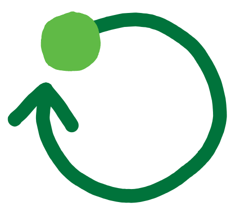
The sorrows of a smooth digraph: the first hardness criterion for infinite directed graph-colouring problems



Johanna Brunar
TU Wien

joint work with M. Kozik, T. Nagy, M. Pinsker

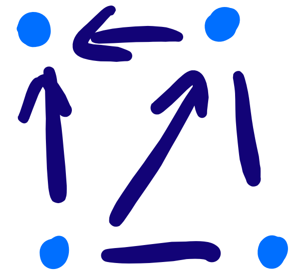
ÖAW ÖSTERREICHISCHE AKADEMIE DER WISSENSCHAFTEN

erc European Research Council Established by the European Commission

Bnglobal Algebra & Logic Seminar Spring '25

ON THE ROAD TO THE CSP-DICHOTOMY

$\mathbb{A} = (A; (R_i)_{i \in I})$ REL. STRUCTURE WITH SIGNATURE τ



CSP($\mathbb{A}$):

- INPUT: $\mathbb{X}$ FINITE τ-STRUCTURE
- QUESTION: DOES THERE EXIST A HOMOMORPHISM $\mathbb{X} \to \mathbb{A}$

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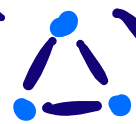
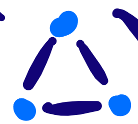
EXAMPLES:

(i) CSP() = 3-COLOURING

$\mathrm{CSP}(\mathbb{A})$:

- INPUT: $\mathbb{X}$ FINITE τ-STRUCTURE
- QUESTION: DOES THERE EXIST A HOMOMORPHISM $\mathbb{X} \to \mathbb{A}$

<u>EXAMPLES</u>:

(i) CSP() = 3-COLOURING

(ii) $\mathrm{CSP}(\mathbb{Q};<)$ = DIGRAPH ACYCLICITY

CSP($\mathbb{A}$):

- INPUT: $\mathbb{X}$ FINITE τ-STRUCTURE
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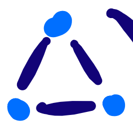
EXAMPLES:

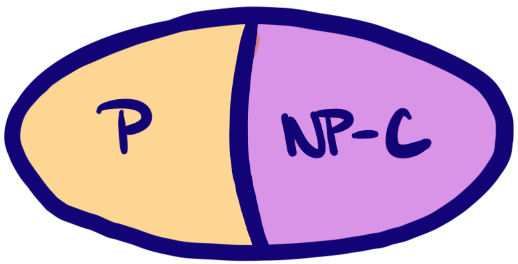
(i) CSP() = 3-COLOURING

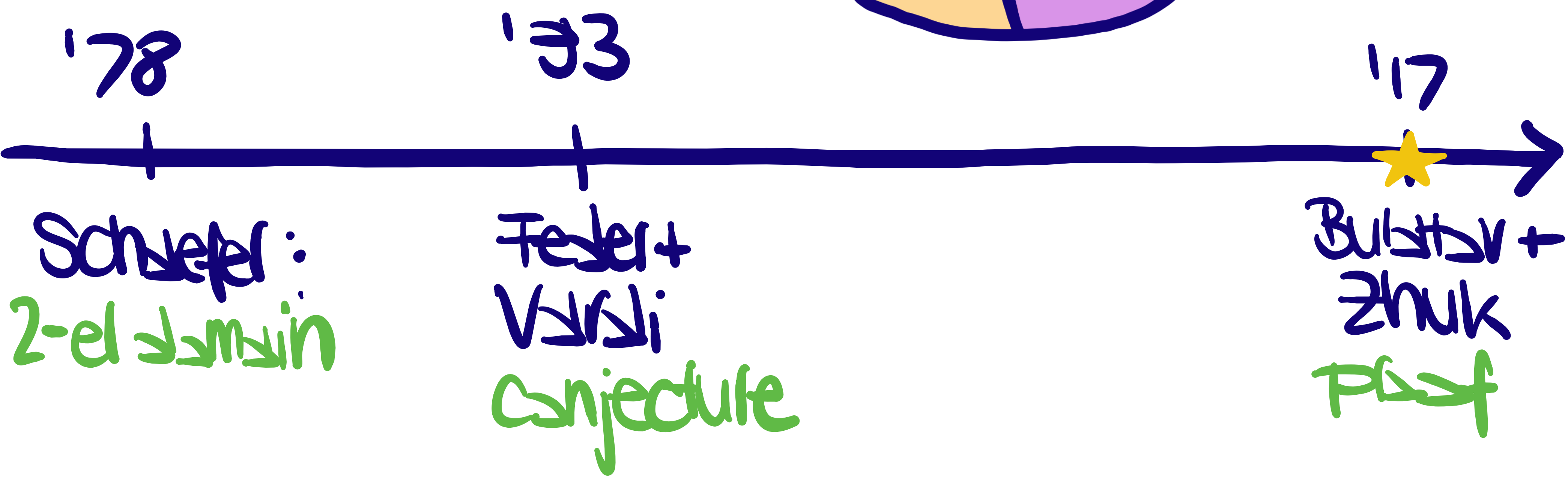
(ii) CSP($\mathbb{Q}$; $<$) = DIGRAPH ACYCLICITY

(iii) CSP($\mathbb{Z}$; $\{0\}$, $\{1\}$, $+$, $\cdot$) = DIOPHANTINE EQU.

