

# Hereditary First-Order Logic and Extensional ESO

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joint work with Manuel Bodirsky

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TU Dresden

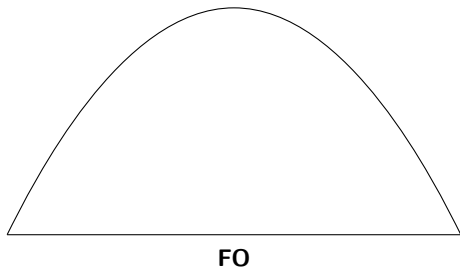
87th Theorietag, Jena, 2025



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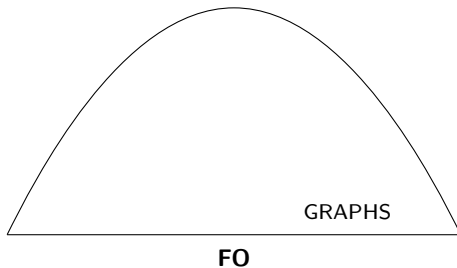
# First-Order Logic



# First-Order Logic

- ▶ Loopless symmetric digraphs (a.k.a. simple graphs)

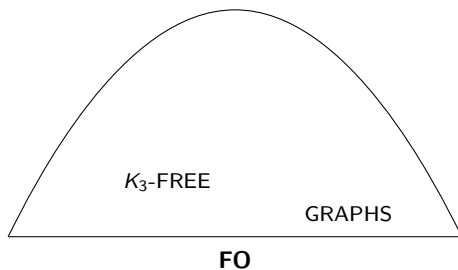
$$\forall x. \neg E(x, x) \wedge \forall x, y. E(x, y) \iff E(y, x)$$



# First-Order Logic

- ▶ Triangle-free graphs

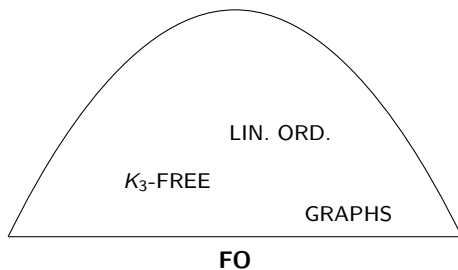
$$[\dots] \wedge \forall x, y, z. \neg E(x, y) \vee \neg E(y, z) \vee \neg E(z, x)$$



# First-Order Logic

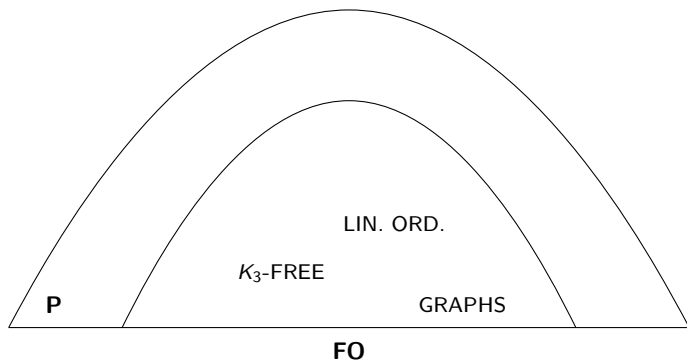
- ▶ (Strict) linear orders

$$\forall x. \neg E(x, x) \wedge \forall x, y. x = y \vee E(x, y) \vee E(y, x) \wedge E \text{ is transitive}$$



