A non-finitely related minimal Taylor algebra

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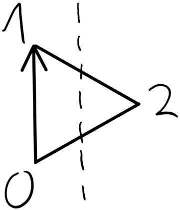

Definition: minimal Taylor

Definition

A finite idempotent algebra $\mathbb{A}$ is called *Taylor*, if

- $\mathbb{A}$ satisfies a non-trivial Maltsev condition
- (Maroti, McKenzie) $\mathrm{Clo}(\mathbb{A})$ contains a weak-NU operation

$$w(y, x, \ldots, x) \approx w(x, y, x, \ldots, x) \approx \cdots \approx w(x, \ldots, x, y)$$

- (Barto, Kozik) $\mathrm{Clo}(\mathbb{A})$ contains cyclic operations for all primes $p > |\mathbb{A}|$

$$c_p(x_1, x_2, \ldots, x_p) \approx c_p(x_p, x_1, \ldots, x_{p-1})$$

- $\mathrm{HS(P)}(\mathbb{A})$ does not contain a naked set

Definition

A Taylor algebra $\mathbb{A}$ is called *minimal*, if any other algebra $\mathbb{B}$ with $\mathrm{Clo}(\mathbb{B}) \subsetneq \mathrm{Clo}(\mathbb{A})$ is not Taylor.

Definition: finitely related

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An algebra $\mathbb{A}$ is called *finitely related* if there is a finite set of relations $\{R_1, \dots R_n\}$ on the domain of $\mathbb{A}$, such that

$$f \in \mathrm{Clo}(\mathbb{A}) \iff f \text{ preserves all } R_i$$

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Every minimal Taylor algebra is finitely related.

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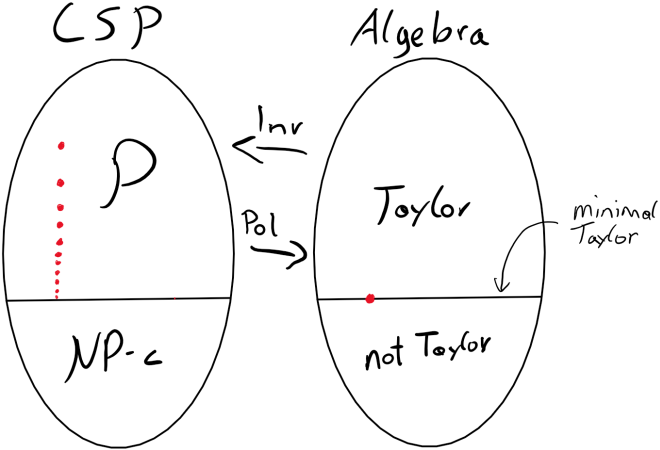
Theorem (H.)

There is a minimal Taylor algebra that is not finitely related.

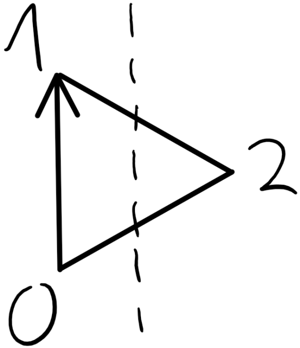
Connection to constraint satisfaction problems



The algebra

$\mathbb{A} = (\{0, 1, 2\}, m)$ $\quad m$ ternary

- $m|_{\{0,2\}} = m|_{\{1,2\}} = \text{maj}$
- $m|_{\{0,1\}}(x, y, z) = x \vee y \vee z$
- $m(0, 1, 2) = \cdots$
  $\cdots = m(2, 1, 0) = 1$



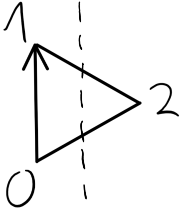
Theorem (H.)

The algebra $\mathbb{A}$ is minimal Taylor, but not finitely related.

Understanding the algebra

- m is idempotent
- m is symmetric $\implies \mathbb{A}$ Taylor
- $\theta = (\, 0, 1 \mid 2\,)$ is congruence
- $m/\theta = \text{maj}$

$\implies$ We understand $\mathrm{Clo}(\mathbb{A})$ modulo θ!



