GMSNP and finite-template PCSPs

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Motivation

Conjecture (Brakensiek and Guruswami)

For every pair of non-bipartite finite graphs $G_1 \to G_2$ the promise CSP with template (G_1, G_2) is NP-hard.

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In particular, if H is a (possibly infinite) non-bipartite graph with $\chi(H) \in \mathbb{Z}^+$, then CSP(H) is NP-hard.

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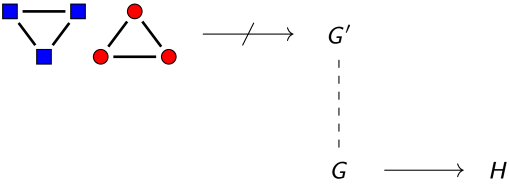

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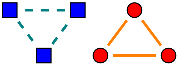

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Theorem: Not with these attempts.

Tools

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Smallest finite factor

A finite structure $\mathbb{C}$ is the *smallest finite factor* of a structure $\mathbb{S}$ if for every finite structure $\mathbb{B}$

$$\mathbb{S} \to \mathbb{B} \text{ if and only if } \mathbb{C} \to \mathbb{B}$$

- **Ex. 1** Every finite structure has a smallest finite factor.
- **Ex. 2** $(\mathbb{Q}, <)$ has a smallest finite factor (the loop).
- **Ex. 3** If $\mathbb{S}$ has a (vertex) Ramsey expansion, then $\mathbb{S}$ has a smallest finite factor.
- **Ex. 4** The infinite directed path does not have a smallest finite factor.

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Finite-domain up to high girth

The CSP of a structure $\mathbb{S}$ is *finite-domain up to high girth* if there is a finite structure $\mathbb{C}$ and a positive integer ℓ such that for every finite $\mathbb{B}$ of girth larger than ℓ

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Obs. II CSP($\mathbb{C}$) reduces in polynomial time to CSP($\mathbb{S}$).

Tools

Duality Theorems for Finite Structures (Nešetřil and Tardif)

For every finite set of trees $\mathcal{T}$ there is a finite structure $\mathbb{D}_{\mathcal{T}}$ such that $\mathrm{CSP}(\mathbb{D}_{\mathcal{T}}) = \mathrm{Forb}(\mathcal{T})$

$$\mathcal{T} \not\to \mathbb{B} \to \mathbb{D}_{\mathcal{T}} \quad \text{for every finite} \quad \mathbb{B}$$

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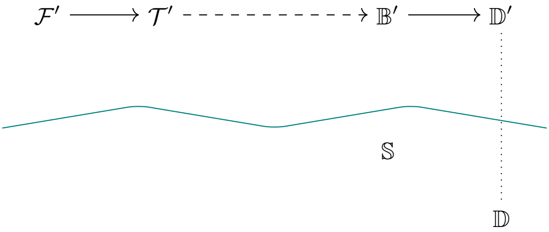

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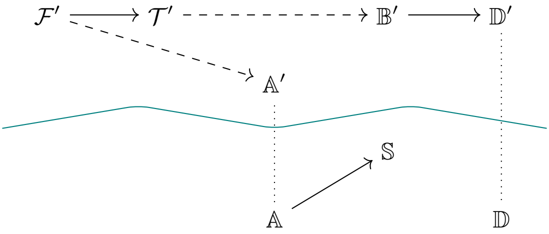

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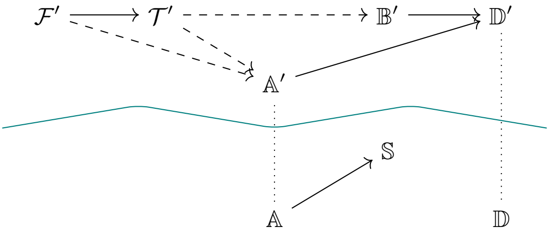

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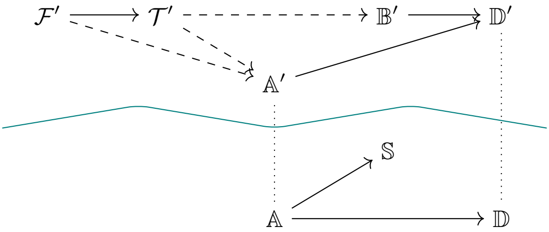

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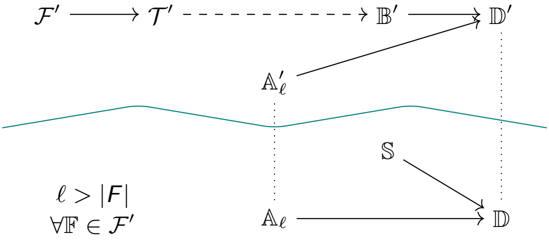

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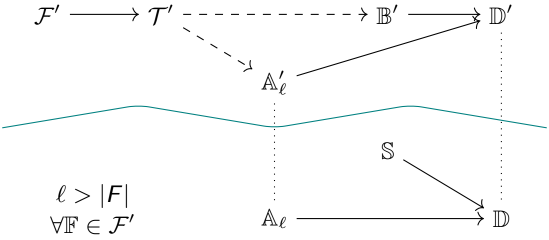

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If CSP($\mathbb{S}$) is in GMSNP, then there is a finite structure $\mathbb{C}$ such that

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Corollary

If $\mathbb{H}$ is a non-bipartite graph with finite chromatic number, and CSP($\mathbb{H}$) is in GMSNP, then CSP($\mathbb{H}$) is NP-complete.

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Corollary

If the tractability of a finite-template PCSP is explained by a GMSNP sandwich, then it is explained by a finite sandwich.

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Qst II. Does every ω-categorical structure have a vertex Ramsey expansion?

Thank you for your attention!