Kőnig = Ramsey, A compactness lemma for Ramsey categories

Max Hadek

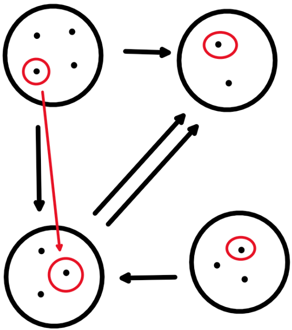
Charles University

Oberwolfach 2025

Funded by the European Union (ERC, POCOCOP, 101071674). Views and opinions expressed are however those of the author only and do not necessarily reflect those of the European Union or the European Research Council Executive Agency. Neither the European Union nor the granting authority can be held responsible for them.

Solving diagrams

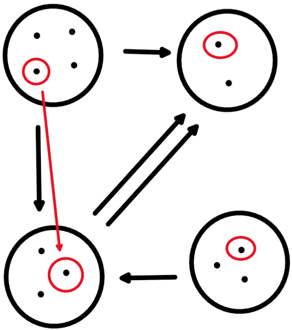
Input: diagram := some sets and maps



Task: pick one element out of each set, compatibly.

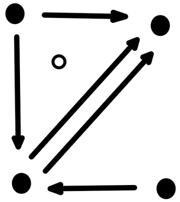
Solving diagrams

Input: diagram := some sets and maps



Task: pick one element out of each set, compatibly.

Shape of a diagram: underlying category / multigraph

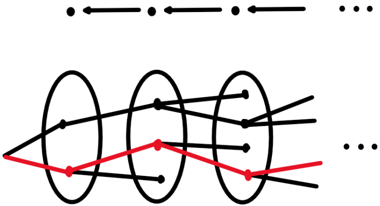


Question: For what shapes does every diagram have a solution?

Compactness

Kőnig's tree lemma

Shape:

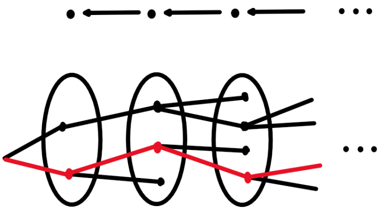


if sets are **finite** and **nonempty**
$\implies \exists$ solution

Compactness

Kőnig's tree lemma

Shape:

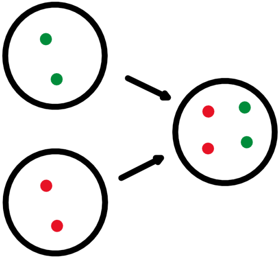


if sets are **finite** and **nonempty**
$\Longrightarrow \exists$ solution

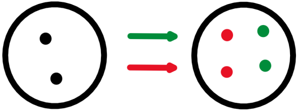
Other shapes?

What can go wrong?

Forks

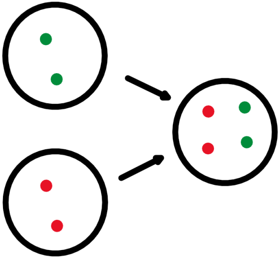


Parallel maps

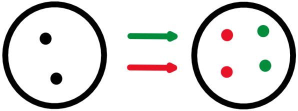


What can go wrong?

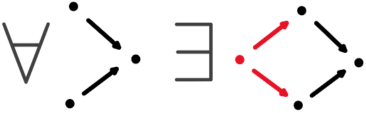
Forks

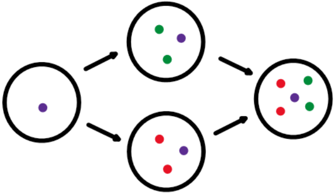


Parallel maps



Fixing forks:
call a shape **co-confluent** if



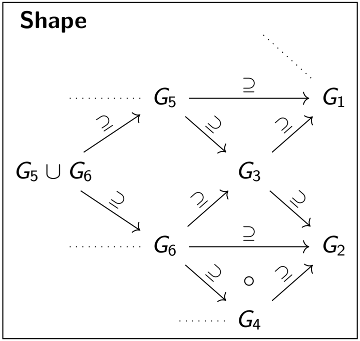


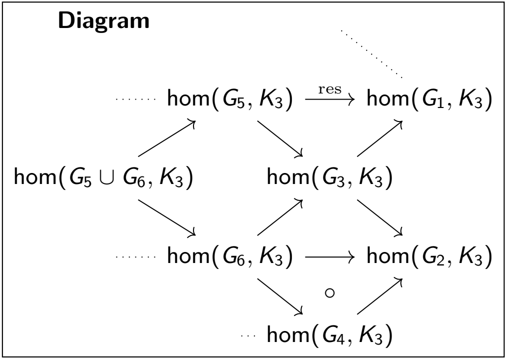
$\mathcal{S}$ confluent + no parallel maps
$\implies$ Kőnigs lemma for $\mathcal{S}$

