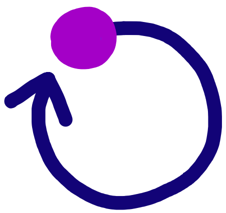
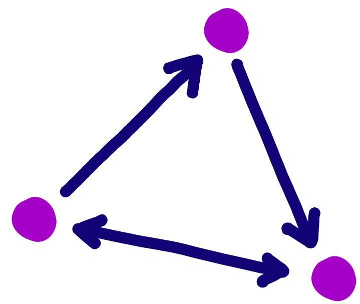
The sorrows of a smooth digraph: from finite to infinite



Johanna Brunar

TU Wien

joint work with M. Kozik, T. Nagy, M. Pinsker



Dagstuhl, May 2025

ON THE ROAD TO THE FINITE-DOMAIN CSP-DICHOTOMY

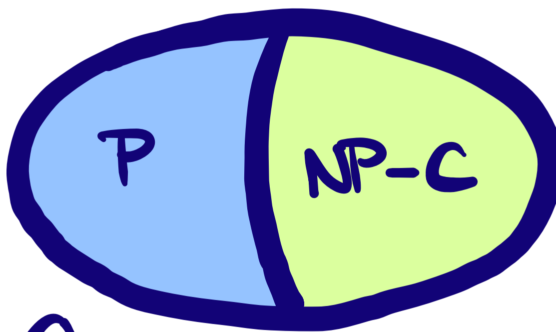
'78 — SCHAEFER — BOOLEAN

'93 — FEDER+VARDI — CONJECTURE

'17 — BULATOV, ZHUK — PROOF

THM (BULATOV, ZHUK '17)

$\mathbb{A}$ FINITE. CSP($\mathbb{A}$) IS SOLVABLE IN P-TIME OR NP-C.



ON THE ROAD TO THE FINITE-DOMAIN CSP-DICHOTOMY

'78 — SCHAEFER: BOOLEAN

'93 — FEDER+VARDI: CONJ.