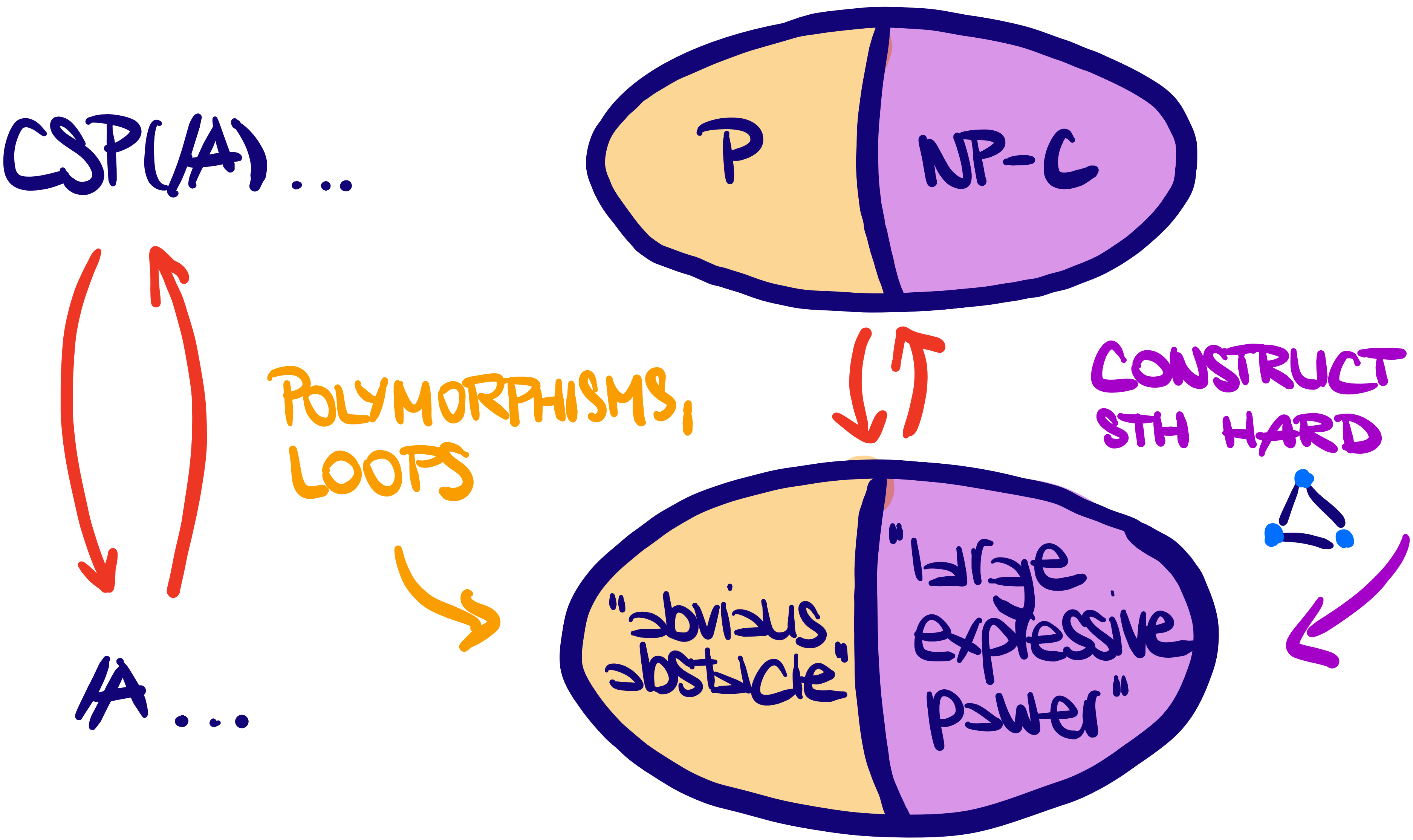
DICHOTOMY-THM (BULATOV, ZHUK '17)

$\mathbb{A}$ FINITE. THEN CSP($\mathbb{A}$) IS TRACTABLE OR NP-COMPLETE.





