A compactness lemma for Ramsey categories

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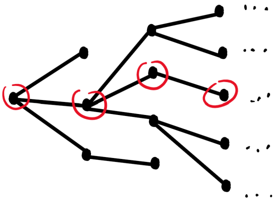
Kőnig's tree lemma

Lemma (Kőnig, 1927)

Any finitely branching, infinite tree contains an infinite path.



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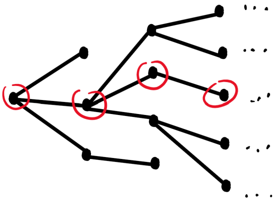
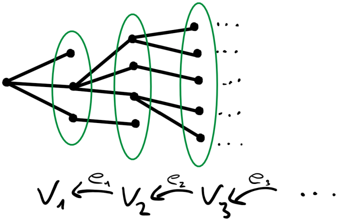




$$\{\text{infinite paths}\} = \{(v_n \in V_n \mid e_n(v_{n+1}) = v_n\} = \lim_{n\in\mathbb{N}} V_n$$

Lemma (Kőnig, 1927), rephrased

Any functor $\mathcal{F} : (\mathbb{N}, \leq)^{\mathrm{op}} \to \mathrm{Set}$ where where $\mathcal{F}n$ is finite and non-empty for all $n \in \mathbb{N}$, has non-empty limit.

Example: 3-colouring

Example:

A graph G is 3-colourable if all its finite subgraphs are 3-colourable.

Proof.

If G countable:

finite

$$G_1 \subseteq G_2 \subseteq \cdots \subseteq G$$

finite non-empty

$$\mathcal{F} : (\mathbb{N}, \leq)^{\mathrm{op}} \to \mathrm{Set},\ n \mapsto \mathrm{Hom}(G_n, K_3)$$

König

$$\lim \mathcal{F} = \mathrm{Hom}(G, K_3)$$

non-empty

□

Improvements?

Lemma (Kőnig, 1927), rephrased

For any functor $\mathcal{F} : (\mathbb{N}, \leq)^{\mathrm{op}} \to \mathrm{Set}$ we have

$$\mathcal{F}n \neq \emptyset \text{ finite } \forall n \in \mathbb{N} \implies \lim \mathcal{F} \neq \emptyset$$

Replace $(\mathbb{N}, \leq)$ with an arbitrary category $\mathcal{C}$:

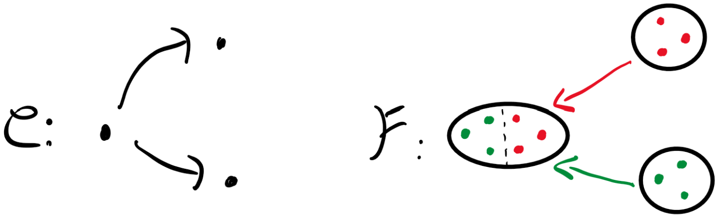
Definition

Call $\mathcal{C}$ *Kőnig* if for every functor $\mathcal{F} : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ we have

$$\mathcal{F}C \neq \emptyset \text{ finite } \forall C \in \mathcal{C} \implies \lim \mathcal{F} \neq \emptyset$$

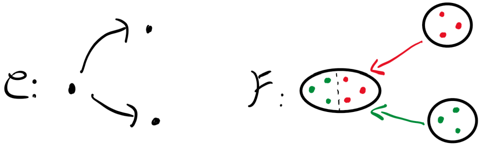
Obstacles for being Kőnig

Forks



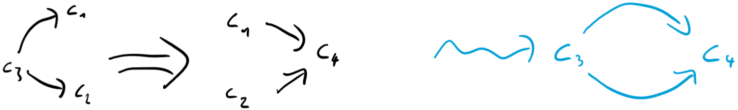
Parallel arrows

Obstacles: Forks



Definition

A category is called *confluent* if every two objects with a common lower bound also have a common upper bound.



Remark

If $\mathcal{C}$ consists of finite structures and embeddings, then JEP $\implies$ confluence.

Posets

Lemma

A poset is Kőnig if and only if it is confluent.

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Example

A graph G is 3-colourable if all its finite subgraphs are 3-colourable.

Proof.

confluent

$$\mathcal{C} := (\text{all finite subgraphs of } G\text{, ordered by inclusion})$$

$G_1 \subseteq G_1 \cup G_2$
$G_2 \subseteq$

$$\mathcal{F} : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set},\ (G_i \subseteq G) \mapsto \mathrm{Hom}(G_i, K_3)$$

finite
non-empty

Kőnig

$$\lim \mathcal{F} = \mathrm{Hom}(G, K_3)$$

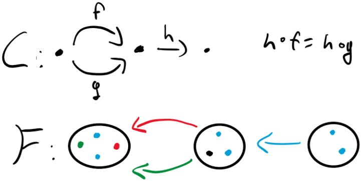
non-empty

□

Parallel arrows

Potential fix

Require that for any two arrows f, g there is h "coequalizing" them.