'05 — BULATOV+JEAVONS + KROKHIN

'17 — BULATOV+ZHUK: PROOF

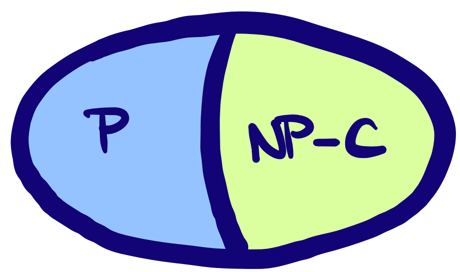
- TCT
- ABSORPTION

← ALGEBRAIC APPROACH → $\mathrm{Pol}(\mathbb{A})$

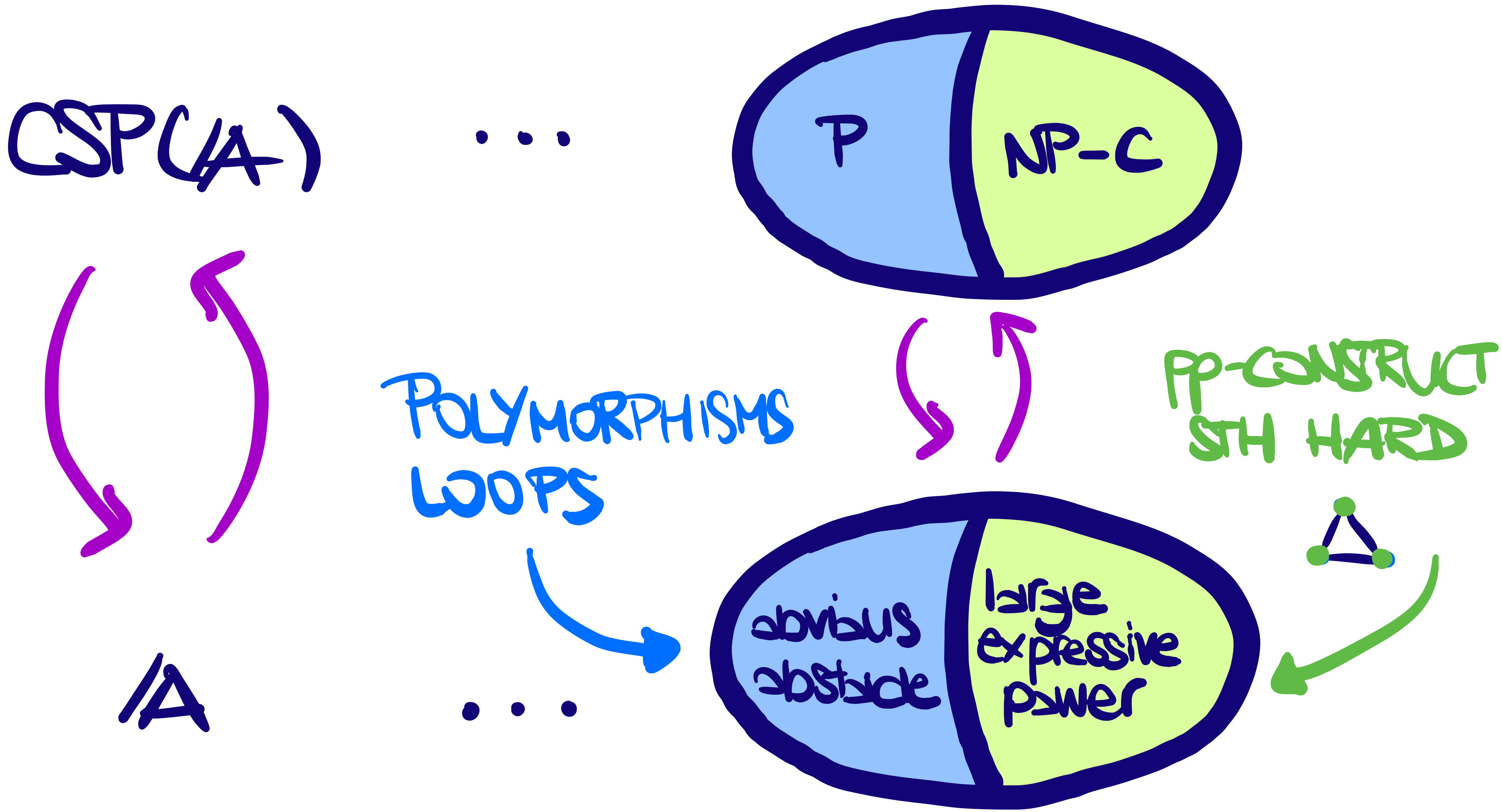
<u>THM</u> (KEARNES+MARKOVIC+MCKENZIE '15, BULATOV, ZHUK '17)

$\mathbb{A}$ FINITE. IF $\mathrm{Pol}(\mathbb{A}) \models$ SIGGERS, THEN $\mathrm{CSP}(\mathbb{A})$ IN P, NP-C O/W.

$s(a,r,e,a) \approx s(r,a,r,e)$



ON THE ROAD TO THE FINITE-DOMAIN CSP-DICHOTOMY



LIBOR'S DREAM

$\boxed{\text{Dream Theorem}}$ For $\underbrace{P, P'}_{\text{computational problems}}$ $\quad \underbrace{*}_{\text{nice condition}} \iff \underbrace{P' \leq P}_{\text{poly-time reduction}}$

$\boxed{\text{Real theorems}}$ For $P, P' \in$ Class $\quad * \Rightarrow P' \leq P$

We have
- such theorem for Class = fixed-template finite-domain CSPs
- $*$ weak enough to "explain" NP-hardness:

  $P \in$ Class NP-hard, $P' \in$ Class $\Rightarrow$ $*$ is satisfied

We want
- make Class bigger $\longrightarrow$ ~~this talk~~
- make $*$ weaker $\longrightarrow$ THIS TALK!

FROM FINITE TO INFINITE

<u>Conj:</u> (BODIRSKY + PINSKER '12)

A REDUCT OF COUNTABLE FINITELY BOUNDED HOMOGENEOUS STRUCTURE. THEN CSP(A) IS SOLVABLE IN P-TIME OR NP-COMPLETE.