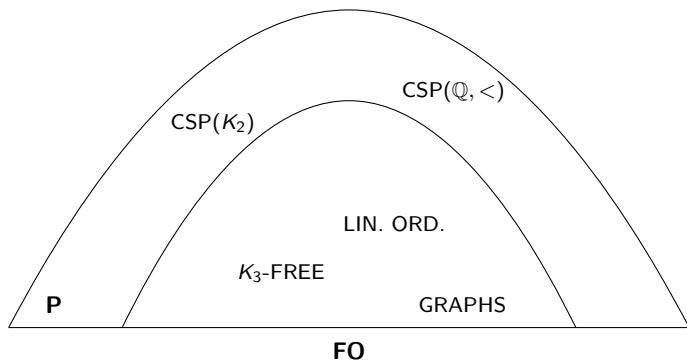
# First-Order Logic

- ▶ If  $\mathcal{C}$  is expressible in FO, then  $\mathcal{C}$  is in P



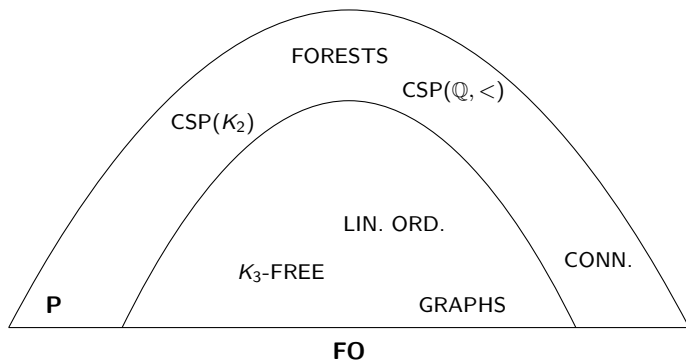
# First-Order Logic

- ▶  $\text{CSP}(H) :=$  class of finite digraphs (structures)  $G$  such that  $G \rightarrow H$



# First-Order Logic

- ▶ Many classes are not expressible in FO





# Existential Second-Order Logic

# Existential Second-Order Logic

- ▶ Bipartite graphs

$$\exists W, B. \forall x, y. (W(x) \Leftrightarrow \neg B(x)) \wedge (E(x, y) \Rightarrow (W(x) \Leftrightarrow B(y)))$$

- ▶ Finite domain CSPs

$$\exists U_1, \dots, U_k \forall x_1, \dots, x_n. (\dots)$$

- ▶ Acyclic digraphs =  $\text{CSP}(\mathbb{Q}, <)$

$$\exists T \forall x, y. (E(x, y) \Rightarrow T(x, y) \wedge T \text{ is a strict linear order})$$

- ▶ Strict NP (SNP)