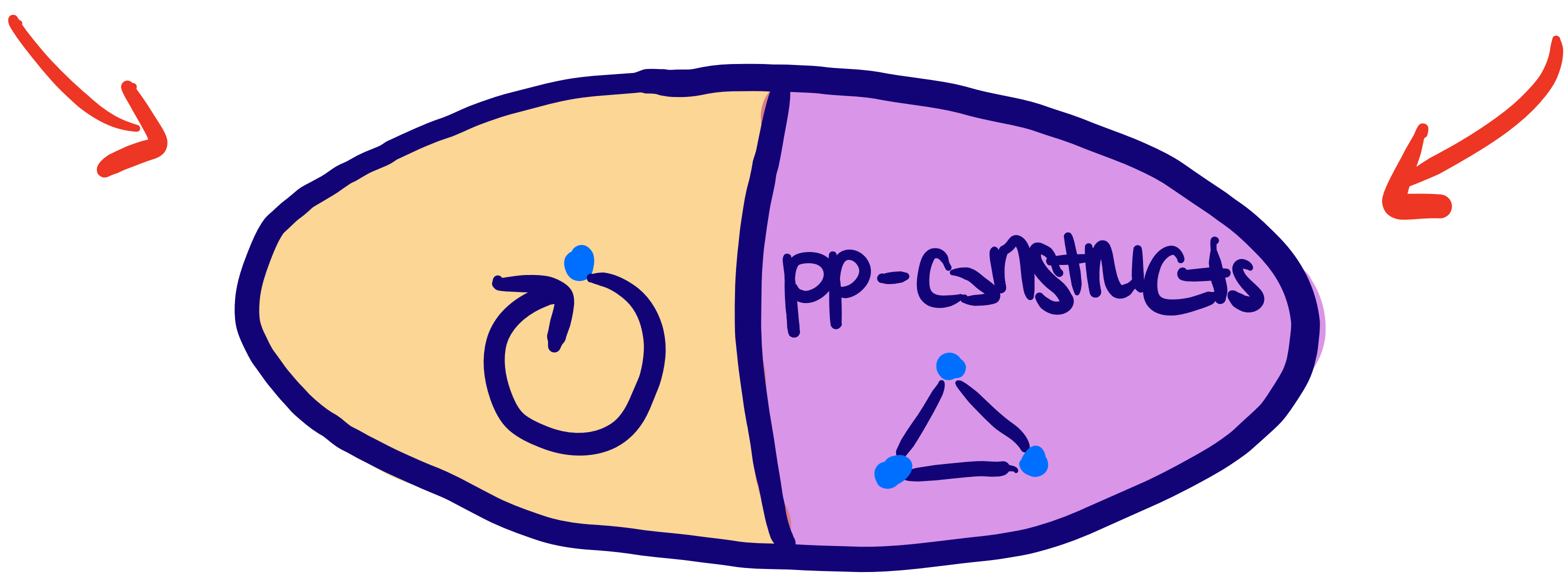
STRUCTURAL DICHOTOMIES



LOOP LEMMATA

<u>Thm</u> (HELL+NEŠETRIL '90)
G FINITE UNDIRECTED NON-BIPARTITE.

<u>Thm</u> (BARTO+KOZIK+NIVEN '09)
G FINITE DIRECTED SMOOTH ALGEBRAIC LENGTH 1.

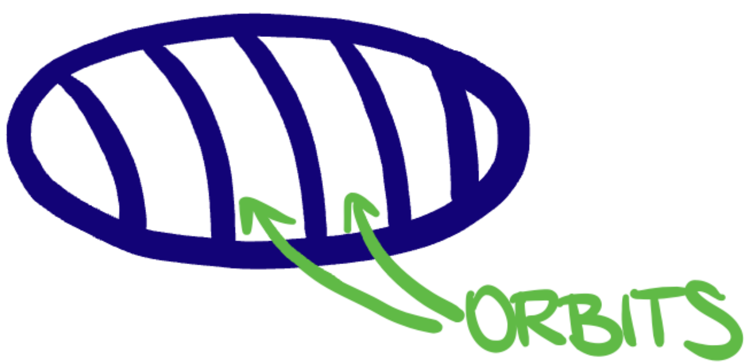


INFINITY

$\Omega \leq \mathrm{Aut}(\mathbb{G})$

- $\mathbb{G}/\Omega$: FACTOR GRAPH MOD 1-Ω-ORBIT EQUIV.
  I.E. $O \to P$ IFF $\exists x \in O, y \in P$: $x \to y$ IN $\mathbb{G}$
- Ω IS OLIGOMORPHIC IFF $\forall k$: G^k/Ω FINITE

$\Omega \curvearrowright G^k$



EXAMPLE:

$(\mathbb{Q}; <)$

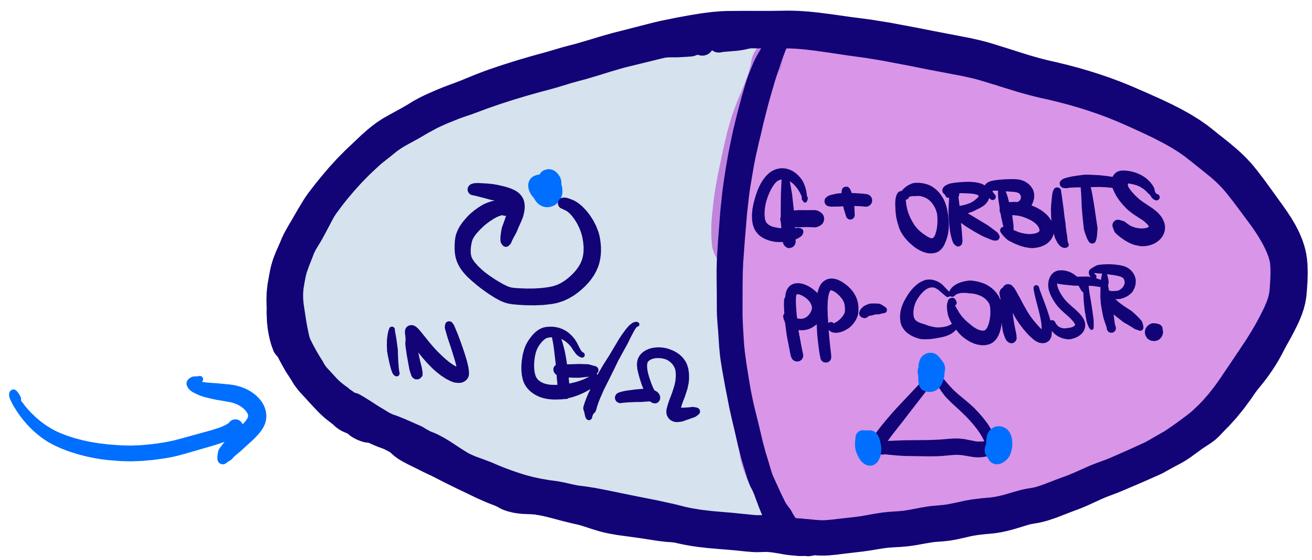
"CLOSE TO FINITE: 1-ORBITS TAKE ROLE OF ELEMENTS"

LIFTING FINITE TO INFINITE

HELL-NEŠETŘIL LIFTED:

<u>Thm</u> (BARTO + BODOR + KOZIK + MOTTET + PINSKER '23)
$\mathbb{G}$ SMOOTH, $\Omega \leq \mathrm{Aut}(\mathbb{G})$ OLIGOMORPHIC, $\mathbb{G}/\Omega$ SYMMETRIC, NON-BIPARTITE.

