Containment for Guarded Monotone Strict NP

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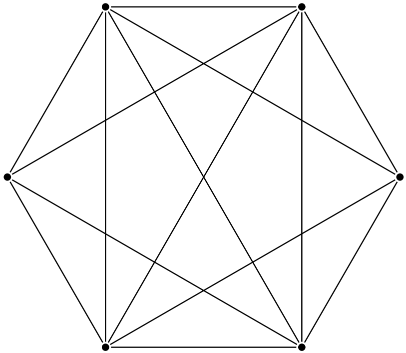
What is Guarded Monotone SNP?

A classic example

No Monochromatic Triangle

Given: a graph (V, E).
Task: to partition E in two classes E_1, E_2 such that neither (V, E_1) nor (V, E_2) contains a triangle.



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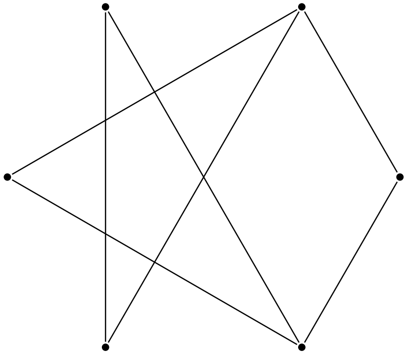

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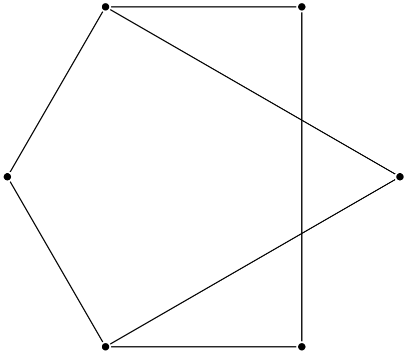
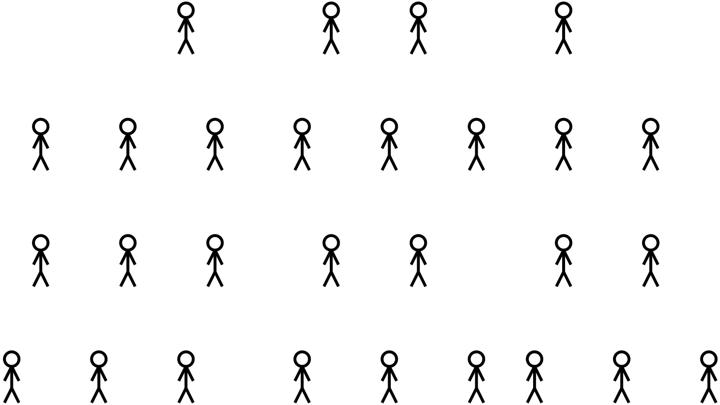


(V, E_2)

