Hereditary First-Order Logic and Extensional ESO

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joint work with Manuel Bodirsky

Institute of Algebra
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Graph modification problems

Graph modification problems

Edge-modification

For a fixed graph class $\mathcal{C}$ and an input graph G determine the minimum number of edge modifications to G so that it belongs to $\mathcal{C}$.

- Edge-deletion problems (Yanakkakis 1981).
- Hardness of Edge-Modification problems (Alon, Stav 2009).
- Dichotomy Results on the Hardness of H-free Edge Modification Problems (Aravind, Sandeep, Sivadasan 2017).
- Hardness of approximation for H-free edge modification problems (Bliznets, Cygan, Komosa, Pilipczuk 2018)

Graph modification problems

Resilience problems

For a fixed query μ, determine the resilience of μ in an input database $\mathbb{D}$.

- The Complexity of Resilience and Responsibility for Self-Join-Free Conjunctive Queries (Freire, Gatterbauer, Immerman, Meliou 2015)
- New Results for the Complexity of Resilience for Binary with Self-Joins (Freire, Gatterbauer, Immerman, Meliou 2020).
- A Unified Approach for Resilience and Causal Responsibility with Integer Linear Programming (ILP) and LP Relaxations. (Makhija, Gatterbauer 2023).
- The Complexity of Resilience Problems via Valued Constraint Satisfaction (Bodirsky, Semanišinová, Lutz 2024).

Graph modification problems

Vertex-deletion

For a fixed graph class $\mathcal{C}$ and an input graph G determine the minimum k so that $G - U$ belongs to $\mathcal{C}$ for some $|U| \leq k$.

- Node-Deletion NP-Complete Problems (Krishnamoorthy, Deo 1979)
- The node-deletion problem for hereditary properties is NP-complete (Lewis, Yanakkakis 1980).
- Finding odd-cycle transversals (Reed, Smith, Vetta 2004)
- On the Descriptive Complexity of Vertex Deletion Problems (Bannach, Chudigiewitsch, Tantau 2024)

Graph modification problems

Modification to first-order logic

Edge-modification: Given a graph G and a positive integer k test whether it is possible to modify at most k edges so that it satisfies ϕ.

Edge-completion: Given a graph G and a positive integer k test whether it is possible to add at most k edges from G so that it satisfies ϕ.

Edge-deletion: Given a graph G and a positive integer k test whether it is possible to remove at most k edges from G so that it satisfies ϕ.

Vertex-deletion: Given a graph G and a positive integer k test whether it is possible to remove at most k vertices from G so that it satisfies ϕ.

On the parameterized complexity of graph modification to first-order logic properties (Fomin, Golovach, Thilikos 2020)

Graph modification problems

Modification to first-order logic (without parameter k)

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Ex. 2 Chordal graphs: $\phi :=$ exists a simplicial vertex (Rose, 1970).

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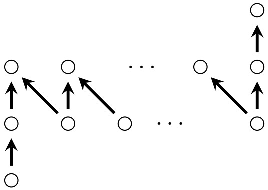
Ex. 4 CSP($\to\to$) is in HerFO.



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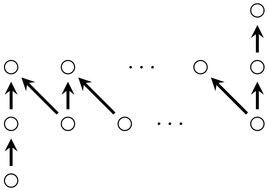

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Lem. If the quantifier prefix of ϕ is of the form $\forall^*\exists\forall^*$ or $\forall^*\exists^*$, then $\mathrm{Her}(\phi)$ is in P and in SNP.

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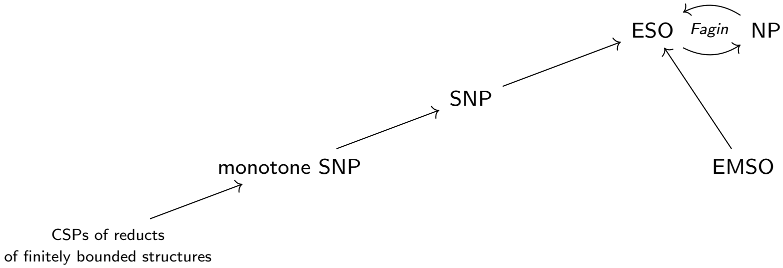
Thm. If the quantifier prefix of ϕ is of the form $\forall^*\exists\forall^*$ or $\forall^*\exists^*$, then Her(ϕ) is in P and in SNP. For every other quantifier prefix $\mathcal{Q}$, there is an FO formula ϕ with quantifier prefix $\mathcal{Q}$ such that Her(ϕ) is coNP-complete.