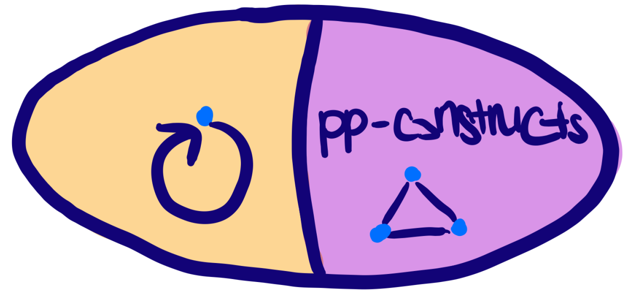
"PSEUDO-LOOP" WRT. Ω



OUR DREAM

Thm (BARTO+KOZIK+NIVEN '09)

G FINITE **DI**RECTED SMOOTH ALGEBRAIC LENGTH 1 . $\Rightarrow$

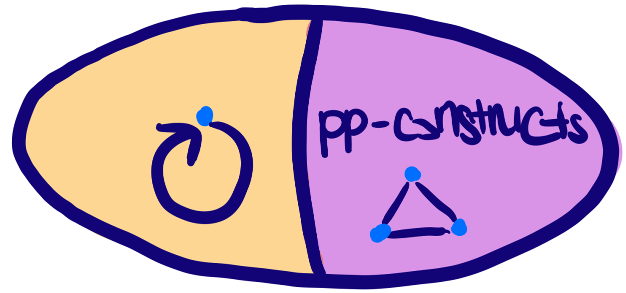


CAN WE LIFT THIS TO THE INFINITE?

OUR DREAM

Thm (BARTO+KOZIK+NIVEN '09)
∄ FINITE DIRECTED SMOOTH ALGEBRAIC LENGTH 1.

⇒



CAN WE LIFT THIS TO THE INFINITE?

CONSERVATIVITY

BULATOV:
CONSERVATIVE TEMPLATES, I.E. ALL SUBSETS ARE UNARY RELS.

CONSERVATIVE GRAPH-COLOURING (= LIST HOMOMORPHISM)

LHOM($\mathbb{G}$):

INPUT: FINITE DIGRAPH $\mathbb{H}$, $L: H \to \mathcal{P}(G)$

QUESTION: DOES THERE EXIST A HOMOMORPHISM $h: \mathbb{H} \to \mathbb{G}$ WITH $h(x) \in L(x)$ $\forall x \in H$?

$\Omega \leq \mathrm{Aut}(\mathbb{G})$ OLIGOMORPHIC $\Rightarrow$ 1-ORBITS OF Ω TAKE ROLE OF ELEMENTS!

K-CONSERVATIVE EXPANSION OF $\mathbb{G}$ WRT. Ω:

$\mathbb{G}$ EXPANDED BY ALL UNARY RELS $O_1 \cup \dots \cup O_k$ WHERE O_i IS 1-ORBIT OF Ω

OUR RESULT: THE FIRST STRUCTURAL DICHOTOMY FOR INFINITE DIGRAPHS

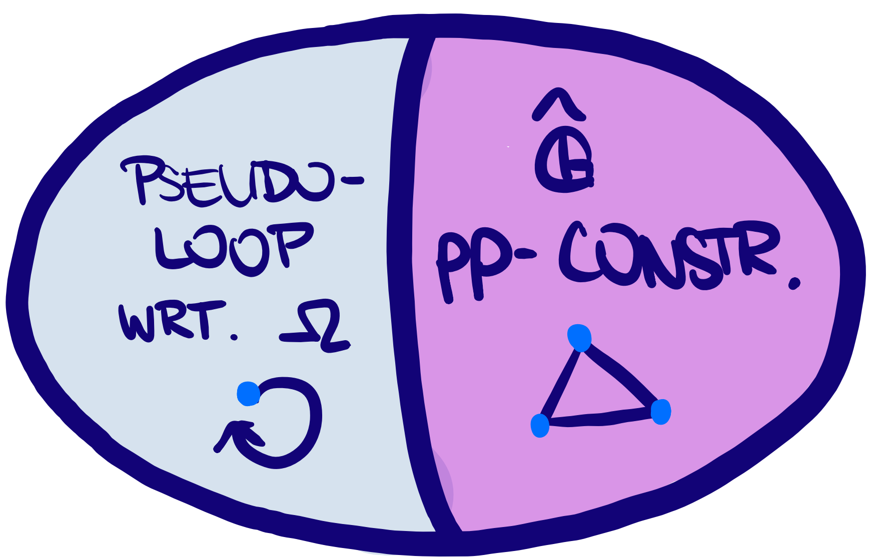
<u>Thm</u> (KOZIK + NAGY + PINSKER + B. '25)

$\mathbb{G}$ SMOOTH DIGRAPH OF ALGEBRAIC LENGTH 1

$\Omega \leq \mathrm{Aut}(\mathbb{G})$ OLIGOMORPHIC.