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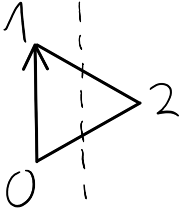


Proposition

Let $f \in \mathrm{Clo}(\mathbb{A})$, then

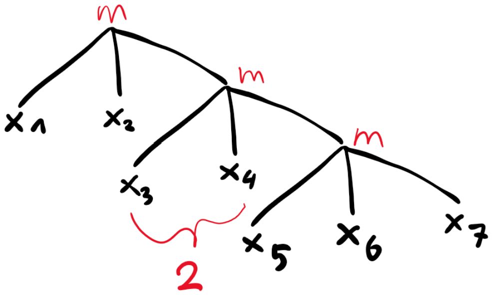
- (monotone) $f(2,\dots,2,2,0,\dots,0) = 0$ implies
  $$f(2,\dots,2,0,0,\dots,0) = 0$$
- (self dual) $f(2,\dots,2,0,\dots,0) = 0$ if and only if
  $$f(0,\dots,0,2,\dots,2) = 2$$

Inside the congruence

Proposition

Let $f \in \mathrm{Clo}(\mathbb{A})$ with $f(2, \ldots, 2, 0/1, \ldots, 0/1) = 0/1$. Then

$$f(2, \ldots, 2, y_1, \ldots, y_k)|_{\{0,1\}^k} = \bigvee \text{some } y_i$$



Deciding finite relatedness: does theory help?

Theorem (Aichinger, Mayr, McKenzie)

If an algebra has a cube term, then it is finitely related.

$$t(x, ?, \dots, ?) = y, \ \dots, \ t(?, \dots, ?, x) = y$$

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Theorem (Barto, Bulin)

If $\exists B : B \triangleleft_J \mathbb{A}$ but $B \not\triangleleft \mathbb{A}$, then $\mathbb{A}$ is not finitely related.

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Both theorems do not apply to $\mathbb{A}$!

How to prove non-finitely related?

Observation

$\mathbb{A}$ is not finitely related iff $\forall n \exists f_n$ of arity $> n$ such that $f_n \notin \mathrm{Clo}(\mathbb{A})$ but all n-ary minors of f_n are in $\mathrm{Clo}(\mathbb{A})$.

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Proof.

Show ($\Longleftarrow$): Let $\{R_1, R_2, \dots\}$ be a relational basis.
Find $f_n \notin \mathrm{Clo}(\mathbb{A})$ where f_n does not preserve some R_i, but all its n-ary minors do.

$$f_n(\underbrace{\bar{x}_1, \bar{x}_2, \bar{x}_3, \dots, \bar{x}_{ar}}_{\in R_i, \text{ more than } n \text{ many}}) \notin R_i$$

Then $|R_i| > n$, hence $|\{R_1, R_2, \dots\}| = \infty$. □

Proof strategy

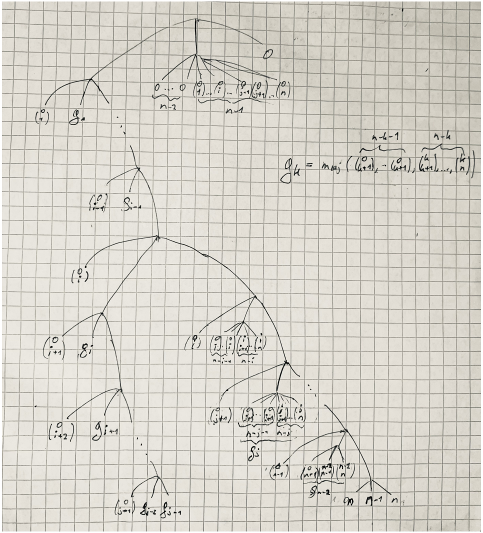
Observation

An algebra $\mathbb{A}$ is not finitely related iff $\forall n \exists f_n$ of arity $> n$ such that all n-ary minors of f_n are in $\mathrm{Clo}(\mathbb{A})$, but $f_n \notin \mathrm{Clo}(\mathbb{A})$.

Strategy

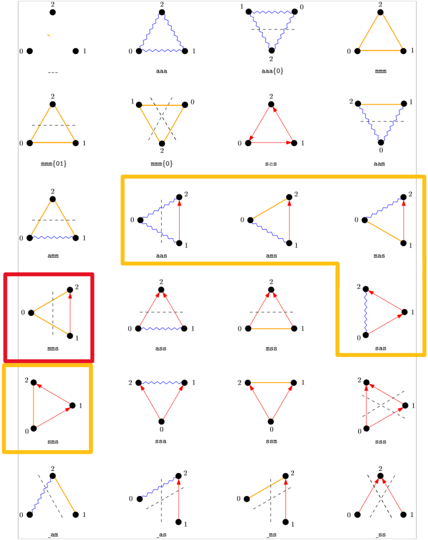
- Find relations R_n of arity n
- Construct f_n for every n of arity $\approx n^2$
- Show f_n does not preserve R_n, i.e $f_n \notin \mathrm{Clo}(\mathbb{A})$
- Show that n-ary minors of f_n are in $\mathrm{Clo}(\mathbb{A})$

A big tree



Minimal Taylor algebras on 3 elements

- (Brady) there are 24
- (Barto, Brady, Jankovec, Vucaj, Zhuk) 18 are "well behaved": "nice" finite relational description
- (H.) one is evil
- 5 are unstudied
- Picture by (Vucaj)



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