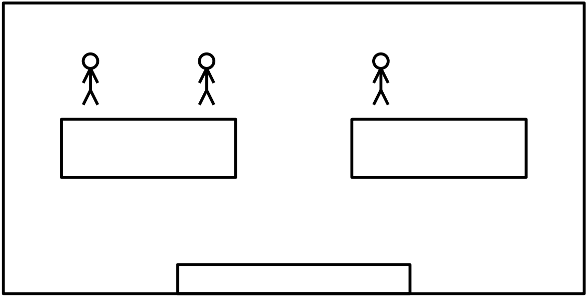
A “real life” example



School students

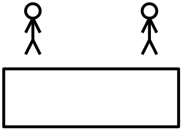
A "real life" example

The classroom

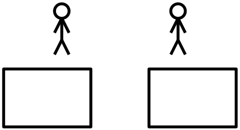


Two desks, three people

A “real life” example



$\Longrightarrow$ always together



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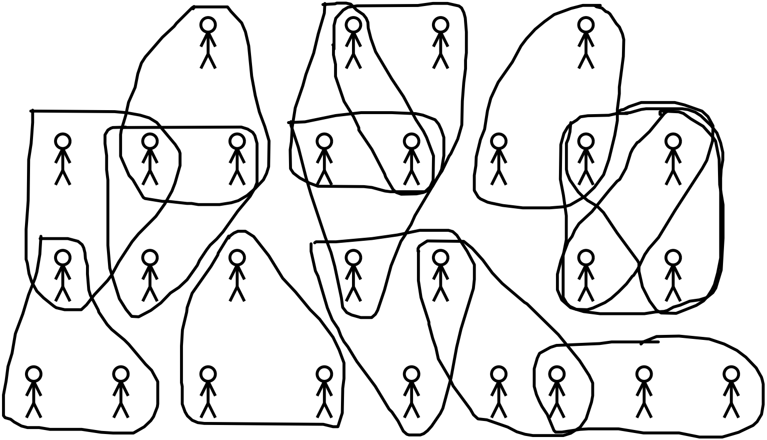

Choose who sits together

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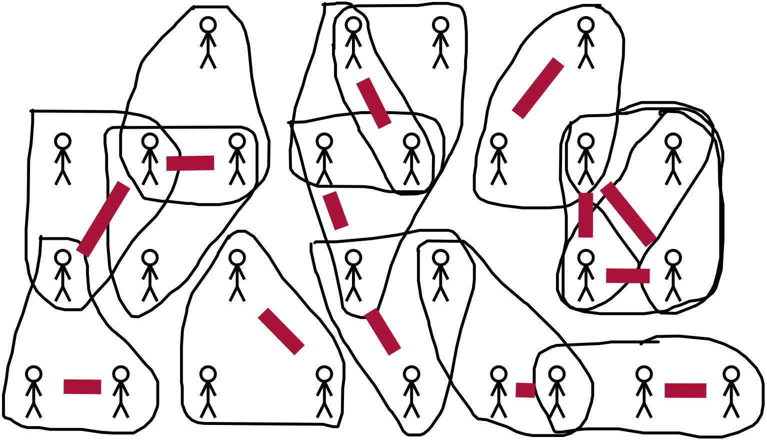

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A formal definition of GMSNP

Given: finite relational structure $\mathbb{A}$ (e.g., graph or ternary "classroom" relation)

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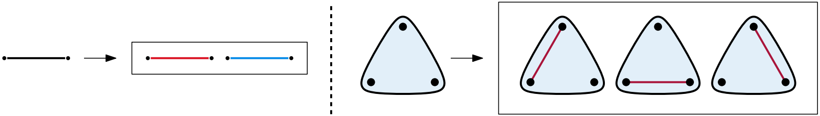

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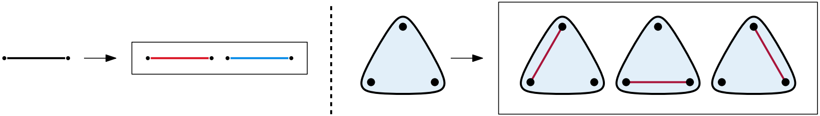
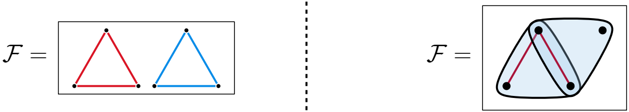
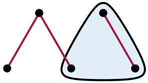


s.t. $\mathbb{A}^{\text{col}}$ is $\mathcal{F}$-free, i.e., for NO $\mathbb{F}$ from finite family $\mathcal{F}$, there is a homomorphism $\mathbb{F} \to \mathbb{A}^{\text{col}}$.



Why "guarded" and why "monotone"?