$\hat{\mathbb{G}} :=$ 2-CONSERVATIVE EXPANSION OF $\mathbb{G}$ WRT. Ω

STRUCTURAL DICHOTOMY:

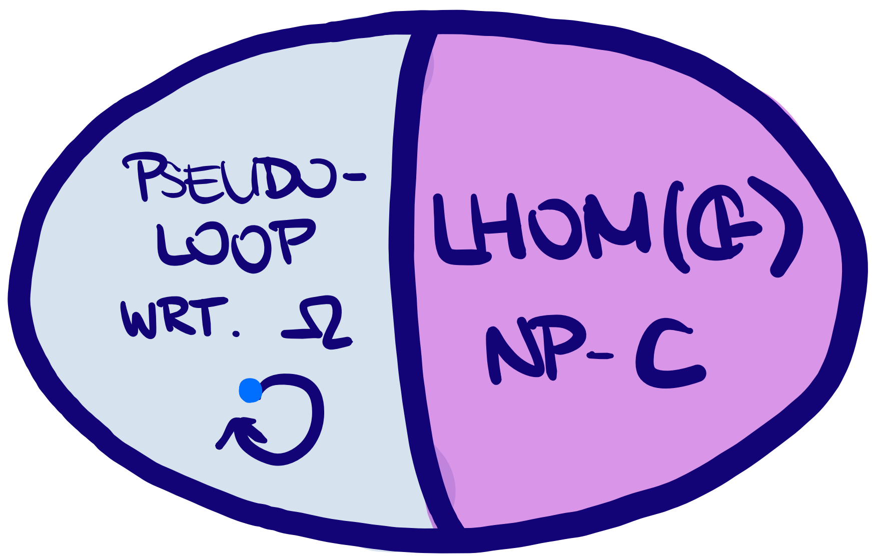


OUR RESULT: THE FIRST HARDNESS CRITERION FOR INFINITE DIGRAPHS

Cor (KOZIK + NAGY + PINSKER + B. '25)

$\mathbb{G}$ SMOOTH DIGRAPH OF ALGEBRAIC LENGTH 1
$\Omega \leq \mathrm{Aut}(\mathbb{G})$ OLIGOMORPHIC.

HARDNESS CRITERION:



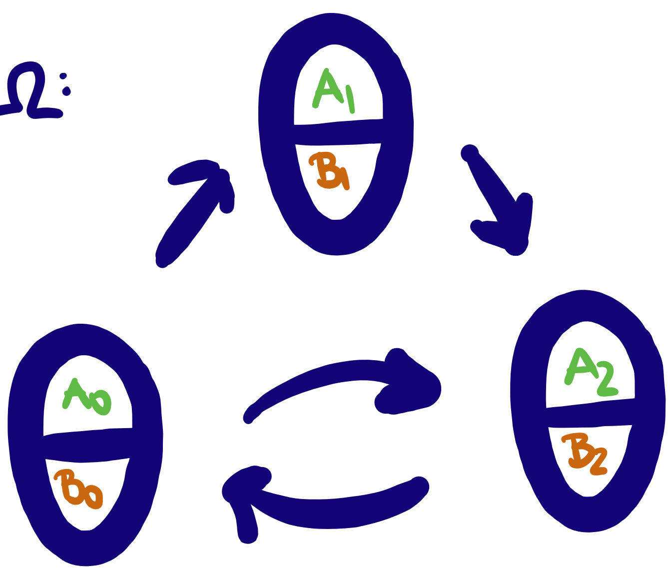
FINITISING $\mathbb{G}$ WRT. Ω

$\mathbb{G} = (G; \rightarrow)$, $\Omega \leq \mathrm{AUT}(\mathbb{G})$

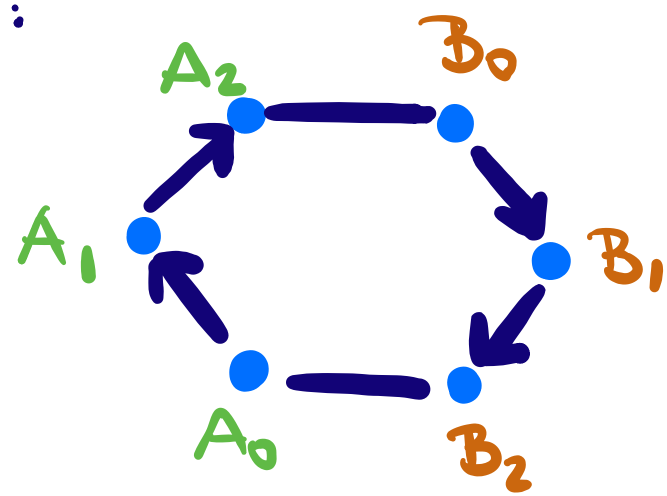
"APPLY ESTABLISHED TECHNIQUES TO FACTOR mod α"

- α IS REFINEMENT OF 1-Ω-ORBIT-EQUIVALENCE
- $\mathbb{G}/\alpha$ FINITE
- α IS Ω-INVARIANT
- $\rightarrow$ IS BIJECTION BETWEEN α-BLOCKS

$\mathbb{G}/\Omega$:



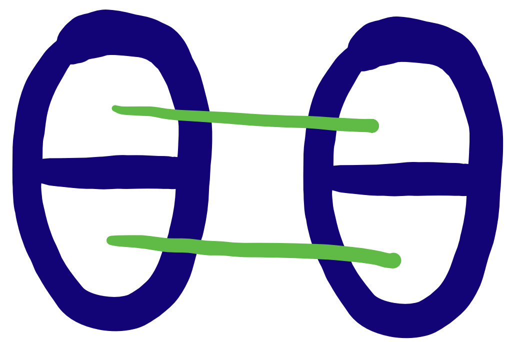
$\mathbb{G}/\alpha$:



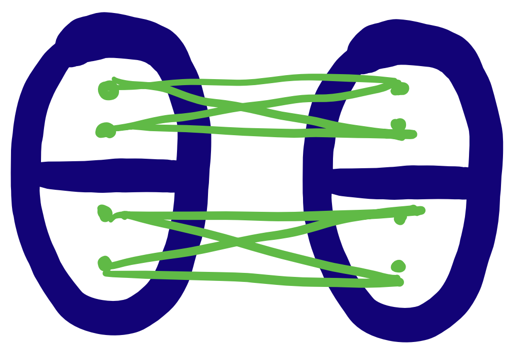
HOW TO LIFT RESULTS FROM $\mathfrak{G}/\alpha$ TO $\mathfrak{G}$?

BLOW-UP RELATIONS IN FACTOR:

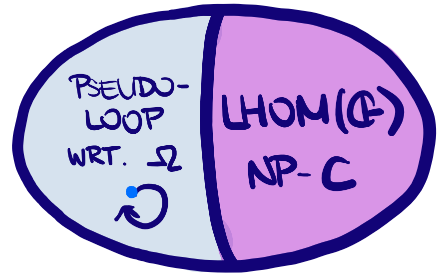
RELATION ON $\mathfrak{G}/\alpha$:



α-BLOW-UP:



THE MASTER PROOF



SPS: $\mathbb{G}$ DOES NOT HAVE PSEUDOLOOP WRT. Ω

SOURCE OF HARDNESS:

$OR(\sigma,\sigma) = \sigma(x,y) \vee \sigma(z,w)$ FOR σ EQUIV. ON A FINITE SET.

IDEA:

- DEFINE α-BLOW-UP OF $OR(\sigma,\sigma)$ ON SUBSET OF $\mathbb{G}/\alpha$
- OTHERWISE, REDUCE TO SUITABLE SUBGRAPH $\mathbb{H}$

THE MASTER PROOF

INDUCTION ON NUMBER OF α-BLOCKS

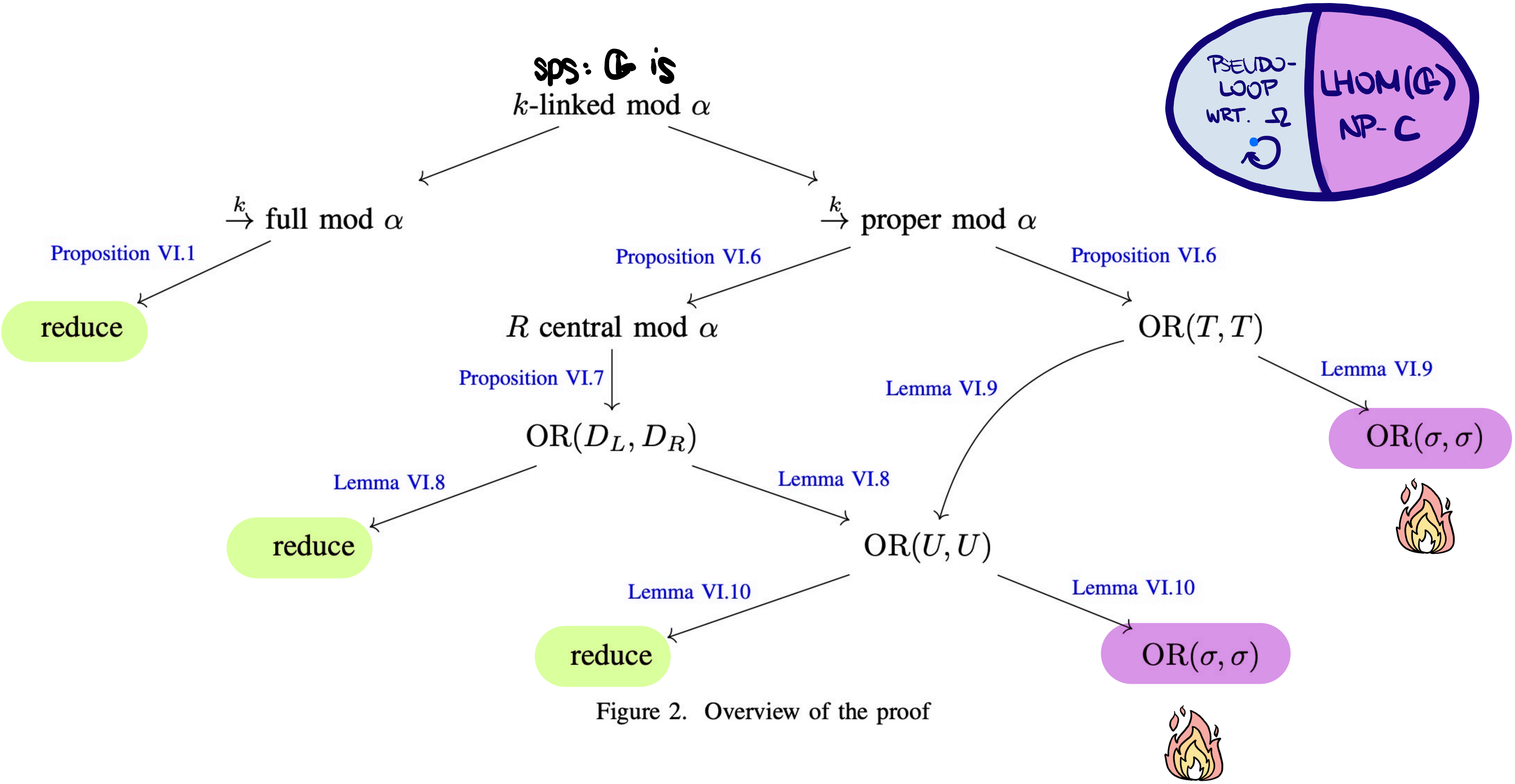
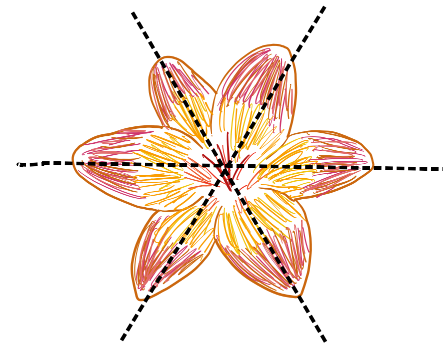