Lemma

this + confluent $\implies$ Kőnig

Remark

If $\mathcal{C}$ consists of finite structures and embeddings, there are no "coequalizers"

Ramsey

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Definition

A category $\mathcal{C}$ is called *Ramsey* if for all $A, B \in \mathcal{C}$ and $k \in \mathbb{N}$, there is $C \in \mathcal{C}$ such that for all $\chi : \mathrm{Hom}(A, C) \to [k]$ there is $h : B \to C$ such that

$$\mathrm{Hom}(A, B) \xrightarrow{h_*} \mathrm{Hom}(A, C) \xrightarrow{\chi} [k]$$

is constant.

Morally

For any property of arrows, there is h such that $h \circ f$ and $h \circ g$ are the same, w.r.t that property.

Theorem

For a small, locally finite category $\mathcal{C}$, TFAE:

1. $\mathcal{C}$ is confluent and Ramsey
2. $\mathcal{C}$ is Kőnig, i.e. for all functors $\mathcal{F} : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ we have

$$\mathcal{F}C \neq \emptyset \text{ finite } \forall C \in \mathcal{C} \implies \lim \mathcal{F} \neq \emptyset$$

Theorem

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How to apply

1. Local to global: "something exists everywhere locally $\implies$ it exists globally"

$$\forall C \in \mathcal{C} : \mathcal{F}C \neq \emptyset \implies \lim \mathcal{F} \neq \emptyset$$

2. Global to local: "globally nothing bad happens $\implies$ somewhere locally nothing bad happens"

$$\lim \mathcal{F} = \emptyset \implies \exists C \in \mathcal{C} : \mathcal{F}C = \emptyset$$

Proving Ramseyness - global to local

Theorem

If $\mathcal{C}$ is locally finite, then Kőnig $\implies$ Ramsey

Proof.

Fix $A, B \in \mathcal{C}$ and $k \in \mathbb{N}$.

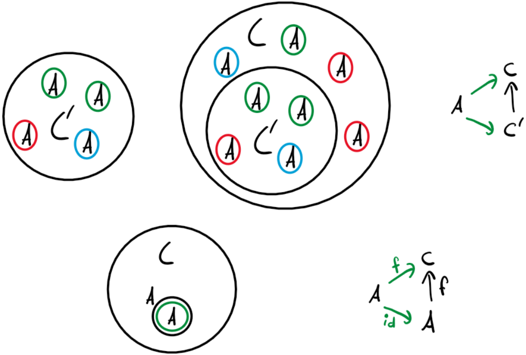
$$\begin{aligned} \mathcal{F} : \mathcal{C}^{\mathrm{op}} &\to \mathrm{Set} \\ C &\mapsto \{\text{bad colourings of } \mathrm{Hom}(A, C)\} = \\ &= \{\chi : \mathrm{Hom}(A, C) \to [k] \mid \forall h : B \to C, \chi \circ h_* \text{ not constant}\} \end{aligned}$$

Show that $\lim \mathcal{F} = \{\text{global bad colourings}\} = \emptyset$. □

Proving Ramseyness - continued

Proof - continued.

$(\chi_C : \mathrm{Hom}(A, C) \to [k])_{C \in \mathcal{C}} \in \lim$ with compatibility:



$\Longrightarrow$ every global colouring is constant □

Application: Canonization

Definition

Let $\mathcal{A}$ and $\mathcal{B}$ be (Fraïssé) classes of finite relational structures. An (injective) *canonical funtion* $\Phi : \mathcal{A} \to \mathcal{B}$ is a functor that preserves domains, i.e. $\forall \mathbb{X} \in \mathcal{A}$

$$|\mathbb{X}| = |\Phi(\mathbb{X})|$$

Theorem (Bodirsky, Pinsker, Tsankov 2011)

Let $\mathbb{A}, \mathbb{B}$ be Fraïssé limits of $\mathcal{A}, \mathcal{B}$ respectively and $f : \mathbb{A} \to \mathbb{B}$ an (injective) function. If $\mathcal{A}$ Ramsey then there is a canonical function $\Phi : \mathcal{A} \to \mathcal{B}$ such that for every $\mathbb{X} \in \mathcal{A}$ there is an embedding $\iota : \mathbb{X} \to \mathbb{A}$ such that

$$\Phi(\mathbb{X}) = \mathbb{B}[f \circ \iota]$$

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$$\Phi(\mathbb{X}) = \mathbb{B}[f \circ \iota]$$

Proof.

Let $\mathcal{F} : \mathcal{A}^{\mathrm{op}} \to \mathrm{Set}$, $\mathbb{X} \mapsto \{\mathbb{B}[f \circ \iota] \mid \iota : \mathbb{X} \to \mathbb{A}\}$. Take $\Phi \in \lim \mathcal{F}$. □