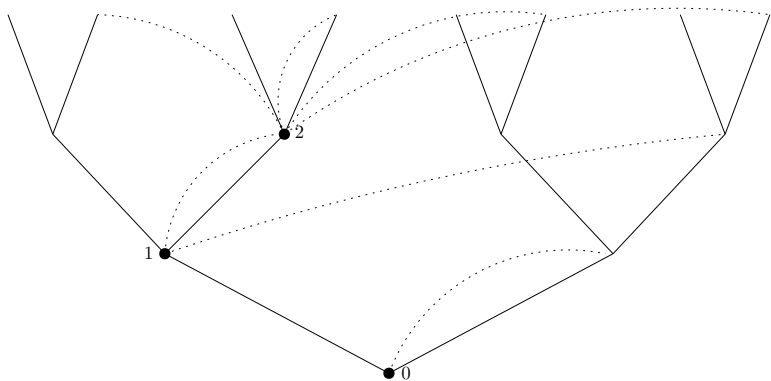
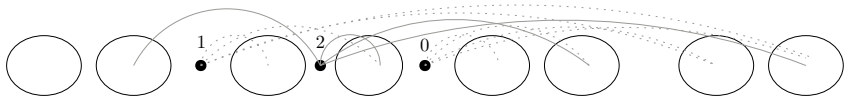
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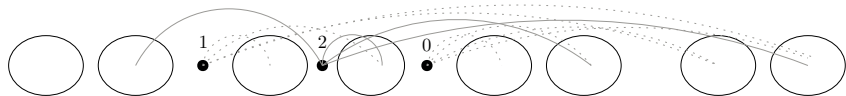
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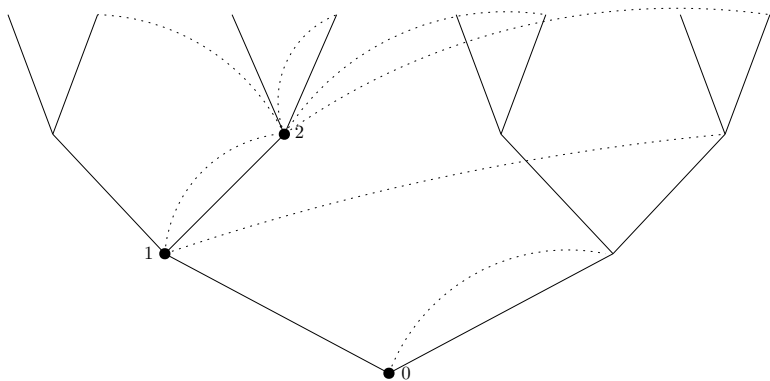
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⋮



Let $<$ be an enumeration of \mathbf{R} . Given $x, y \in \mathbf{R}$, the **meet** (or **split**) of x and y , denoted by $x \wedge y$, is the $<$ -least vertex $z \in \mathbf{R}$ such that $z = x$ or $z = y$ or exactly one of xz and yz is an edge of \mathbf{R} .

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Laflamme, Sauer, and Vuksanovic in 2006 constructed $\lambda: \mathbf{R} \rightarrow \mathbf{R}$ such that the colourings induced by $\xi_{\mathbf{G}}$ on $\lambda[\mathbf{R}]$ are both universal and unavoidable.

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$$\chi(v) = \begin{cases} \text{no} & \text{if there is no } w < v \text{ with } vw \text{ an edge} \\ \text{red} & \text{if the edge } vw \text{ with } w \text{ minimal is red} \\ \text{blue} & \text{if the edge } vw \text{ with } w \text{ minimal is blue.} \end{cases}$$