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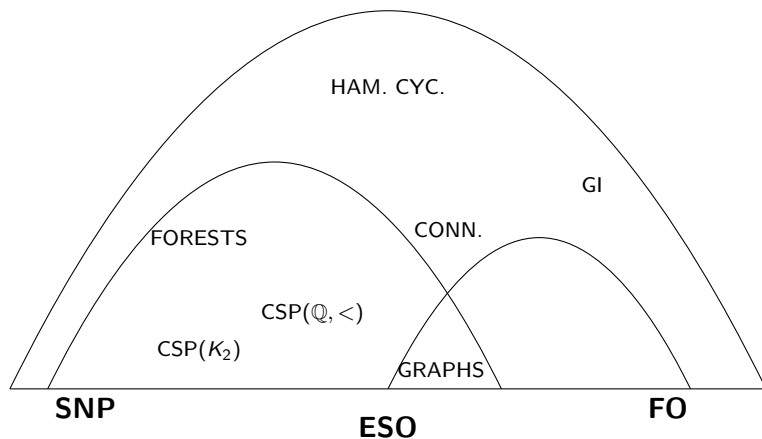
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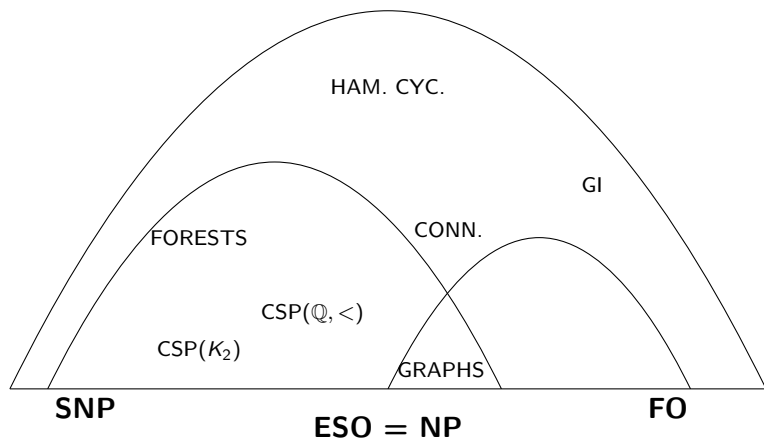
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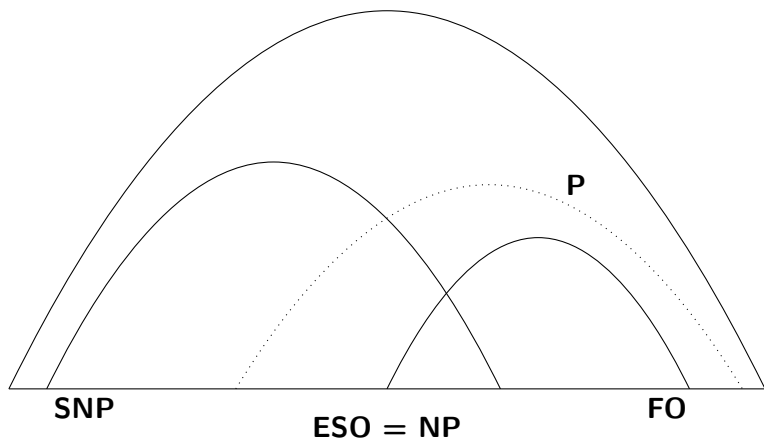
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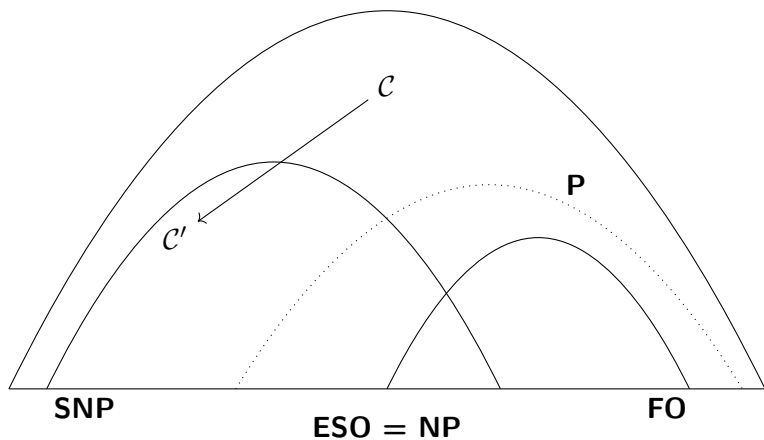


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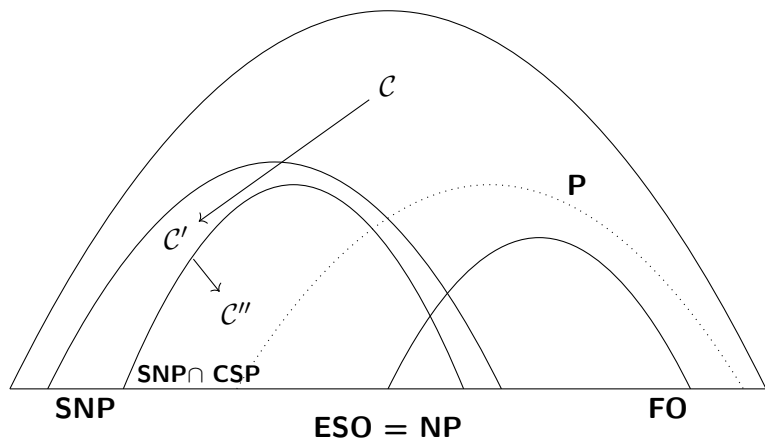




# Existential Second-Order Logic



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# Hereditary First-Order Logic

A structure  $\mathbb{A}$  *hereditarily satisfies*  $\phi$  if every (induced) substructure  $\mathbb{A}'$  of  $\mathbb{A}$  satisfies  $\phi$ .