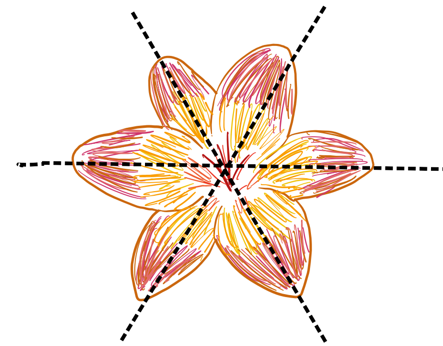
Figure 2. Overview of the proof

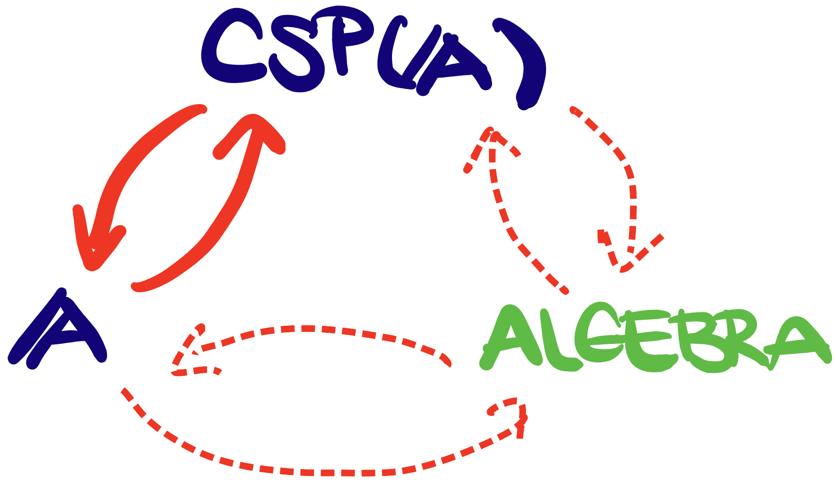
ALGEBRAIC CONSEQUENCES



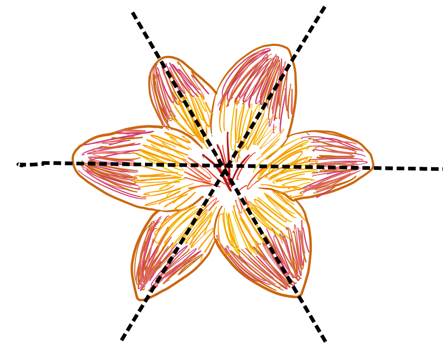
$$\begin{tikzcd} & \mathrm{CSP}(\mathbb{A}) \arrow[dl, bend right] \\ \mathbb{A} \arrow[ur, bend right] & \end{tikzcd}$$