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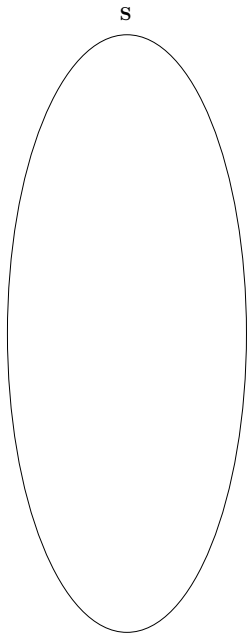
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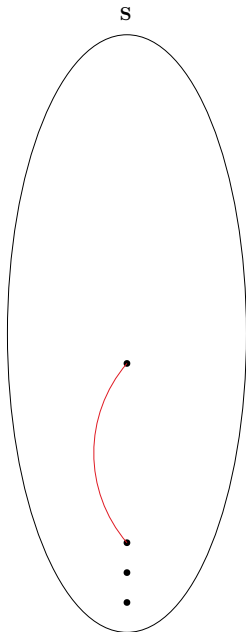
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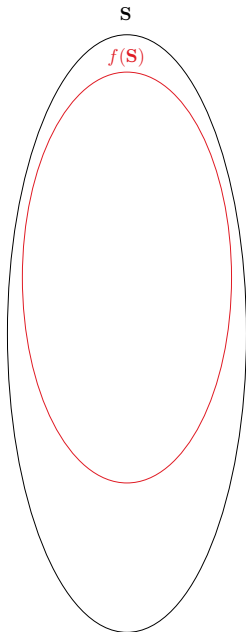
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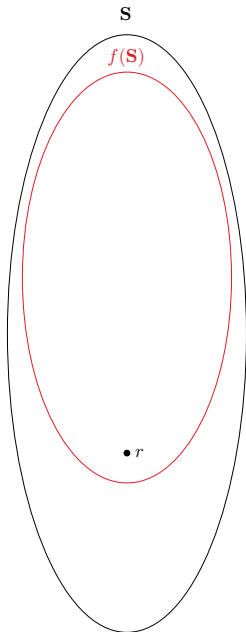


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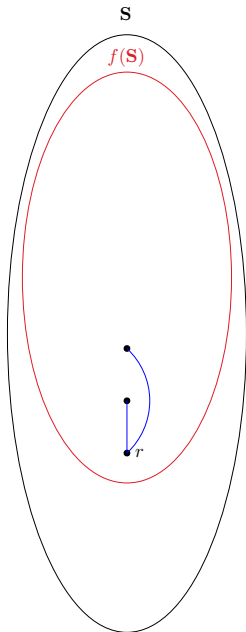
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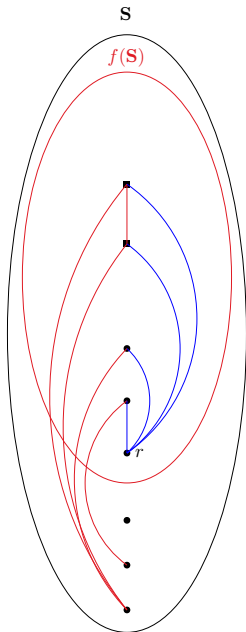
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By, say, the Nešetřil–Rödl theorem, there is a c -monochromatic red edge in X .



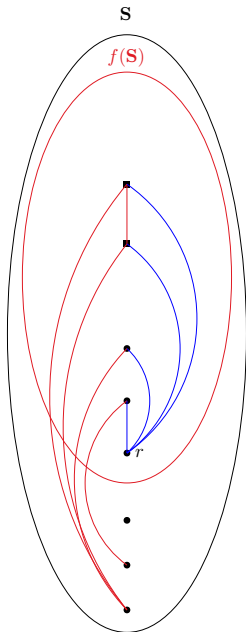
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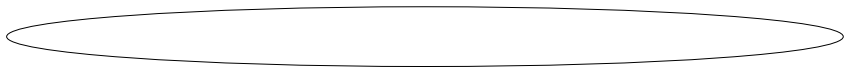


What in the hay bales is goin' on?

Skibidi what?