**Ex. 1** Acyclic digraphs:  $\phi :=$  exists a source.

**Ex. 2** Forests:  $\phi :=$  exists a vertex of degree 1.

**Ex. 3** Chordal graphs:  $\phi :=$  exists a simplicial vertex (Rose, 1970).

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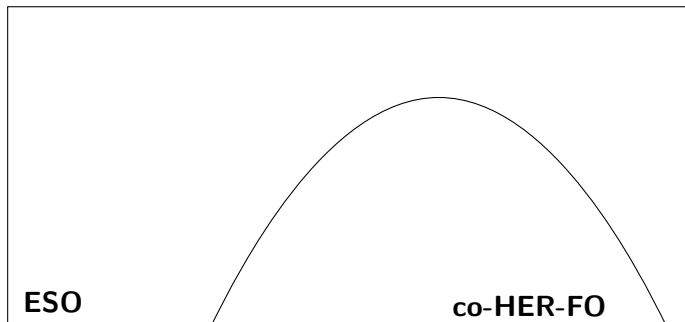
**Ex. 3** Chordal graphs:  $\phi :=$  exists a simplicial vertex (Rose, 1970).

**Two more.**  $\text{CSP}(\vec{P}_3)$ ; and every directed cycle induces a symmetric edge.

# Hereditary First-Order Logic

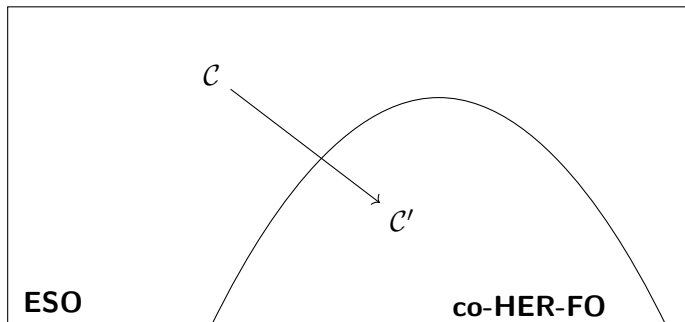
**Obs.** If  $\mathbb{A}$  does not hereditarily satisfy  $\phi$ , then there is a set of vertices  $S$  such that  $\mathbb{A}[S]$  satisfies  $\neg\phi$ . Equivalently,

$$\mathbb{A} \models \exists S. S \neq \emptyset \wedge (\neg\phi)_S$$



# Hereditary First-Order Logic

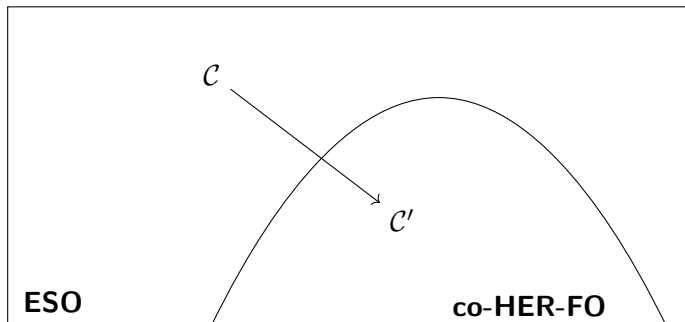
**Qst 1.** Are the complements of HER-FO NP-rich?



# Hereditary First-Order Logic

**Qst 1.** Are the complements of HER-FO NP-rich?

**Qst 2.** Are there NP-intermediate problems in (co-)HER-FO?



# Extensional ESO



# Extensional ESO

**Example.** Acyclic digraphs =  $\text{CSP}(\mathbb{Q}, <)$

$$\exists T \forall x, y. (E(x, y) \Rightarrow T(x, y) \wedge T \text{ is a strict linear order})$$

**Non-example.** 2-COL

$$\exists W, B. \forall x, y. (W(x) \Leftrightarrow \neg B(x)) \wedge (E(x, y) \Rightarrow (W(x) \Leftrightarrow B(y)))$$