ALGEBRAIC METHODS DO NOT LIFT WELL :(

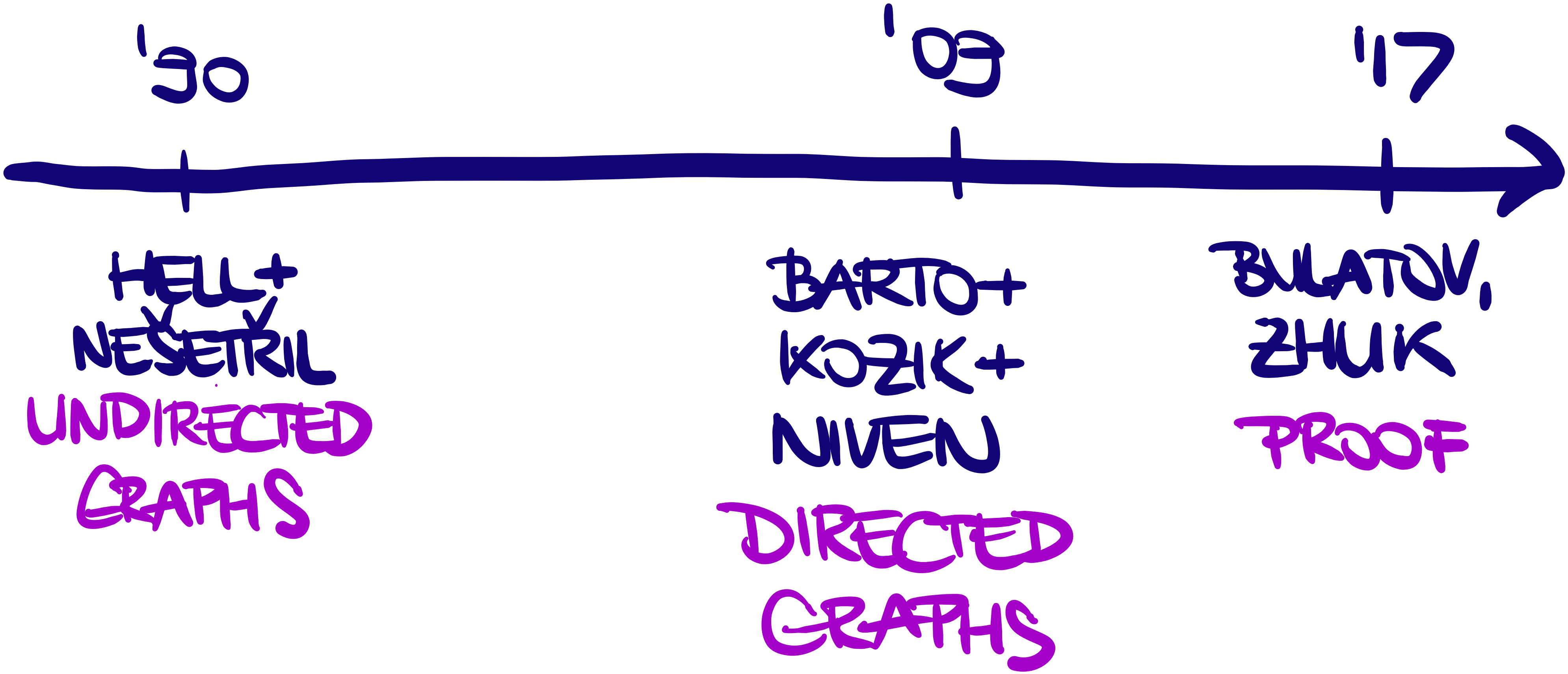
FROM FINITE TO INFINITE

PROGRAMME:

(i) NEW COMBINATORIAL PROOFS OF KNOWN FINITE STRUCTURAL DICHOTOMIES

(ii) LIFT TO INFINITY

ON THE ROAD TO THE FINITE-DOMAIN CSP-DICHOTOMY

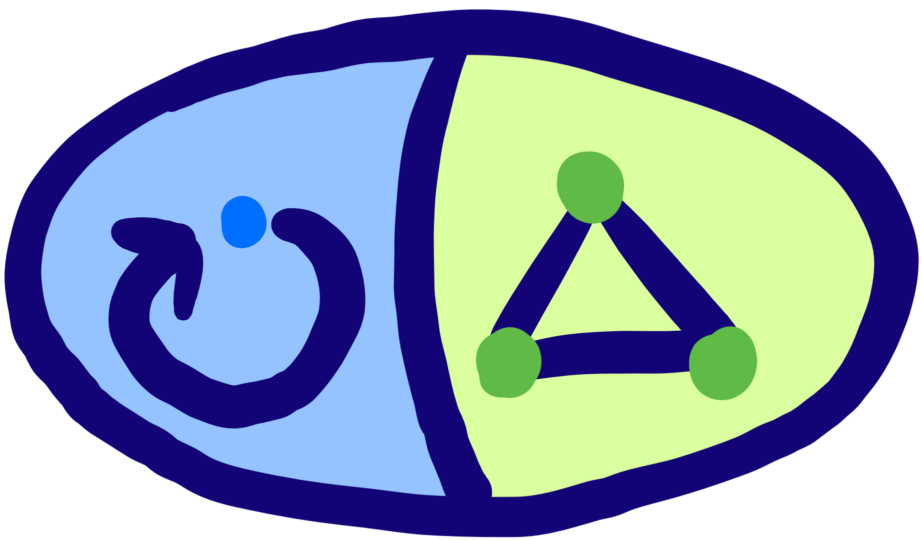


ON THE ROAD TO THE FINITE-DOMAIN CSP-DICHOTOMY

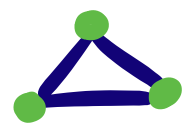
Thm: (HELL+NEŠETŘIL '90)
€ FINITE UNDIRECTED NON-BIPARTITE.

Thm: (BARTO+KOZIK+NIVEN '09)
€ FINITE DIRECTED SMOOTH ALGEBRAIC LENGTH 1.

