Temporal Valued Constraint Satisfaction Problems

Manuel Bodirsky

Institut für Algebra, TU Dresden
Joint work with Édouard Bonnet and Žaneta Semanišinová

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Overview

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4. Complexity Classification Result

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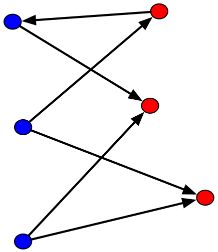

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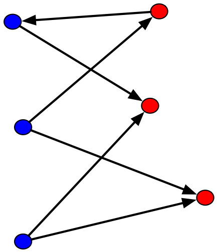

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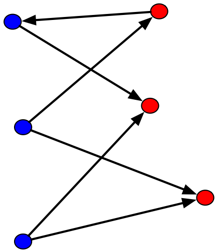

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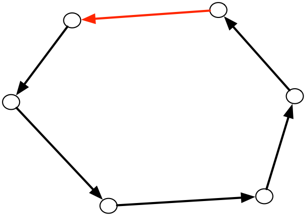

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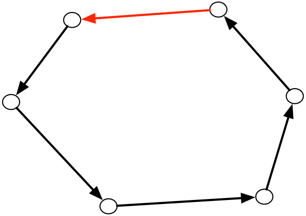

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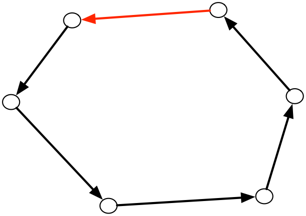

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For which Γ can VCSP(Γ) be solved in polynomial time?

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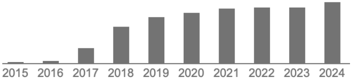

Articles Google Scholar
A proof of the CSP dichotomy conjecture
D Zhuk - Journal of the ACM (JACM), 2020
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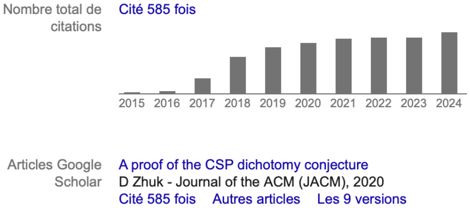

Does not capture Minimum Feedback Arc Set!

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Theorem (Bodirsky+Kara'2010): Every temporal CSP is in P or NP-complete.

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Theorem (Cameron'1976):
A structure $\Gamma = (\mathbb{Q}; R_1, R_2, \dots)$ is isomorphic to a temporal structure
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Theorem (B.+Bonnet+Semanišinová MFCS'25):
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Examples: $\text{Aut}(\mathbb{Q};<)$, $\text{Aut}(\Gamma)$ for temporal valued structures Γ, $\text{Aut}(\text{Rado})$, ...

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- True or false?
  A relation is preserved by the fractional polymorphisms of Γ
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