- *Guarded* – in every $\mathbb{F} \in \mathcal{F}$, "colors" are defined *within* original relational tuples



$\notin \mathcal{F}$

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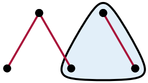
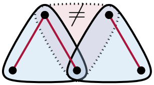


- *Monotone* – $\mathbb{A}^{\mathsf{col}}$ must be $\mathcal{F}$-free *homomorphism*-wise (not embedding, full homomorphism, etc.)



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- **B., Pinsker, Rydval (2025):** containment for GMSNP is decidable

CSP and Dichotomy

Constraint Satisfaction Problems

Let $\mathbb{A} = (\mathsf{A}; \mathsf{R}_1^{\mathbb{A}}, \dots, \mathsf{R}_s^{\mathbb{A}})$ be a relational structure with domain A and signature $\{\mathsf{R}_1, \dots, \mathsf{R}_s\}$ ($\mathsf{R}_i^{\mathbb{A}} \subseteq \mathsf{A}^{k_i}$)

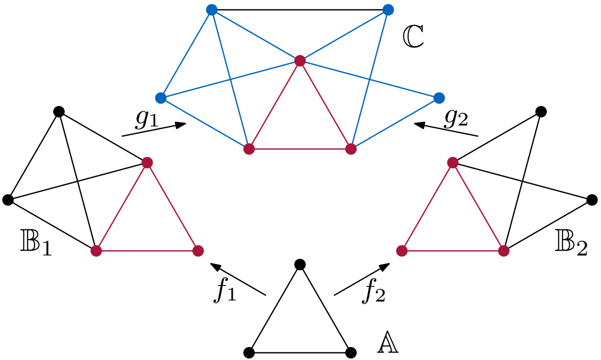
CSP($\mathbb{A}$)

Given: Finite structure $\mathbb{I} = (\mathsf{I}; \mathsf{R}_1^{\mathbb{I}}, \dots, \mathsf{R}_s^{\mathbb{I}})$
Task: Find $\mathsf{h} \colon \mathsf{I} \to \mathsf{A}$ such that $\mathsf{h}(\mathsf{R}_i^{\mathbb{I}}) \subseteq \mathsf{R}_i^{\mathbb{A}}$ for all $i \in [k]$

Example

- $\big(\{0,1\}; \{0\}, \{1\}, \{0,1\}^3 \setminus (1,1,0)\big)$ – Horn-SAT
- $\big(\{R,B\}; \{(R,B),(B,R)\}\big)$ – 2-coloring

Amalgamation Property (AP)



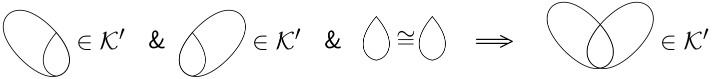
Definition

$\mathcal{K}$ has the *amalgamation* property if, for any $\mathbb{A}, \mathbb{B}_1, \mathbb{B}_2 \in \mathcal{K}$ and embeddings $f_1 \colon \mathbb{A} \to \mathbb{B}_1, f_2 \colon \mathbb{A} \to \mathbb{B}_2$, there exists $\mathbb{C} \in \mathcal{K}$ and embeddings $g_1 \colon \mathbb{B}_1 \to \mathbb{C}, g_2 \colon \mathbb{B}_2 \to \mathbb{C}$ such that $g_1 \circ f_1 = g_2 \circ f_2$.

Amalgamation for GMSNP

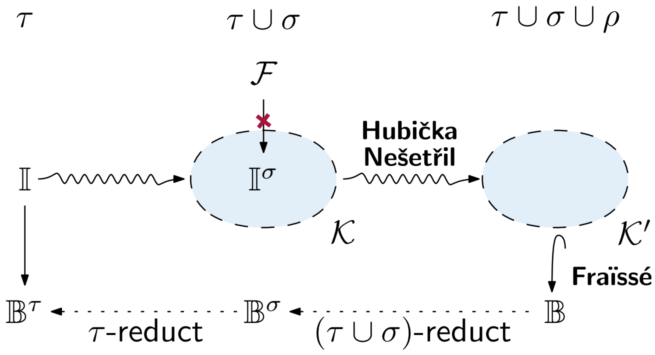
Let $\mathcal{K} :=$ all finite $\mathcal{F}$-free structures (all solutions).
Hubička, Nešetřil: there is $\mathcal{K}'$ obtained from $\mathcal{K}$ by adding finitely many new relations. $\mathcal{K}'$ is closed under taking substructures (HP) and has the amalgamation property (AP).

