Kőnig = Ramsey

Max Hadek[1]

Charles University

Spring School 2025

[1] Funded by the European Union (ERC, POCOCOP, 101071674). Views and opinions expressed are however those of the author only and do not necessarily reflect those of the European Union or the European Research Council Executive Agency. Neither the European Union nor the granting authority can be held responsible for them.

Kőnig vs Ramsey

Dénes Kőnig

- 1884 - 1944
- Budapest, Hungary
- Student of Minkowski

Frank P. Ramsey

- 1903 - 1930
- Cambridge, UK
- Friend of Wittgenstein

Theorem (Kőnig, 1927)

Any finitely branching, infinite tree contains an infinite path.

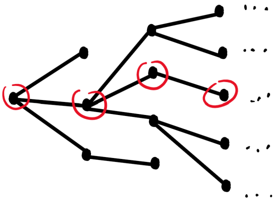
Theorem (Ramsey, 1928)

Something about graph colourings / strict linear orders.

Kőnigs Lemma

Lemma (Kőnig, 1927)

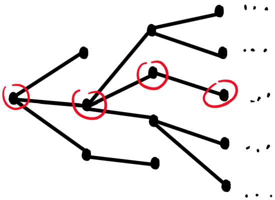
Any finitely branching, infinite tree contains an infinite path.

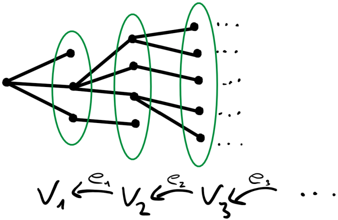


Kőnigs Lemma

Lemma (Kőnig, 1927)

Any finitely branching, infinite tree contains an infinite path.





$$\{\text{infinite paths}\} = \{(v_n \in V_n \mid e_n(v_{n+1}) = v_n\} = \lim_{n\in\mathbb{N}} V_n$$

Lemma (Kőnig, 1927), rephrased

Any functor $\mathcal{F} : (\mathbb{N}, \leq)^{\mathrm{op}} \to \mathrm{Set}$ where where $\mathcal{F}n$ is finite and non-empty for all $n \in \mathbb{N}$, has non-empty limit.

Example:

A graph G is 3-colourable if all its finite subgraphs are 3-colourable.

Proof.

If G countable:

finite

$$G_1 \subseteq G_2 \subseteq \cdots \subseteq G$$

finite non-empty

$$\mathcal{F} : (\mathbb{N}, \leq)^{\mathrm{op}} \to \mathrm{Set}, \; n \mapsto \mathrm{Hom}(G_n, K_3)$$

König

$$\lim \mathcal{F} = \mathrm{Hom}(G, K_3)$$

non-empty

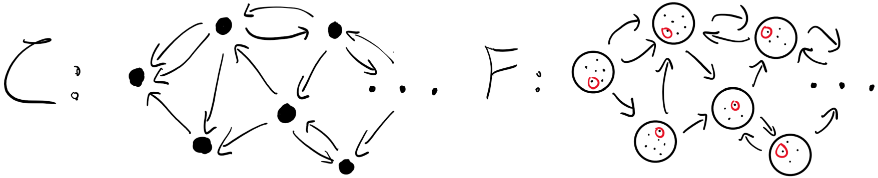
□

Improvements?

Lemma (Kőnig, 1927), rephrased

For any functor $\mathcal{F} : (\mathbb{N}, \leq)^{\mathrm{op}} \to \mathrm{Set}$ we have

$$\mathcal{F}n \neq \emptyset \text{ finite } \forall n \in \mathbb{N} \implies \lim \mathcal{F} \neq \emptyset$$



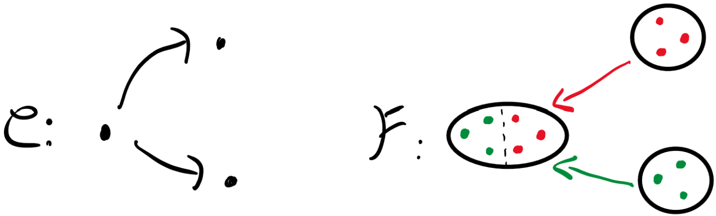
Definition

Call $\mathcal{C}$ *Kőnig* if for every functor $\mathcal{F} : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ we have

$$\mathcal{F}C \neq \emptyset \text{ finite } \forall C \in \mathcal{C} \implies \lim \mathcal{F} \neq \emptyset$$

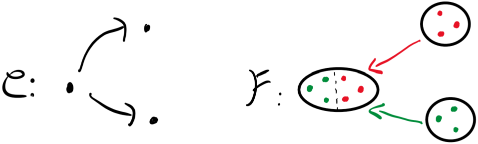
Obstacles for being Kőnig

Forks



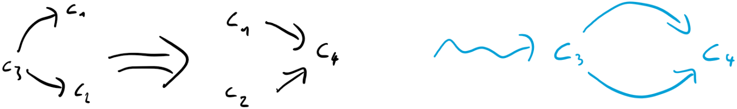
Parallel arrows

Obstacles: Forks



Definition

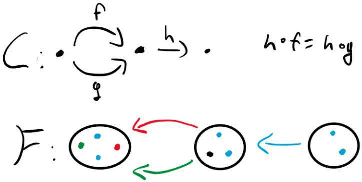
A category is called *confluent* if every two objects with a common lower bound also have a common upper bound.