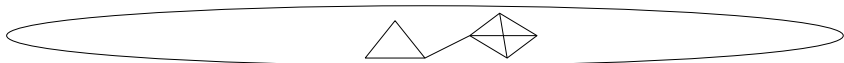
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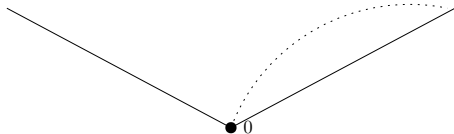
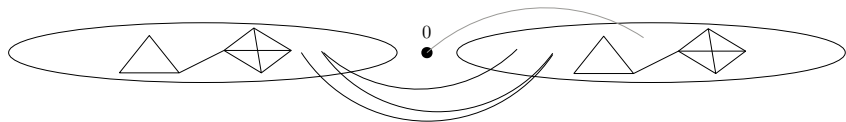
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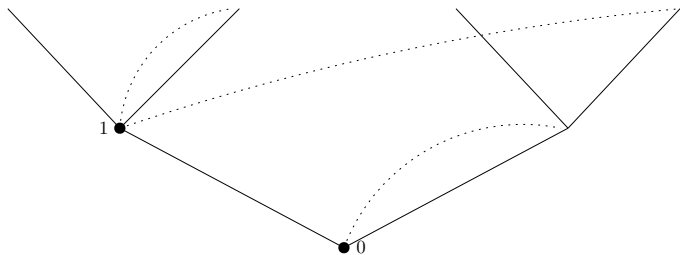
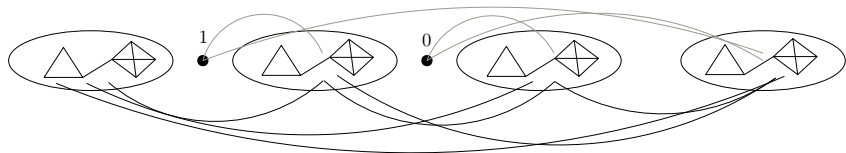
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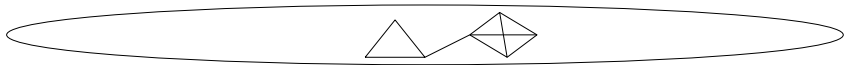


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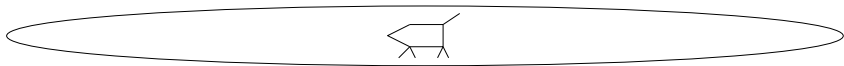
The random graph R



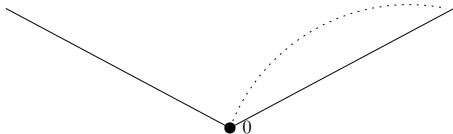
The Henson triangle-free graph \mathbf{H}_3



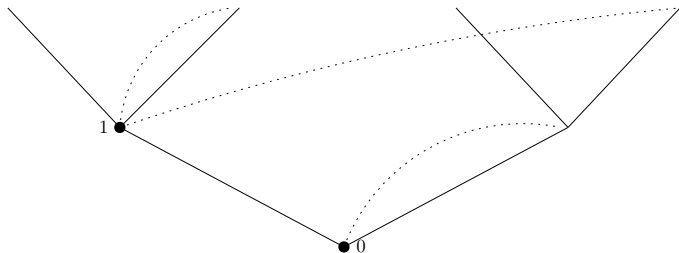
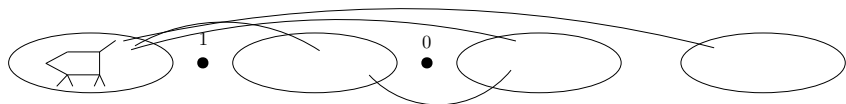
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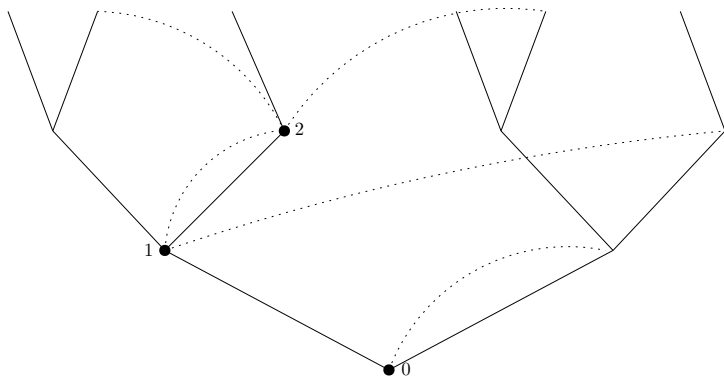
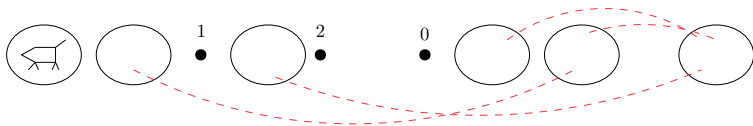
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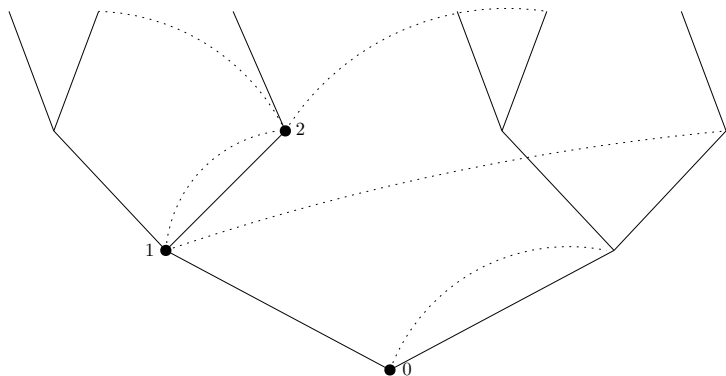
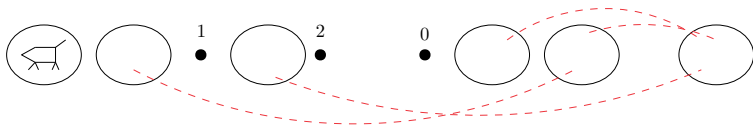
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see also [Dobrinen'17]

