GMSNP

Alexey Barsukov

Plan of the talk

(i) History

(ii) GMSNP as infinite-domain CSP

(iii) GMSNP as finite-domain CSP

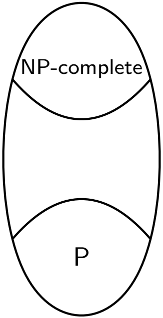
(iv) Decidability of containment

(v) Dichotomy for GMSNP

History

Pre-history (1970s)

Thm (Ladner): If $\mathsf{P} \neq \mathsf{NP}$ then $\mathsf{P} \cup \mathsf{NP}\text{-complete} \subsetneq \mathsf{NP}$



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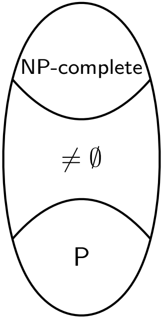

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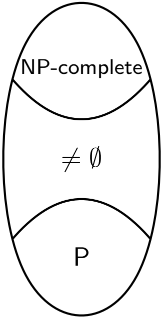

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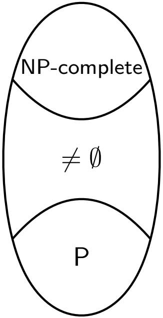

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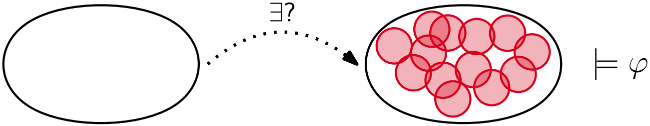

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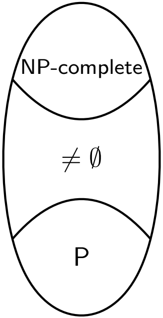

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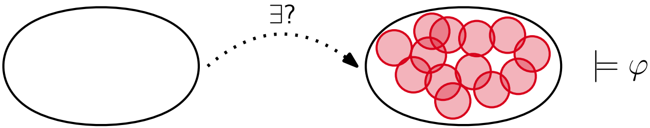

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first-order sentence over the signature $\{R_1, \ldots, S_\ell\}$

MMSNP: definition

Def: *Monotone Monadic Strict NP without "$\neq$"* (MMSNP) consists of the problems of the form:

Input: relational structure $\mathbb{X}$

Q: color elements of $\mathbb{X}$ with Γ s.t. for NO $\mathbb{F} \in \mathcal{F}$, there is hom $\mathbb{F} \to (\mathbb{X}, \Gamma)$

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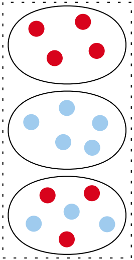

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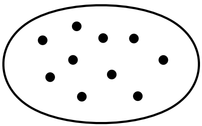

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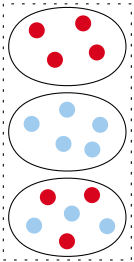

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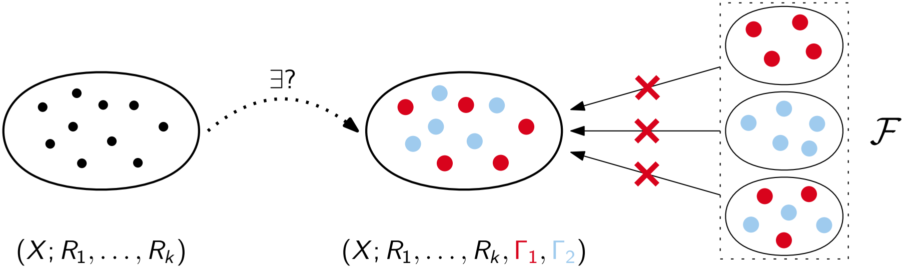

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existential relations are unary

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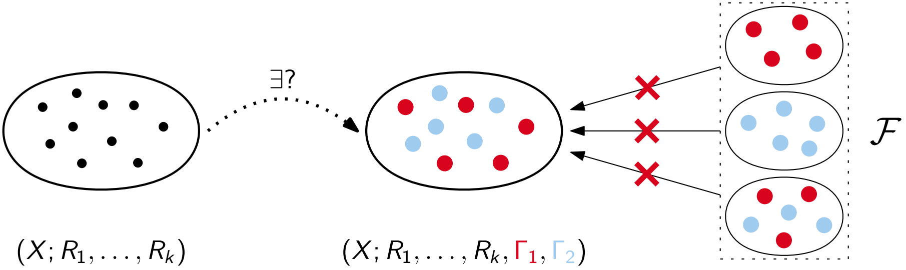

MMSNP: definition

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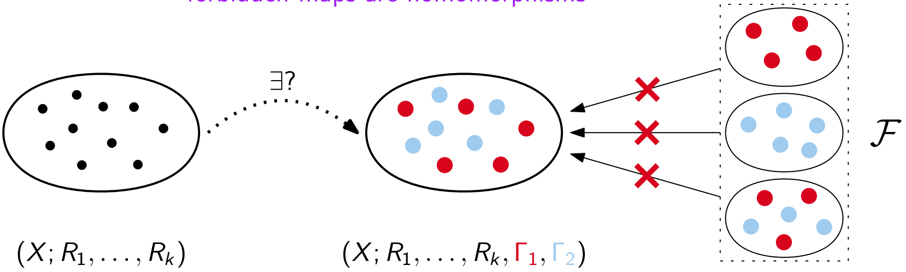
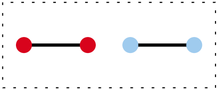