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**Example.**

$$\exists W, B. \forall x, y. W'(x) \Rightarrow W(x) \wedge B'(x) \Rightarrow B(x) \wedge (\dots)$$

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**Example.** Pre-coloured 2-COL

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**Example.** Pre-coloured  $\text{CSP}(H)$

$$\exists U_1, \dots, U_k \left( \forall x \bigwedge_{i=1}^k U'_i(x) \Rightarrow U_k \right) \wedge (\forall x_1, \dots, x_n(\dots))$$

# Extensional ESO

Dear colleagues,

We are pleased to invite you to the online event:

\*Event:\* \*30 Years of Graph Sandwich Problems: A Celebration\*

\*Date:\* March 27, 2025

\*Time:\* 2:00 PM (GMT -3), São Paulo, Brazil

\*Link:\* <https://meet.google.com/sur-pmun-eyy>

In 1995, the publication of the seminal paper:

\*\*M.C. Golumbic, H. Kaplan, R. Shamir, "Graph Sandwich Problems," Journal of Algorithms 19 (1995) 449-473\*\*

opened a rich and extensive research area that continues to inspire publications worldwide.

# Extensional ESO

## Graphs Sandwich Problems ( $\text{GSP}(\Pi)$ )

**Input:** A vertex set  $V$  and sets of edges  $E_1 \subseteq E_2$ .

**Question:** Is there a graph  $(V, E)$  that satisfies property  $\Pi$  and  $E_1 \subseteq E \subseteq E_2$ .

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**Ex 1.  $P_4$ -free graphs:**  $\text{GSP}(\Pi)$  is in P (Golombic, Kaplan, Shamir 1995).

**Ex 2.  $C_4$ -free graphs:**  $\text{GSP}(\Pi)$  is NP-complete (Dantas, de Figueiredo, da Silva, Teixeira 2011).

**Ex 3.  $(K_4 - e)$ -free graphs:**  $\text{GSP}(\Pi)$  is in P (Dantas, de Figueiredo, da Silva, Teixeira 2011).

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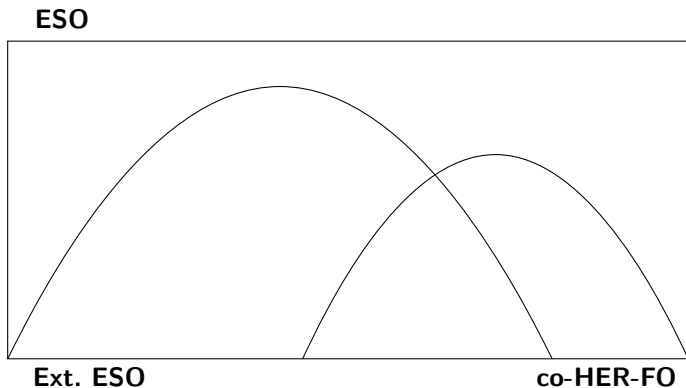
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**Ext. ESO:** GSP( $\Pi$ ) is in Extensional ESO whenever  $\Pi$  is in FO. For instance,  $\Pi := F$ -free.