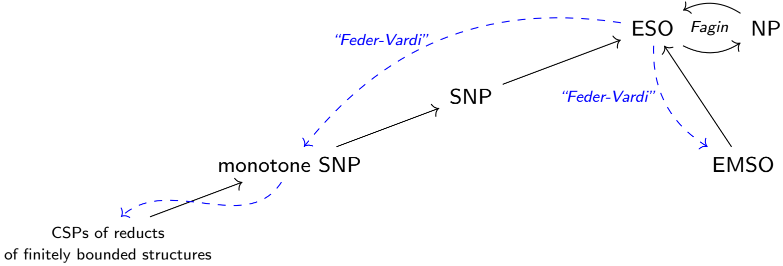
Hereditary first-order logic

“Qst.” What is the computational power of HerFO?

Existential Second-Order Logic



Existential Second-Order Logic



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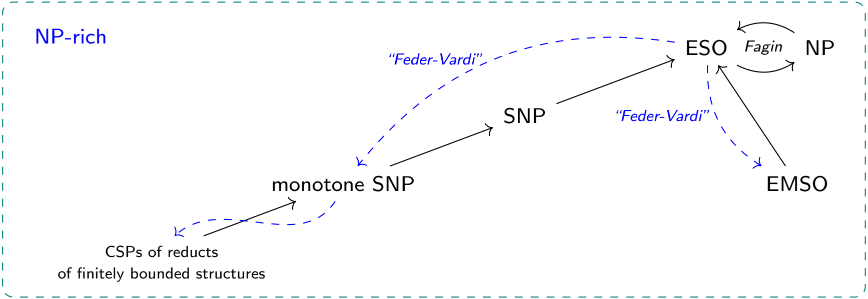
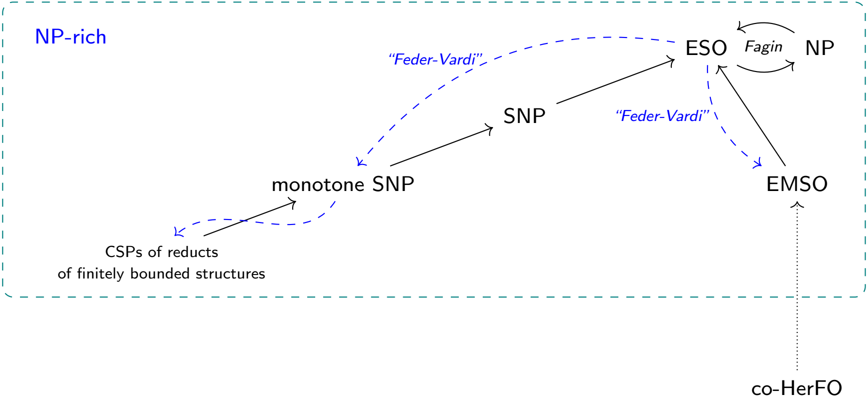


Existential Second-Order Logic



Extensional ESO

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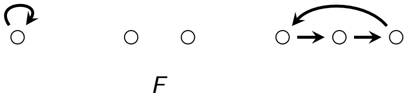
F-free edge-completion: Given a graph G test whether it is possible to add edges to G so that it becomes F-free.

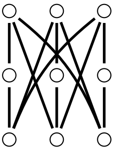
Ex 1. Acyclic digraphs: extend the edge relation to a (strict) linear order.



Extensional ESO

Edge-completion to ϕ: Given a graph G test whether it is possible to add edges so that it satisfies ϕ.