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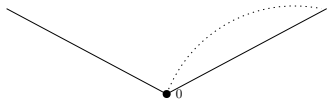
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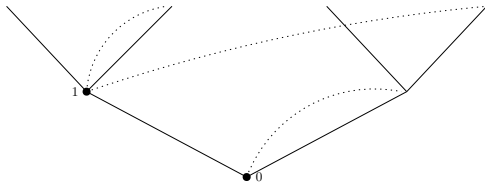
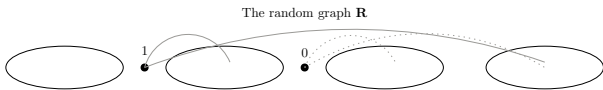


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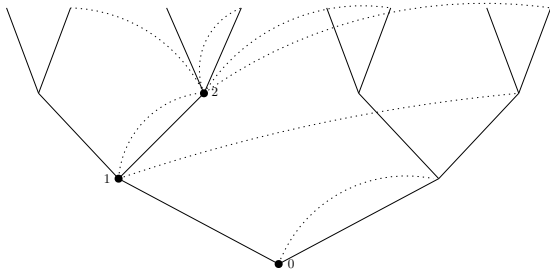
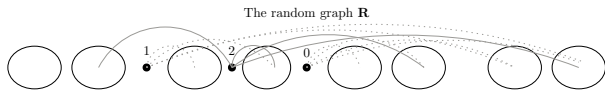
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Ages

Let \mathbf{M} be a structure with vertex set ω . For distinct $[x_1]_{\simeq^l}^{\mathbf{M}}, \dots, [x_k]_{\simeq^l}^{\mathbf{M}} \in \mathbf{M} / \simeq^l$ we denote by $\text{Age}([x_1]_{\simeq^l}^{\mathbf{M}}, \dots, [x_k]_{\simeq^l}^{\mathbf{M}})$ the class of all finite isomorphism types of substructures induced by vertices of $[x_1]_{\simeq^l}^{\mathbf{M}} \cup \dots \cup [x_k]_{\simeq^l}^{\mathbf{M}}$ in \mathbf{M} with vertices of $[x_i]_{\simeq^l}^{\mathbf{M}}$ being labelled by i .

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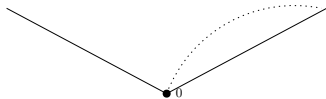
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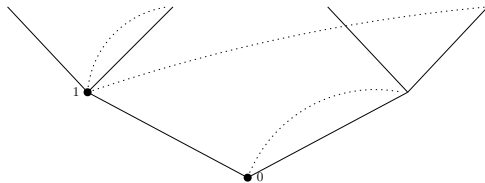
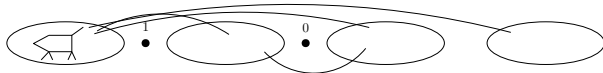
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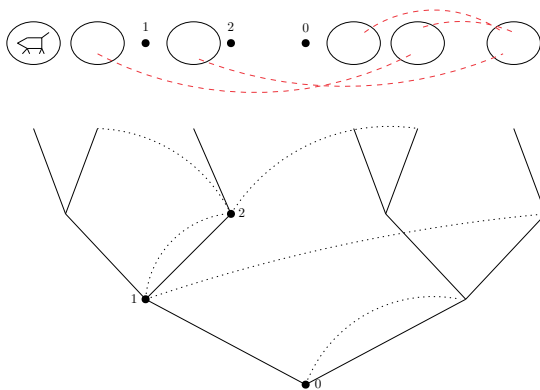
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Level structures