€ HAS LOOP



€ PP-CONSTRUCTS

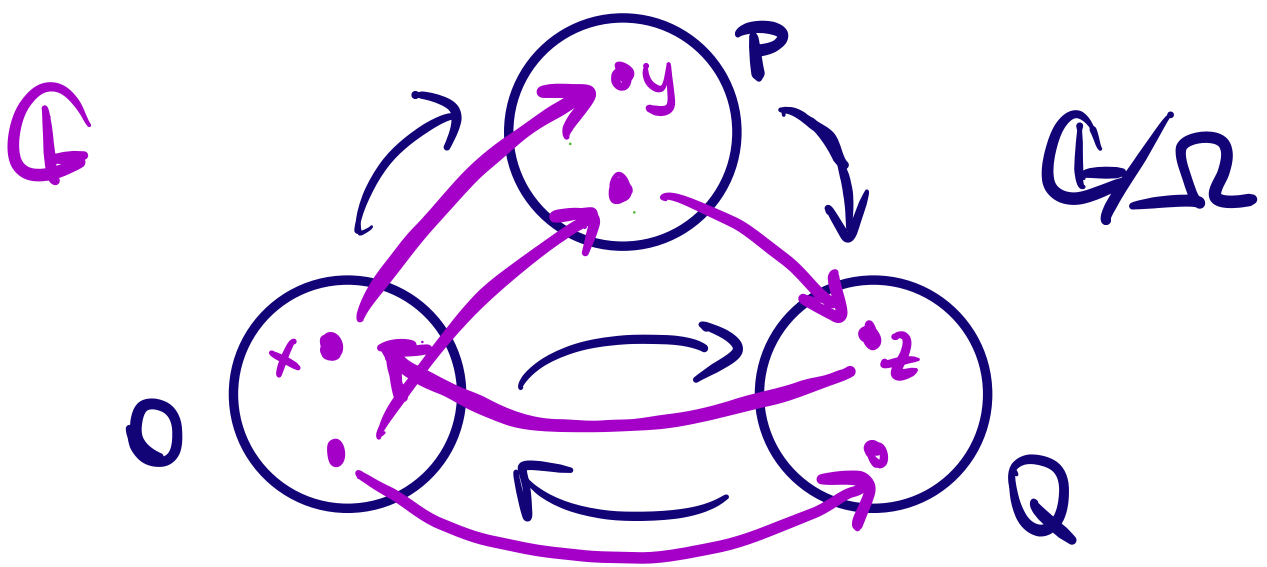


FROM FINITE TO INFINITE

$\Omega \leq \mathrm{Aut}(\mathbb{G})$ OLIGOMORPHIC

$\boxed{\mathbb{G}/\Omega}$

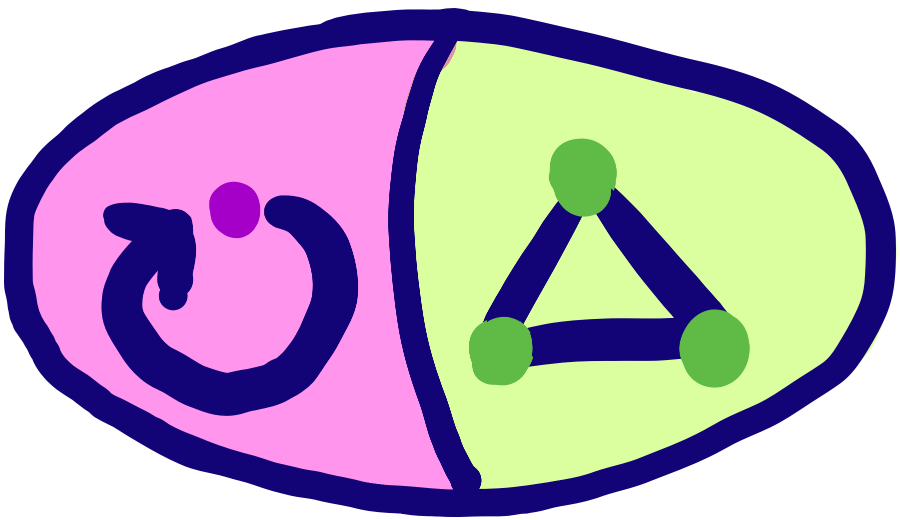
- VERTICES: 1-ORBITS OF $\Omega \curvearrowright \mathbb{G}$
- EDGES: $O \to P$ IFF $\exists x \in O, y \in P$: $x \to y$ in $\mathbb{G}$



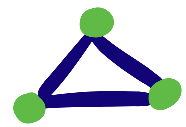
INFINITE HELL-NEŠETŘIL

Thm (BARTO + BODOR + KOZIK + MOTTET + PINSKER '23)
$\mathbb{G}$ SMOOTH, $\Omega \leq \mathrm{Aut}(\mathbb{G})$ OLIGOMORPHIC,
$\mathbb{G}/\Omega$ SYMMETRIC, NON-BIPARTITE.

$\mathbb{G}/\Omega$ HAS LOOP
"PSEUDO-LOOP"



$\mathbb{G}$ + ORBITS
PP-CONSTRUCT



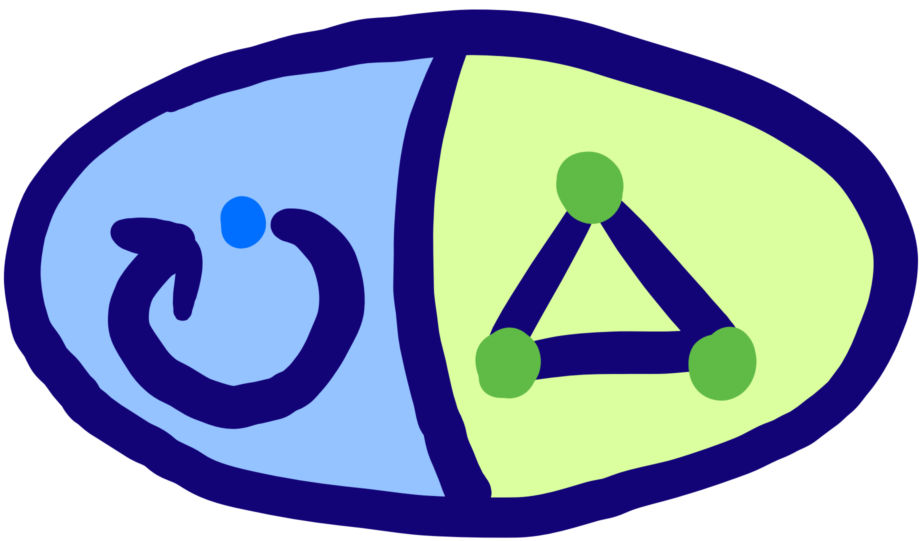
OUR DREAM

PURELY COMBINATORIAL PROOF BY BARTO + BODOR + KOZIK + MOTTET + PINSKER '23

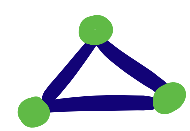
<u>Thm</u> (BARTO + KOZIK + NIVEN '09)

$\mathbb{G}$ FINITE DIGRAPH, SMOOTH, ALGEBRAIC LENGTH 1.

$\mathbb{G}$ HAS LOOP



$\mathbb{G}$ PP-CONSTRUCTS



CAN WE LIFT THIS TO THE INFINITE?