AP :



Fraïssé: if $\mathcal{K}'$ is closed under disjoint unions, has HP and AP, then there is homogeneous (very symmetric) countably infinite structure $\mathbb{B}$ such that $\text{Age}(\mathbb{B}) = \mathcal{K}'$.

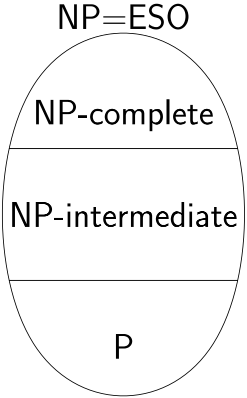
GMSNP seen as a CSP



Observation (BKS'20)

Input $\mathbb{I}$ has $\mathcal{F}$-free σ-expansion $\mathbb{I}^\sigma$ ($\mathbb{I} \in \text{GMSNP}(\mathcal{F})$) if and only if $\mathbb{I}$ homomorphically maps to $\mathbb{B}^\tau$ ($\mathbb{I} \in \text{CSP}(\mathbb{B}^\tau)$).

The dichotomy question



Ladner: If P $\neq$ NP, then NP has problems that are neither in P nor NP-complete.
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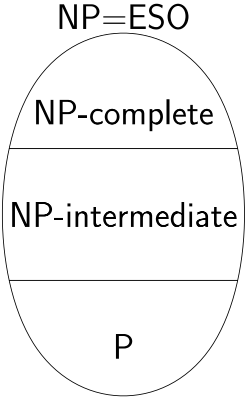

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Question

For a given logic $\mathcal{L} \subset$ ESO, is $\mathcal{L}$ a subset of (P $\cup$ NP-complete)?

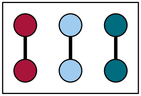
The dichotomy question

finite CSP $\subset$ MMSNP $\subset$ GMSNP $\subset$ CSPs of first-order reducts of finitely bounded homogeneous structures $\subset$ Monotone SNP

Given: finite relational structure

Task: assign to every vertex one of the several colors

such that the result is $\mathcal{F}$-free

$\mathcal{F} =$ 

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Feder, Vardi: Every problem in MMSNP is P-time equivalent to CSP with finite domain

Zhuk, Bulatov: Finite CSPs have dichotomy that is characterized by algebraic properties of the template.

The dichotomy question

finite CSP $\subset$ MMSNP $\subset$ GMSNP $\subset$ CSPs of first-order reducts of finitely bounded homogeneous structures $\subset$ Monotone SNP

$\mathbb{A}$ is **homogeneous** if every isomorphism between its finite substructures extends to an automorphism of $\mathbb{A}$.

$\mathbb{A}$ is **finitely bounded** if for some finite family $\mathcal{F}$

$\forall\, \mathbb{B}$ finite $(\mathbb{B} \subset \mathbb{A} \Leftrightarrow \forall\, \mathbb{F} \in \mathcal{F} \;\; \mathbb{F} \not\to \mathbb{B})$ (Age($\mathbb{A}$) is $\mathcal{F}$-free)

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$\mathbb{B}$ is a **first-order reduct** of $\mathbb{A}$ if $\mathbb{B}$ has the same domain as $\mathbb{A}$ and if every relation of $\mathbb{B}$ is first-order definable in $\mathbb{A}$.

Conjecture (Bodirsky, Pinsker): CSPs of such structures have dichotomy characterized by algebraic properties of the template.