Let \mathbf{M} be a structure with vertex set ω . Given $S \subseteq M$, the **level structure of S of level ℓ in \mathbf{M}** , denoted by $\mathbf{L}_{\mathbf{M}}^{\ell}(S)$ has vertex set S , relation $\simeq^{\ell} \upharpoonright_S$, and for every $s_1, \dots, s_k \in S$ such that they are pairwise \simeq^{ℓ} -non-equivalent, we put (s_1, \dots, s_k) in a relation describing $\text{Age}([s_1]_{\simeq^{\ell}}^{\mathbf{M}}, \dots, [s_k]_{\simeq^{\ell}}^{\mathbf{M}})$.

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For \mathbf{H}_3 , the **first neighbour** of a vertex, and the **first common neighbour** of a pair of vertices are also interesting events.

Paths in age posets

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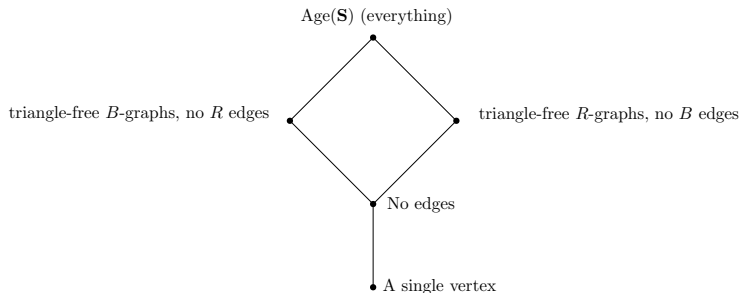
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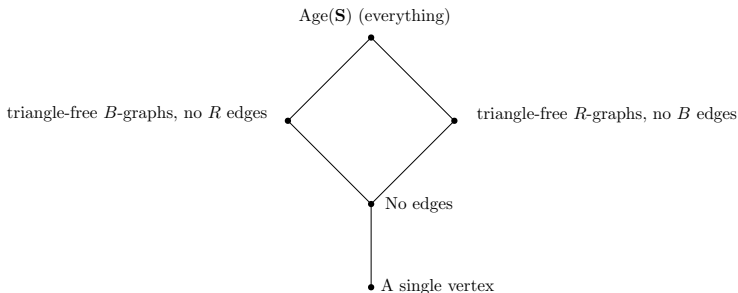


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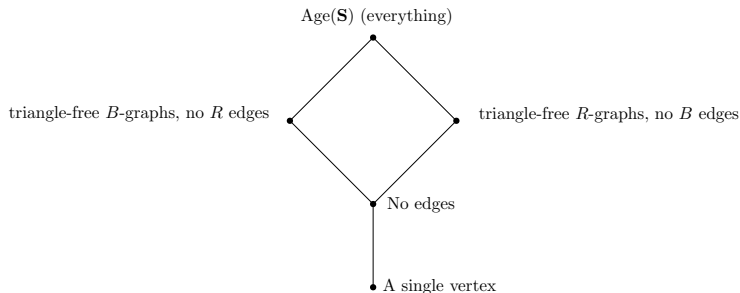
Sauer's colouring: Given $v \in S$, consider $\mathbf{E} = \overline{\{v\}}$. Colour v by the colour of the edge to its first neighbour in \mathbf{E} .

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Theorem (Balko, Chodounský, Dobrinen, Hubička, Konečný, Vena, Zucker'23)

Colouring by maximal paths in the age poset is unavoidable.