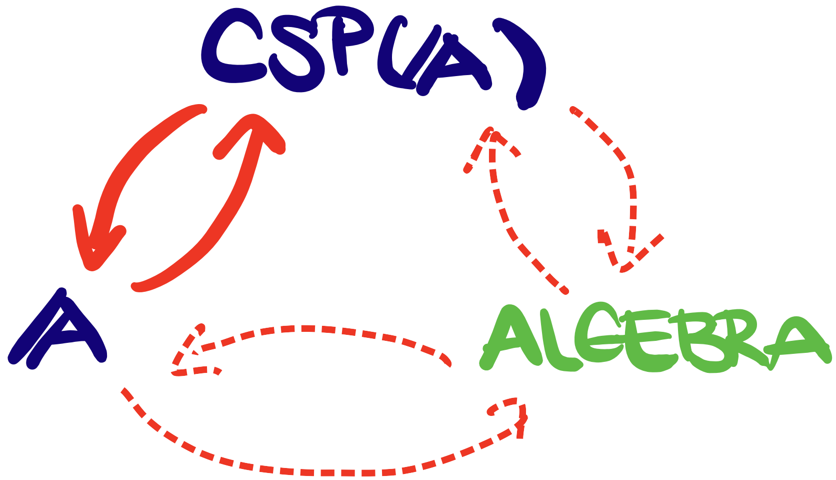
ALGEBRAIC CONSEQUENCES





ALGEBRAIC CONSEQUENCES

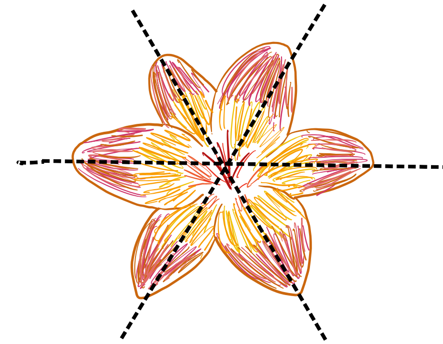


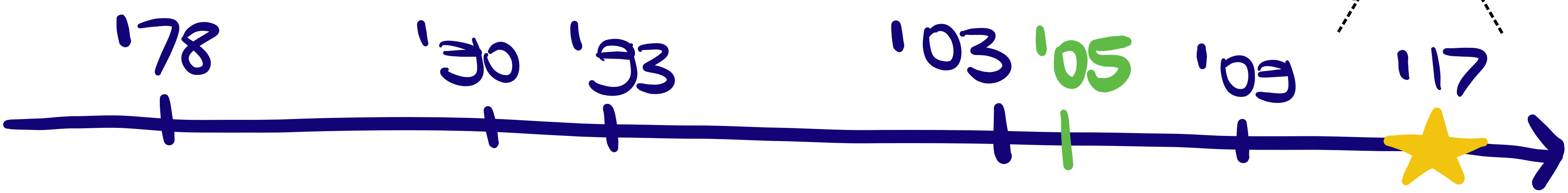


$$\boxed{\mathrm{Pol}(\mathbb{A}) = \bigcup_{n\geq 1} \mathrm{Hom}(\mathbb{A}^n, \mathbb{A})}$$

WHAT IDENTITIES ARE SATISFIED BY OPERATIONS FROM $\mathrm{Pol}(\mathbb{A})$?

ALGEBRAIC CONSEQUENCES



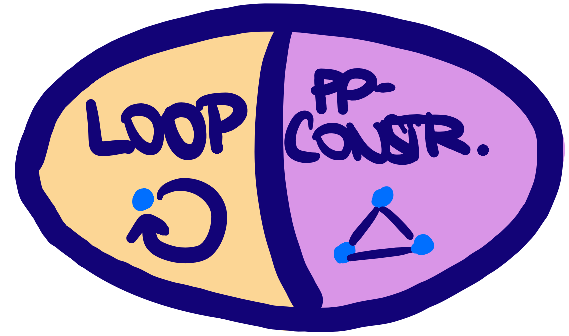


Thm (BULATOV + JEAVONS + KROKHIN '05)

COMPLEXITY OF CSP(A) IS COMPLETELY DETERMINED BY THE EQUATIONAL VARIETY GENERATED BY POL(A)

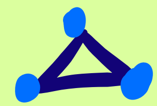
⇒ ALGEBRAIC APPROACH TO CSP

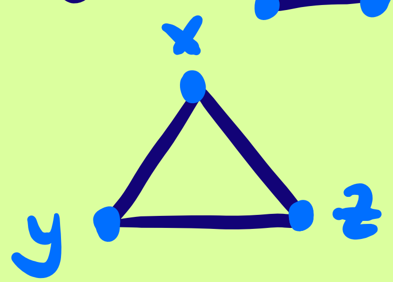
LOOP LEMMATA



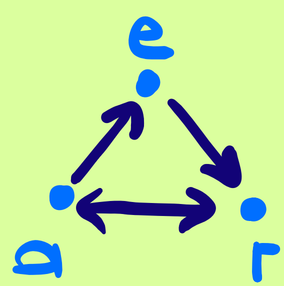
IDENTITIES SATISFIED BY OPERATIONS IN $\mathrm{Pol}(\mathbb{A})$ CAN BE DERIVED FROM LOOPS IN DIGRAPHS:

Thm: $\mathbb{A}$ FINITE, ORBITS PP-DEFINABLE. TFAE

- $\mathbb{A}$ DOES NOT PP-CONSTRUCT 
- $\mathrm{Pol}(\mathbb{A}) \models s(x,y,x,z,y,z) \approx s(y,x,z,x,z,y)$ SIGGERS (6-ARY)



- $\mathrm{Pol}(\mathbb{A}) \models s(a,r,e,a) \approx s(r,a,r,e)$ SIGGERS (4-ARY)



- $\mathrm{Pol}(\mathbb{A}) \models w(y,x\dots x) \approx w(x,y,x\dots x) \dots w(x\dots x,y)$ WNU

PSEUDOLOOP-LEMMATA

PSEUDOLOOPS GIVE PSEUDO-IDENTITIES!

6-ARY PSEUDO-SIGGERS (BARTO+PINSKER '16)

$$u \circ s(x,y,x,z,y,z) \approx v \circ s(y,x,z,x,z,y)$$

Cor (KOZIK+NAGY+PINSKER+B. '25)

Ω OLIGOMORPHIC, $\hat{\mathbb{A}}$ EXPANSION OF $\mathbb{A}$ BY ALL $P \cup Q$ FOR P, Q k-ORBITS. EITHER

- $\hat{\mathbb{A}}$ PP-CONSTRUCTS $\triangle$ OR
- $\mathrm{Pol}(\hat{\mathbb{A}}) \models u \circ s(a,r,e,a) \approx v \circ s(r,a,r,e)$

A graph set forth, one day, to find
A finite thing of any kind
Directed 'twas, yet it was smooth
And at length one still in its youth
With pairs of orbits at his hand
It strived to build things on its land
But when it failed to build K3
A pseudoloop it had to see.

MP