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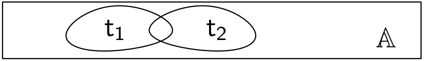
Given: finite relational structure $\mathbb{A}$

Task: assign a color to each k-element subset of $\mathbb{A}$ (k is fixed)

s.t. the colors assigned to intersecting subsets are compatible



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Feder, Vardi: Every problem in NP is P-time equivalent to a problem in Monotone SNP

Containment for GMSNP

The containment question

Given: two decision problems Φ and Ψ
Task: to check whether every YES instance of Φ is a YES instance of Ψ, denoted $\Phi \subseteq \Psi$

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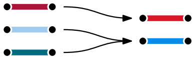
Remark

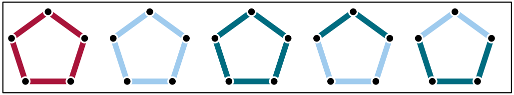
Undecidable: Datalog **(Shmueli)**, FO **(Trakhtenbrot)**
Decidable: finite-domain CSP and MMSNP **(Feder, Vardi)**

Recoloring for GMSNP

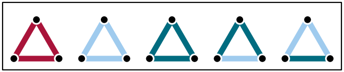
r: {colors of Φ} $\to$ {colors of Ψ} is a **recoloring** from Φ to Ψ



if the preimage $r^{-1}(\mathcal{F}_\Psi)$ has no $\mathcal{F}_\Phi$-free structures



$\mathcal{F}_\Phi$



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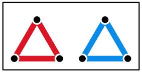
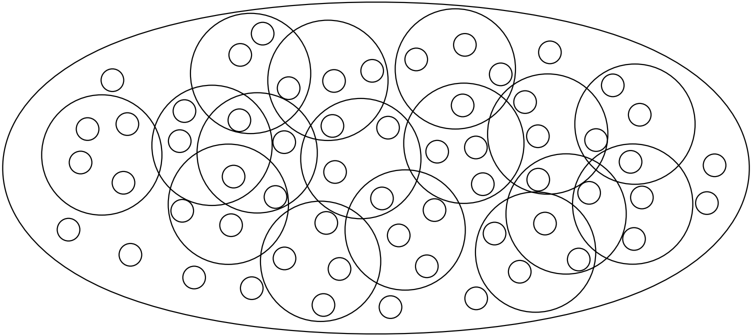


$\mathcal{F}_\Psi$

recoloring $\Rightarrow$ containment

Ramsey property

Class $\mathcal{K}$ is **Ramsey** if for all $\mathbb{A}, \mathbb{B} \in \mathcal{K}$ and all $n \in \mathbb{N}$ there is $\mathbb{C} \in \mathcal{K}$ s.t. for all $\chi\colon \binom{\mathbb{C}}{\mathbb{A}} \to [n]$ there is $\mathbb{B}_0 \in \binom{\mathbb{C}}{\mathbb{B}}$ s.t. χ is constant on $\binom{\mathbb{B}_0}{\mathbb{A}}$



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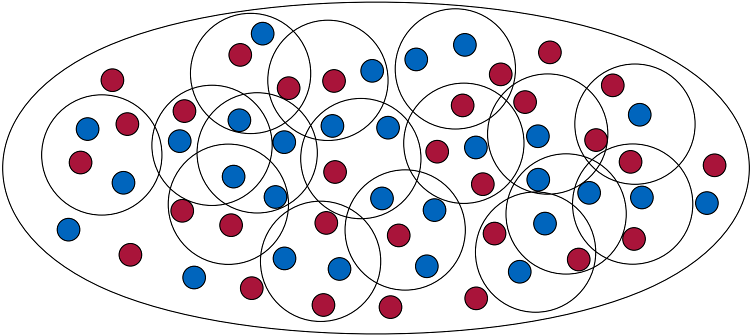