Increasing the arity (see Honza's talk)

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Let \mathbf{G}_3 be the countable random 3-uniform hypergraph and assume that $G_3 = \omega$. Given vertices $x, y, z, w \in G_3$ such that $x = \min(x, y, z, w)$ and $z < w$, define

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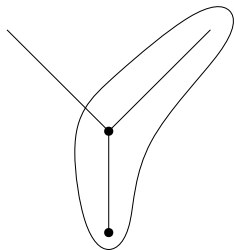
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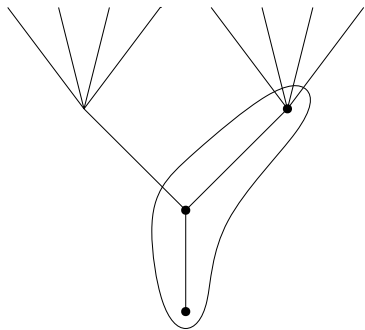
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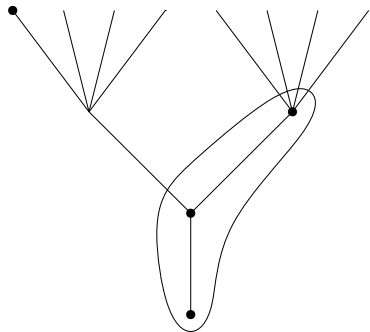
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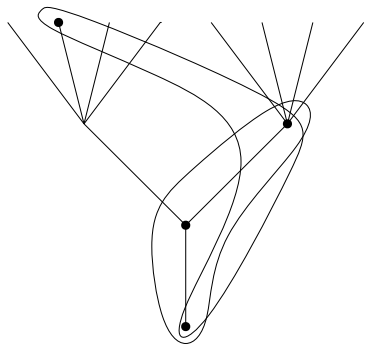
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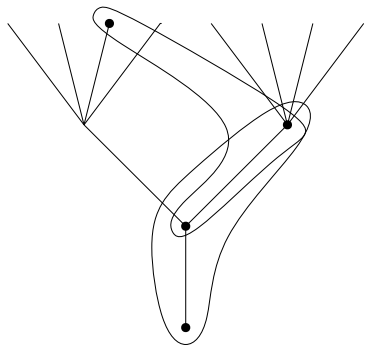
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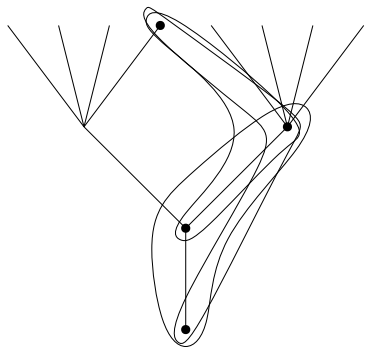
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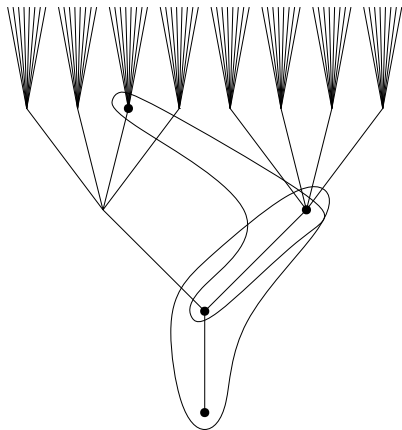