Parallel arrows

Potential fix

Require that for any two arrows f, g there is h "coequalizing" them.



Proposition

this + confluent $\implies$ Kőnig

Posets

Example

A graph G is 3-colourable if all its finite subgraphs are 3-colourable.

Proof.

confluent

$$\mathcal{C} := (\text{all finite subgraphs of } G\text{, ordered by inclusion})$$

$G_1 \subseteq G_1 \cup G_2$
$G_2 \subseteq$

$$\mathcal{F} : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set},\ (G_i \subseteq G) \mapsto \mathrm{Hom}(G_i, K_3)$$

finite
non-empty

Kőnig

$$\lim \mathcal{F} = \mathrm{Hom}(G, K_3)$$

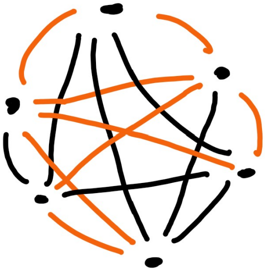
non-empty

□

Ramsey's Theorem

Theorem (Ramsey 1928)

For any k there exists n such that any edge coloring of the complete n element graph contains a monochromatic clique of size k.

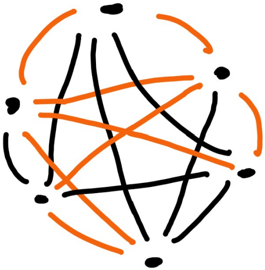


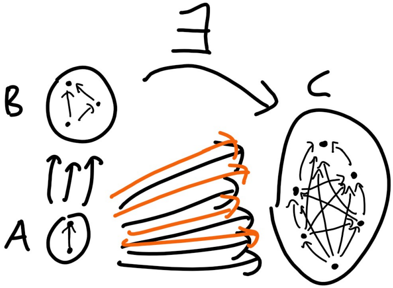
$k = 3, n = 6$

Ramsey's Theorem

Theorem (Ramsey 1928)

For any k there exists n such that any edge coloring of the complete n element graph contains a monochromatic clique of size k.

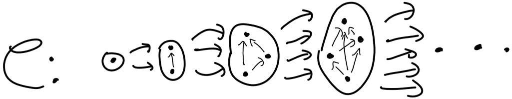




$k = 3, n = 6$

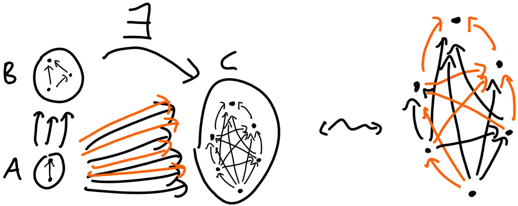
Ramsey's Theorem

Let $\mathcal{C}$ be the category of finite linear orders and embeddings.



Morally

Arrows cannot be "coequalized", but for any property of arrows, there is h such that $h \circ f$ and $h \circ g$ are the same, w.r.t that property.



The Ramsey Property

Definition

A category $\mathcal{C}$ is called *Ramsey* if for all $A, B \in \mathcal{C}$, there is $C \in \mathcal{C}$ such that for all $\chi : \mathrm{Hom}(A, C) \to \{0, 1\}$ there is $h : B \to C$ such that

$$\mathrm{Hom}(A, B) \xrightarrow{h_*} \mathrm{Hom}(A, C) \xrightarrow{\chi} \{0, 1\}$$

is constant.

Theorem (Ramsey, 1928)

The category of finite linear orders and embeddings is Ramsey.

Theorem (H.)

For a small, locally finite category $\mathcal{C}$, TFAE:

1. $\mathcal{C}$ is confluent and Ramsey
2. $\mathcal{C}$ is Kőnig, i.e. for all functors $\mathcal{D} : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ we have

$$\mathcal{D}C \neq \emptyset \text{ finite } \forall C \in \mathcal{C} \implies \lim \mathcal{D} \neq \emptyset$$

Theorem (H.)

For a small, locally finite category $\mathcal{C}$, TFAE:

1. $\mathcal{C}$ is confluent and Ramsey
2. $\mathcal{C}$ is Kőnig, i.e. for all functors $\mathcal{D} : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ we have

$$\mathcal{D}C \neq \emptyset \text{ finite } \forall C \in \mathcal{C} \implies \lim \mathcal{D} \neq \emptyset$$

$=$