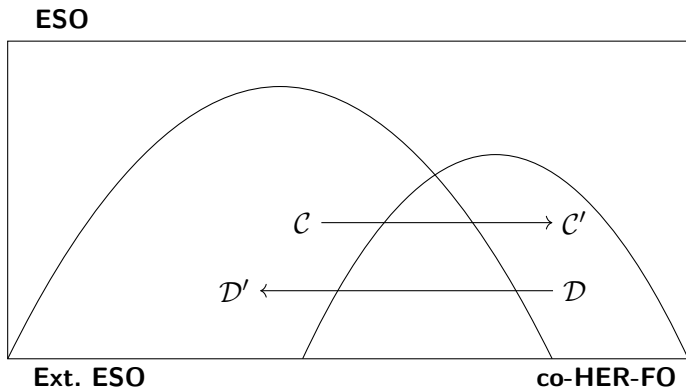


# Extensional ESO



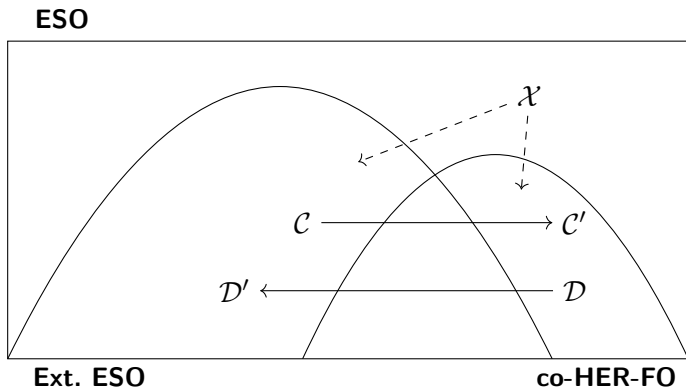
# Extensional ESO

**Theorem.** Extensional ESO and complements of HER-FO have the same computational power (up to P-time equivalence).



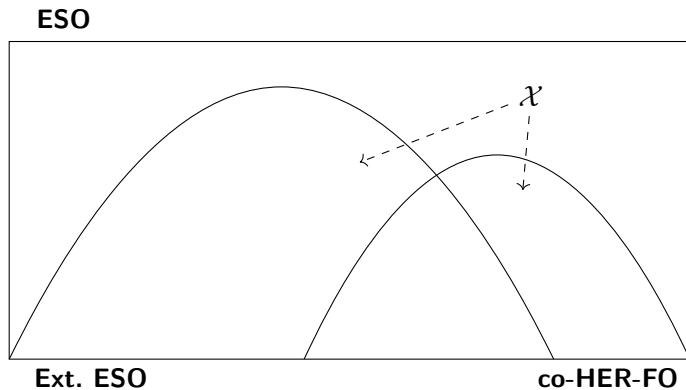
# Extensional ESO

**Theorem.** Extensional ESO and complements of HER-FO have the same computational power (up to P-time equivalence). Moreover, these classes are not NP-rich (unless  $E = NE$ ).



# Extensional ESO

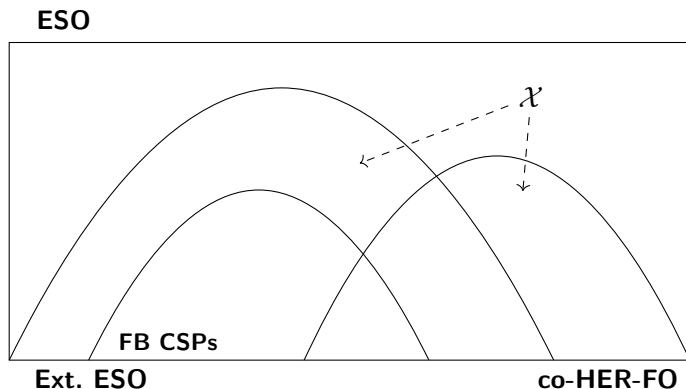
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# Extensional ESO

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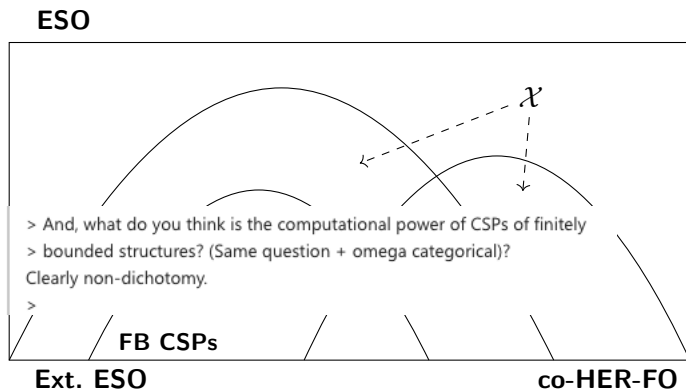
**Lemma.**  $\text{CSP}(H)$  is expressible in ext. ESO whenever  $H$  is finitely bounded.



# Extensional ESO

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Thank you for your attention!