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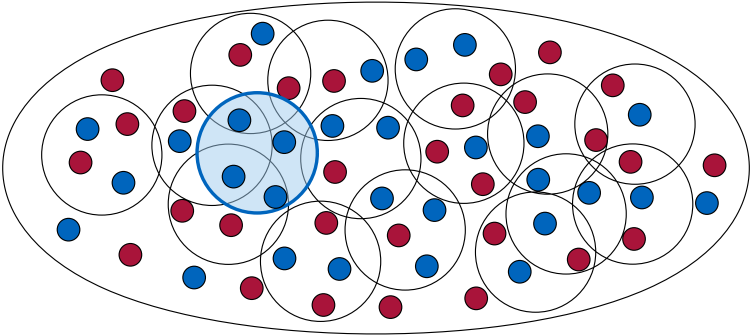
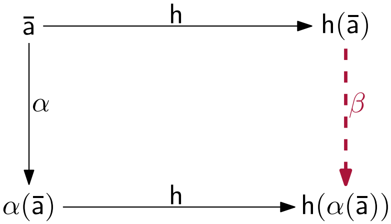


Canonical mappings

$h\colon \mathbb{A} \to \mathbb{B}$ is **canonical** w.r.t. $\mathrm{Aut}(\mathbb{A})$ and $\mathrm{Aut}(\mathbb{B})$

if for every n and every $\bar{a} \in A^n$ and every $\alpha \in \mathrm{Aut}(\mathbb{A})$

there is $\beta \in \mathrm{Aut}(\mathbb{B})$ s.t.



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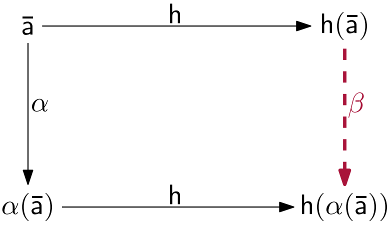

h sends n-colors of $\mathbb{A}$ to n-colors of $\mathbb{B}$!

Containment $\Rightarrow$ recoloring

$$\Phi \subseteq \Psi \quad \Longrightarrow \quad \mathrm{CSP}(\mathbb{B}_{\Phi}^{\tau}) \subseteq \mathrm{CSP}(\mathbb{B}_{\Psi}^{\tau}) \quad \Longrightarrow \quad \exists h\colon \mathbb{B}_{\Phi}^{\tau} \to \mathbb{B}_{\Psi}^{\tau}$$

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Canonical mapping h well-defines a recoloring

$$\mathrm{r}\colon \{\text{colors of } \Phi'\} \to \{\text{colors of } \Psi'\}$$

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$\mathbb{B}^\tau_\Phi$ has a homogeneous Ramsey expansion $\mathbb{B}_\Phi$ $\Longrightarrow$ h can be made canonical w.r.t. $\mathrm{Aut}(\mathbb{B}_\Phi)$ and $\mathrm{Aut}(\mathbb{B}_\Psi)$

Hubička, Nešetřil: $\mathbb{B}^\tau_\Phi$ has such an expansion $\mathbb{B}_\Phi$!

B., Pinsker, Rydval: Φ transforms to an equivalent Φ' s.t.

$$\{\text{colors of } \Phi'\} \xleftrightarrow{\text{bij}} \{\text{orbits of } \tau\text{-tuples in } \mathbb{B}_{\Phi'}\}$$

Canonical mapping h well-defines a recoloring

$$r\colon \{\text{colors of } \Phi'\} \to \{\text{colors of } \Psi'\}$$

containment $\Rightarrow$ recoloring

Conclusion

- For GMSNP, $\Phi \subseteq \Psi \Leftrightarrow \Phi' \xrightarrow{\text{rec}} \Psi'$

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Future work

1. To extend the decidability of containment on bigger classes
2. To prove decidability for GMSNP as it is done for MMSNP
3. To study approximation (promise) GMSNP

Thank You!