Example

$\mathbf{K}_{K_3}$: A graph G is 3-colorable iff all finite subgraphs are 3-colorable

Proof:

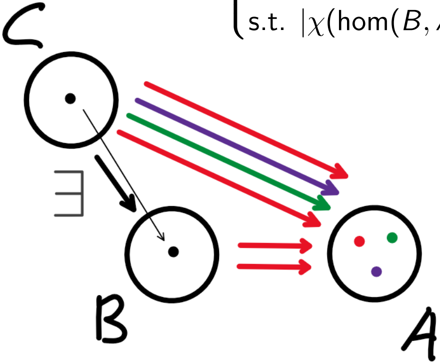




solutions give homomorphisms $G \to K_3$ □

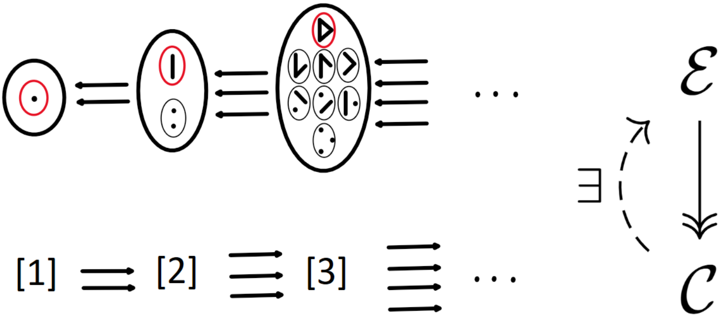
Parallel edges need Ramsey

Def: call a shape $\mathcal{S}$ **dual Ramsey** if
$$\begin{cases} \forall A, B \in \mathcal{S}, r \text{ finite set} \\ \exists C \in \mathcal{S} \\ \forall \chi\colon \ \hom(C, A) \to r \\ \exists f : C \to B \\ \text{s.t. } |\chi(\hom(B, A) \circ f)| = 1 \end{cases}$$



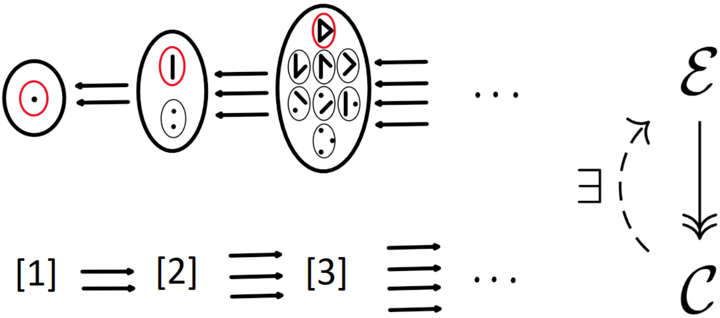
THM (H): Let $\mathcal{S}$ be a locally finite category. Then:

Kőnig's lemma holds for $\mathcal{S}$ $\iff$ $\mathcal{S}$ is co-confluent and dual Ramsey



THM (H): Let $\mathcal{S}$ be a locally finite category. Then:

$$\text{Kőnig's lemma holds for } \mathcal{S} \iff \mathcal{S}^{\mathrm{op}} \text{ is confluent and Ramsey}$$



Applications

Ramsey expansions: a Ramsey expansion is "optimal" iff its a core w.r.t. homomorphisms of diagrams

Ramsey transfers: old and new

products adding constants blowups parts of partite construction	} Grothendieck opfibrations

Short proof of "canonization lemma" [Bodirsky, Pinsker, Tsankov]

The dunce cap

Every locally finite, infinite dimensional simplicial set contains the infinite dunce cap.

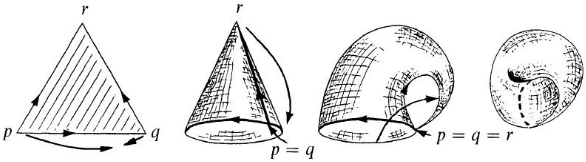


Figure I-6. The dunce cap.

The dunce cap

Every locally finite, infinite dimensional simplicial set contains the infinite dunce cap.

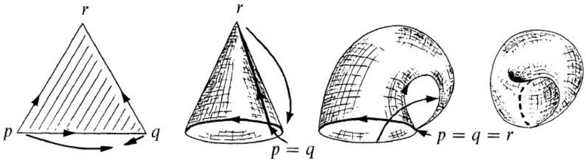


Figure I-6. The dunce cap.

Thank you!