CONSERVATIVE DIGRAPH-COLOURING (LIST HOMOMORPHISM-PROBLEM)

FOR EVERY VERTEX, ADD LIST OF ALLOWED COLOURS:

LHOM($\mathbb{G}$):

- INPUT: FINITE DIGRAPH $\mathbb{H}$, $L: H \to \mathcal{P}(G)$
- QUESTION: DOES THERE EXIST A HOMOMORPHISM $h: \mathbb{H} \to \mathbb{G}$ WITH $h(x) \in L(x)$ $\forall x \in H$?

CONSERVATIVE DIGRAPH-COLOURING (LIST HOMOMORPHISM-PROBLEM)

INFINITE DIGRAPHS WITH $\Omega \leq \mathrm{Aut}(\mathbb{G})$

OLIGOMORPHIC: 1-ORBITS TAKE ROLE OF COLOURS!

$\mathrm{LHOM}(\mathbb{G}, \Omega)$:

- INPUT: FINITE DIGRAPH $\mathbb{H}$ $\quad L : H \to \mathcal{P}(\text{1-ORBITS})$
- QUESTION: DOES THERE EXIST A HOMOMORPHISM $h : \mathbb{H} \to \mathbb{G}$ WITH $h(x) \in L(x)$ $\forall x \in H$?

OUR RESULT: THE FIRST HARDNESS CRITERION FOR INFINITE DIGRAPHS

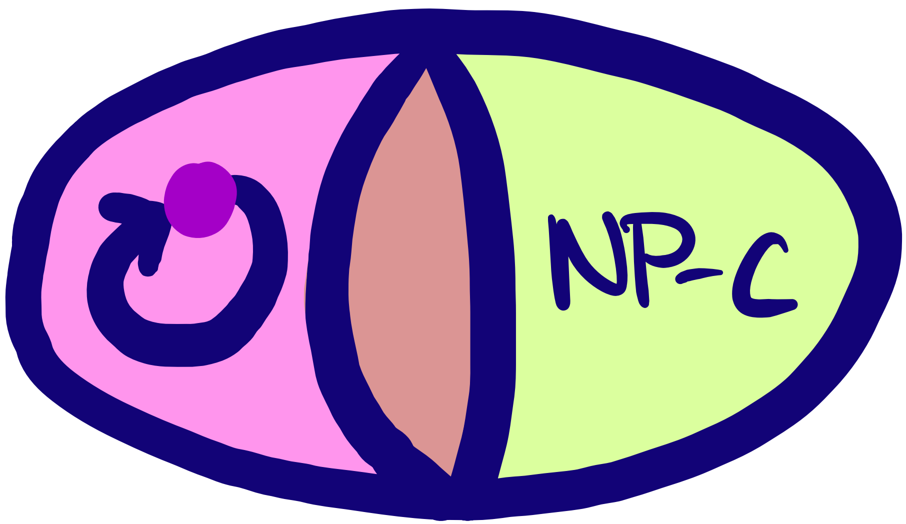
<u>Thm</u> (B.+ KOZIK+ NAGY+ PINSKER '25)

$\mathbb{G}$ SMOOTH DIGRAPH OF ALGEBRAIC LENGTH 1

$\Omega \leq \mathrm{Aut}(\mathbb{G})$ OLIGOMORPHIC.

$\mathbb{G}/\Omega$ HAS LOOP

"PSEUDO-LOOP"



LHOM$(\mathbb{G},\Omega)$ NP-C

OUR RESULT: THE FIRST HARDNESS CRITERION FOR INFINITE DIGRAPHS

Thm (B.+ KOZIK+ NAGY+ PINSKER '25)