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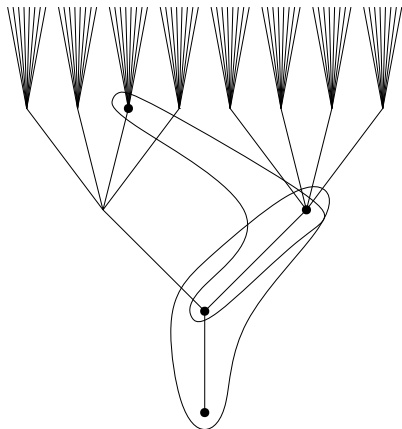
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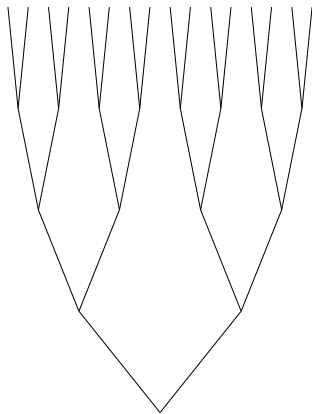
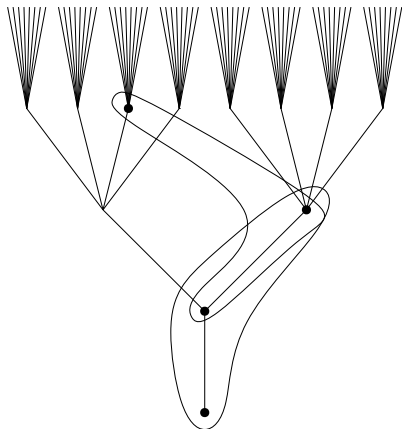
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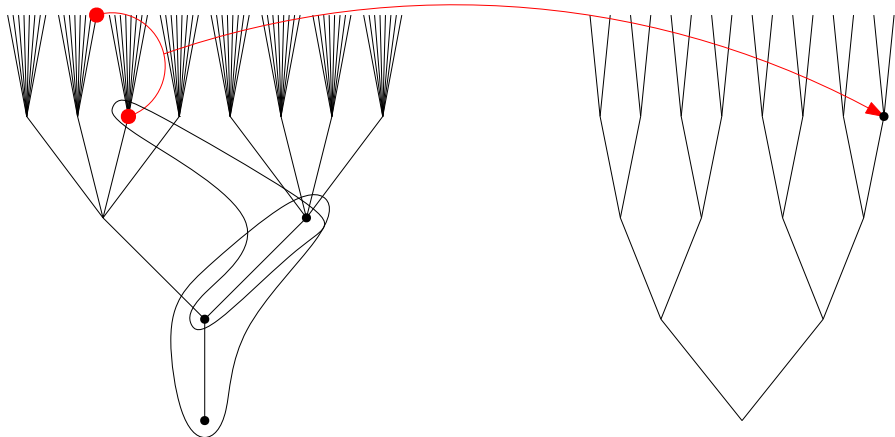
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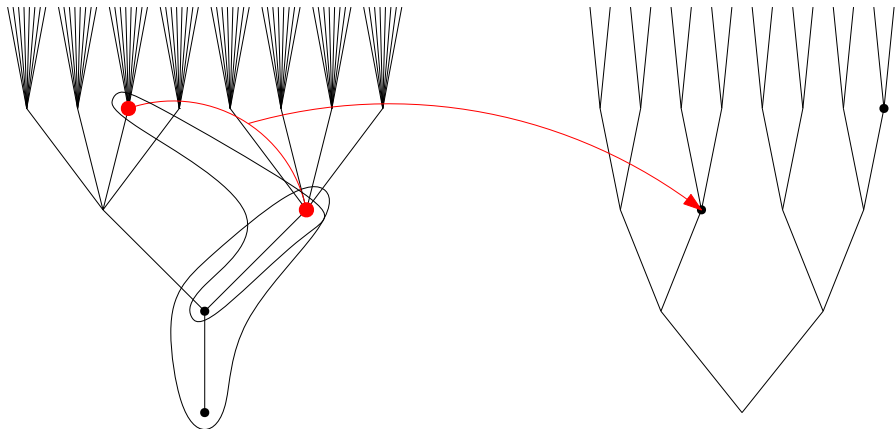
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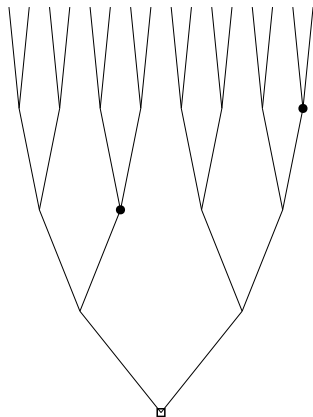
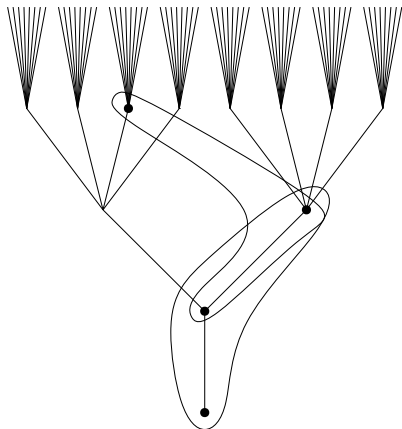
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When colouring substructures of size s in a structure with maximum arity a , it is enough to only consider n -types for every $n \leq \min(s, a - 1)$. [Braunfeld, Chodounský, de Rancourt, Hubička, Kawach, Konečný'24]

Thank you!

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Sorry?