Ex 2. (Pach, 1971) A graph G has circular chromatic number < 3 iff G can be extended to a maximal triangle-free graphs that avoids:



Petersen minus vertex

Extensional ESO

Extensional ESO is the fragment of ESO

$$\exists R_1 \ldots, R_k. \left[\bigwedge_{i \in [n]} \forall \bar{x} R_i'(x) \implies R(\bar{x}) \right] \wedge \phi$$

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Non-Ex $\exists W, B. \forall x \; (W(x) \Leftrightarrow \neg B(x)) \wedge \mathrm{Ind}(W) \wedge \mathrm{Ind}(B)$

Ex 3. $\exists W, B. \forall x. \; W'(x) \Rightarrow W(x) \;\wedge\; B'(x) \Rightarrow B(x) \;\wedge\; \phi.$

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Ex 3. Pre-coloured CSP($\mathbb{A}$) is in extensional ESO for every finite $\mathbb{A}$.

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Ex 4. A graph G is k-colourable iff it has a $\{K_1 + K_2, K_{k+1}\}$-free extension.

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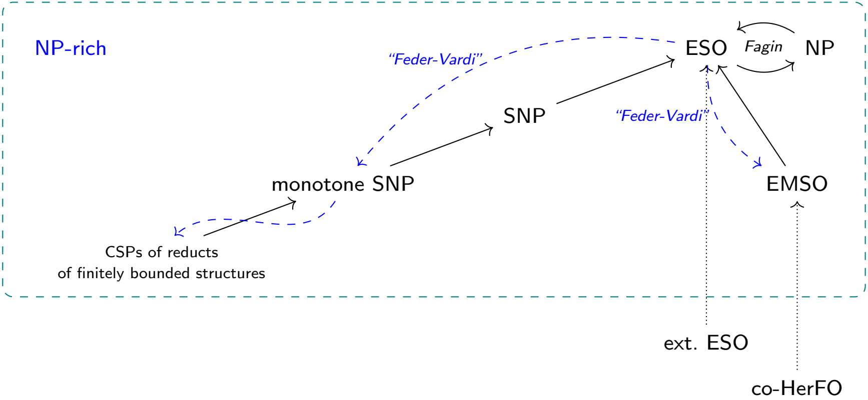
Ex 3. Pre-coloured CSP($\mathbb{A}$) is in extensional ESO for every finite $\mathbb{A}$.

Ex 4. CSP($\mathbb{A}$) and surj-CSP($\mathbb{A}$) are in extensional ESO for every finite $\mathbb{A}$.

Extensional ESO



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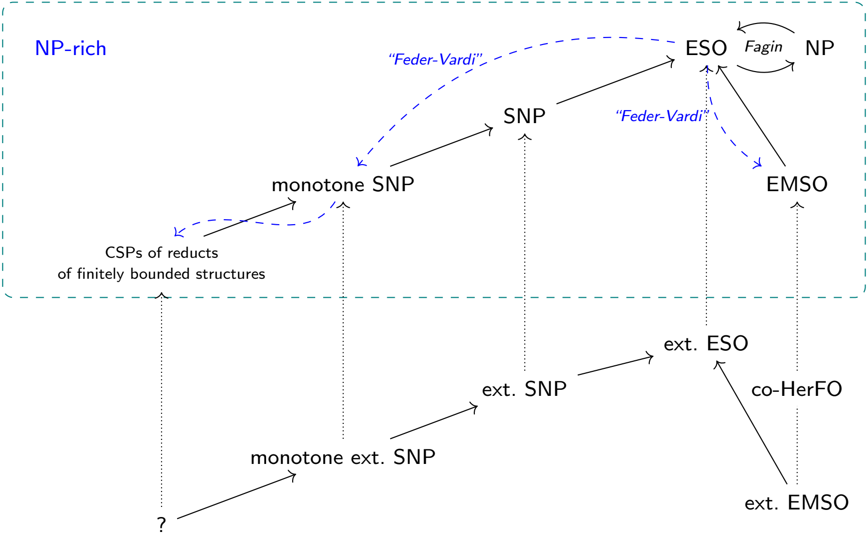