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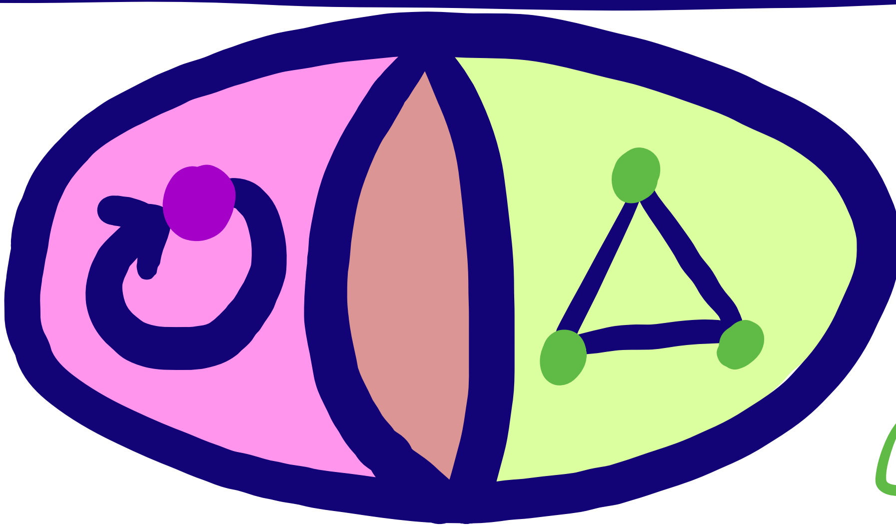
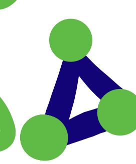


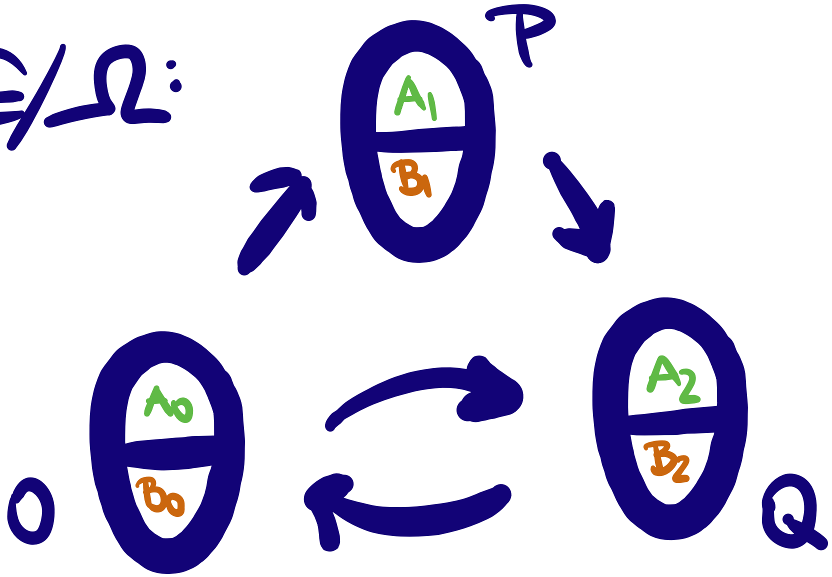
CONSERVATIVE EXPANSION OF $\mathbb{G}$ PP-CONSTRUCTS



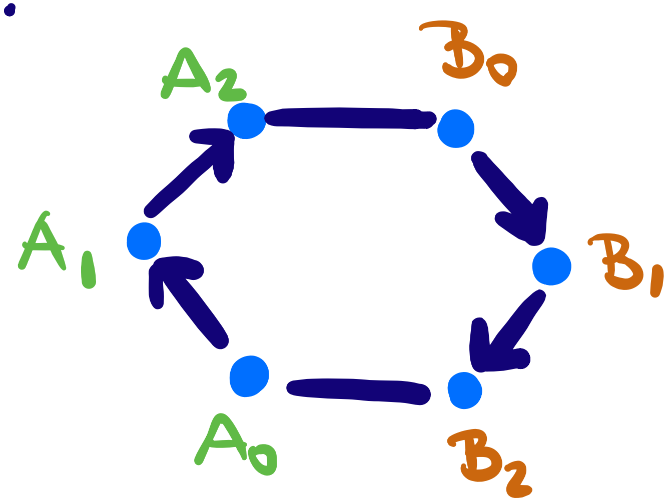
FINITISING $\mathfrak{C}$ WRT. Ω

REFINE FACTOR:

$\mathfrak{C}/\Omega$:



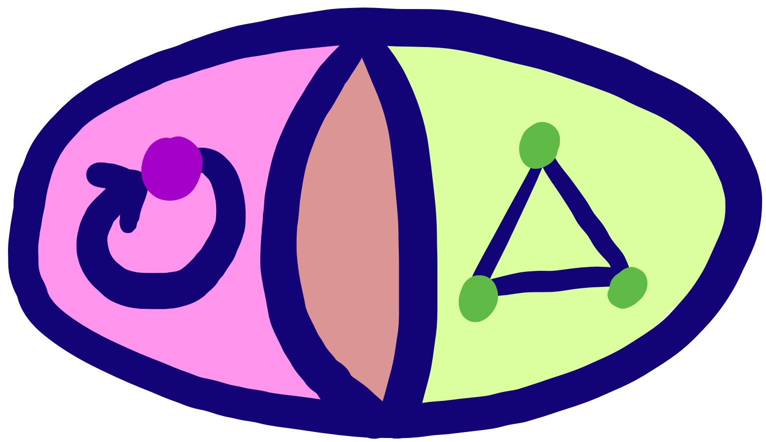
$\mathfrak{C}/\alpha$:

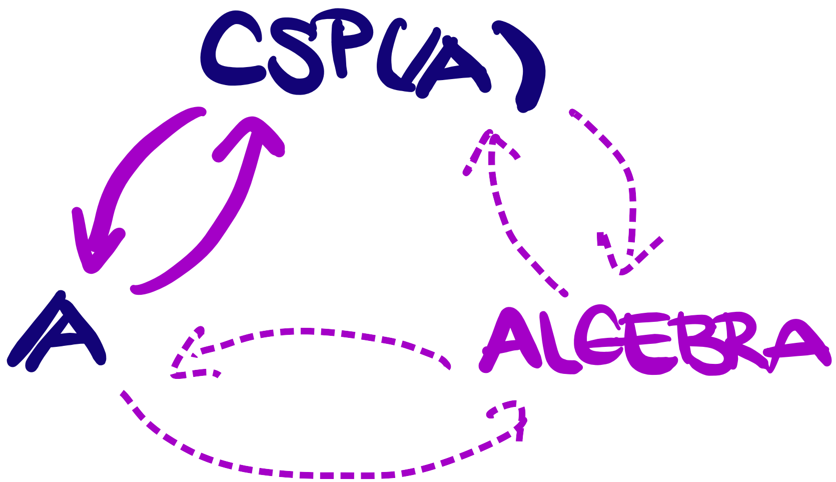


p .. path pattern $(\rightarrow \rightarrow \leftarrow)$

$\alpha_p(x,y)$ iff $\exists z$: $x \overset{p}{\rightsquigarrow} z \overset{p}{\leftsquigarrow} y$

PSEUDO-LOOPS?

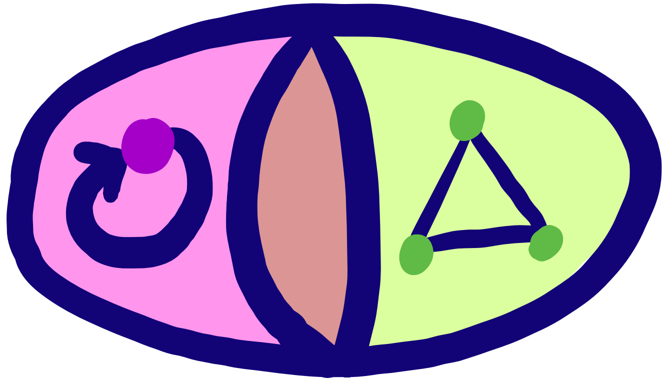




LOOPS $\rightsquigarrow$ IDENTITIES

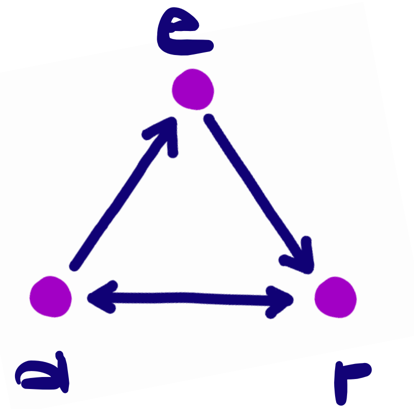
PSEUDO-LOOPS $\rightsquigarrow$ PSEUDO-IDENTITIES

PSEUDO-SIGGERS?



QUESTION $\Omega \leq \mathrm{Aut}(\mathbb{A})$ OLIGOMORPHIC.

- $\mathbb{A}$ pp-CONSTRUCTS K_3 OR
- $\mathrm{Pol}(\mathbb{A}) \models \alpha \circ s(\trianglelefteq, r, e, \trianglelefteq) \approx \mu \circ s(r, \trianglelefteq, r, e)$



WOULD BE IMPLIED BY FULL LIFTING OF BARTO-KOZIK-NIVEN!