Boolean on Top of Temporal Constraint Satisfaction Problems

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Constraint Satisfaction Problems

Let $\mathfrak{B}$ be a relational structure in finite signature τ.

A first-order τ-formula $\phi(x_1, \ldots, x_n)$ is primitive positive if it is of the form

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The Constraint Satisfaction Problem for $\mathfrak{B}$ (CSP($\mathfrak{B}$)) is the computational problem to decide whether a given pp-sentence ϕ is true in $\mathfrak{B}$.

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Example

3SAT

INSTANCE: A propositional formula in conjunctive normal form (CNF) with at most three literals per clause.
QUESTION: Is there a Boolean assignment for the variables such that in each clause at least one literal is true?

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Example

Complete graph on 3 vertices

$$K_3 = (\{0, 1, 2\}; \neq)$$

CSP(K_3) *is the 3-colourability problem for graphs*

Complexity Dichotomy

Theorem (Bulatov (2017), Zhuk (2017))

For every finite structure $\mathfrak{B}$ *with finite signature,* $\mathrm{CSP}(\mathfrak{B})$ *is in P or NP-complete.*

Conjecture (Bodirsky, Pinsker (2011))

For a reduct $\mathfrak{B}$ *of a finitely bounded homogeneous structure,* $\mathrm{CSP}(\mathfrak{B})$ *is in P or NP-complete.*

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Γ *reduct of f.-b. hom. structure with maximal arity k*

The number of orbits of n-element subsets grows with $O(2^{n^k})$

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These structure have been classified by Falque and Thiéry in 2018.

The classification of their CSPs is open.

Temporal CSPs

Theorem (Bodirsky, Kára (2009, 2010))

Let $\mathfrak{B}$ be a structure with a first-order definition in $(\mathbb{Q}; <)$. Then exactly one of the following holds:

- $\mathrm{Pol}(\mathfrak{B})$ *contains one of the operations* min, mi, mx, ll *or their duals. In this case, the CSP of every finite-signature reduct of $\mathfrak{B}$ is in P.*
- $\mathrm{Pol}(\mathfrak{B})$ *has a uniformly continuous minor-preserving map to* $\mathrm{Pol}(K_3)$*. In this case, $\mathfrak{B}$ has a finite-signature reduct whose CSP is NP-complete.*

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First step towards research goals: $(\mathbb{Q}, <)$ with two unary predicates.

Boolean on Top of Temporal Structures

We consider: first-order expansions of $(\mathbb{Q}; <_N, <_P, N, P, \sim)$

- N and P two unary relations, convex bipartition of $\mathbb{Q}$
- $<_N$ the standard order on N
- $<_P$ the standard order on P
- $x \sim y$ iff $(N(x) \wedge N(y)) \vee (P(x) \wedge P(y))$

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Example

$\Gamma \coloneqq (\mathbb{Q}; <_N, <_P, N, P, \sim, F)$ *with*

$$F(x, y, z) :\Leftrightarrow (x \leq_N y \wedge y = z) \vee \\ ((x = y \wedge y <_P z) \vee (x = z \wedge z <_P y) \vee (z = y \wedge y <_P x))$$

Polymorphisms

An operation $f\colon B^k \to B$ is a polymorphism *of a structure $\mathfrak{B}$ if for every relation R of $\mathfrak{B}$ and for all tuples $\bar{r}_1, \ldots, \bar{r}_k \in R$ also $f(\bar{r}_1, \ldots, \bar{r}_k) \in R$.*
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A relation $R \subseteq B^n$ is preserved by all polymorphisms of an ω-categorical structure $\mathfrak{B}$ iff R has a pp-definition in $\mathfrak{B}$.

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Observe: *For* Γ *a first-order expansions of* $(\mathbb{Q}; <, N, P, \sim)$

- $\mathrm{Pol}(\Gamma)|_N$ *and* $\mathrm{Pol}(\Gamma)|_P$ *are polymorphism clones of temporal structures*
- $\mathrm{Pol}(\Gamma)/_{\sim}$ *is a polymorphism clone of a two-element structure*

Some observations about $(\mathbb{Q}; <, N, P, \sim)$

Observation

Γ *first-order expansion of* $(\mathbb{Q}; <, N, P, \sim)$

- *Assume any of* $\mathrm{Pol}(\Gamma)|_N$, $\mathrm{Pol}(\Gamma)|_P$, *or* $\mathrm{Pol}(\Gamma)/_{\sim}$ *has a uniformly continuous minor-preserving map to* $\mathrm{Pol}(K_3)$
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Example

The relation F *from the previous example is preserved by*

$$f(x, y) := \begin{cases} \mathrm{mx}(x, y) \text{ , } \textit{if } x, y \in P \\ \min(x, y) \text{ , } \textit{else.} \end{cases},$$

where $f/_\sim = \min$.

Main Result and Conjecture

Conjecture

Let Γ be a first-order expansion of $(\mathbb{Q}; <_N, <_P, N, P, \sim)$. Then exactly one of the following holds:

- *There are polymorphisms $f_1, f_2, g \in \mathrm{Pol}(\Gamma)$ such that each of $f_1|_N$ and $f_2|_P$ is one of the temporal operations* min, mi, mx, ll *or their duals, and $g/_\sim$ is one of the boolean operations* min, max, majo, mino. *In this case, the CSP of every finite-signature reduct of Γ is in P.*
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- Have tractability results for some combinations of polymorphisms
- Have candidates for general algorithm, proof involves many case distinctions and relies on diagonal canonosation

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- From there, classify complexity of CSPs of structures with polynomial orbit growth

Thank you for your attention

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