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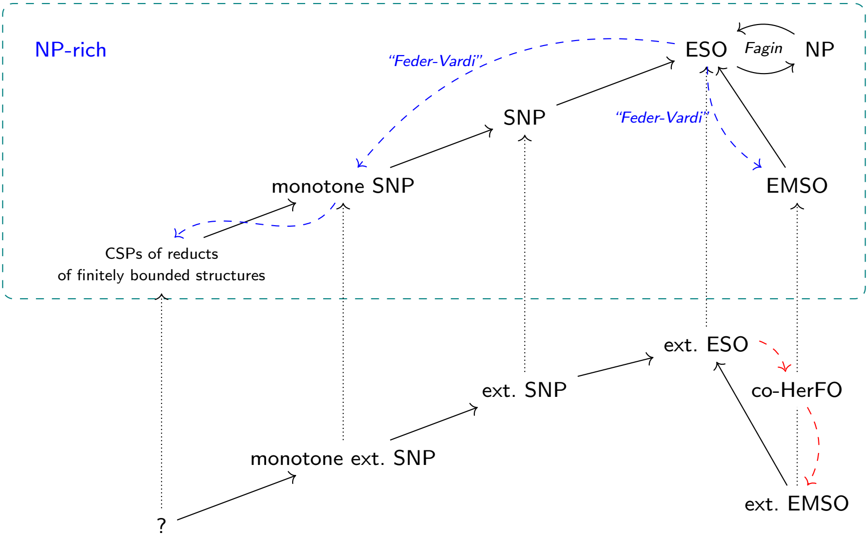

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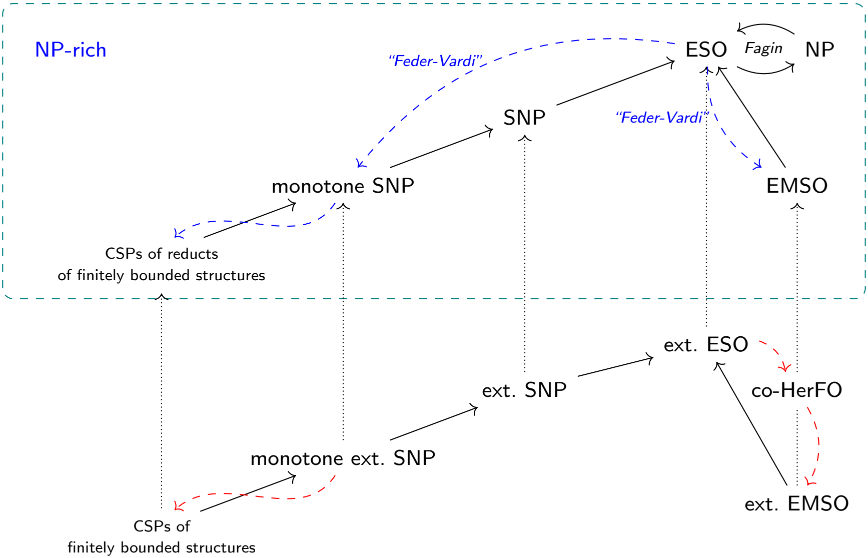

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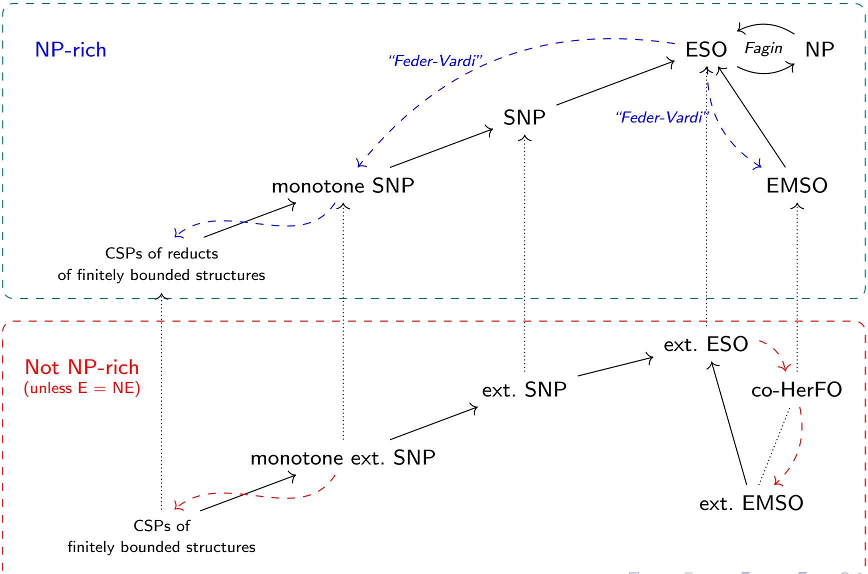

Thank you for your attention!