MMSNP: examples

$\Gamma = \{Red(\cdot), Blue(\cdot)\}$  **2-col**

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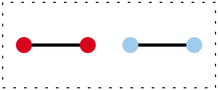

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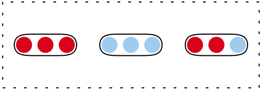

1in3

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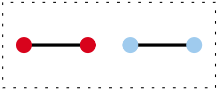
$\Gamma = \{\color{red}{Red(\cdot)}, \color{lightblue}{Blue(\cdot)}\}$  **2-col**

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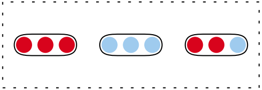
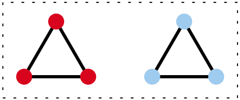
$\Gamma = \{\color{red}{Red(\cdot)}, \color{lightblue}{Blue(\cdot)}\}$  **1in3**

(ii) No-Monochromatic-Triangle

$\Gamma = \{\color{red}{Red(\cdot)}, \color{lightblue}{Blue(\cdot)}\}$  $\mathcal{F}$

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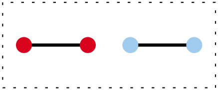

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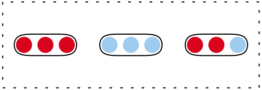

1in3

(ii) No-Monochromatic-Triangle

provably not in "finite-domain CSP"

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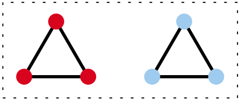

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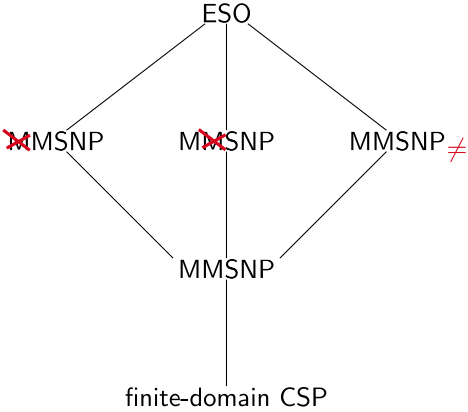
MMSNP: a logic for finite-domain CSP



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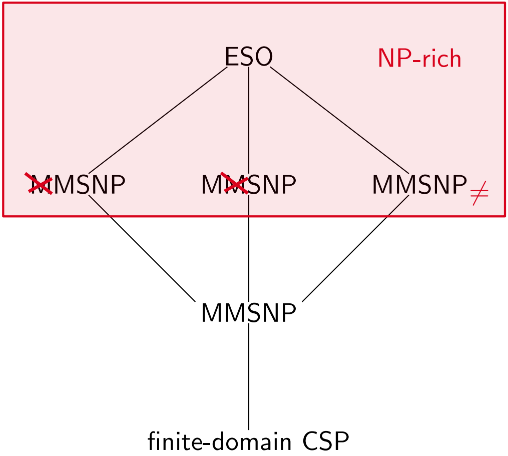

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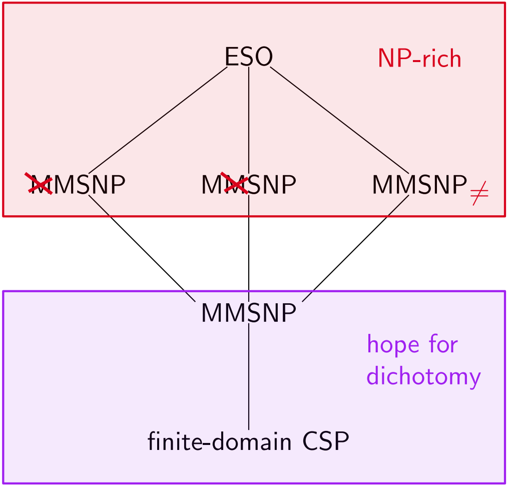

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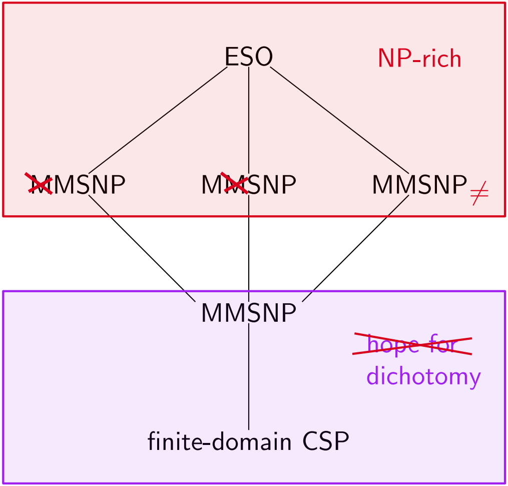

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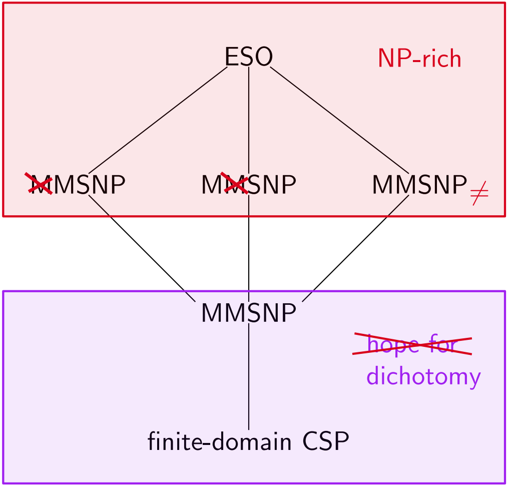

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Q: Is there smth above MMSNP that has a dichotomy?

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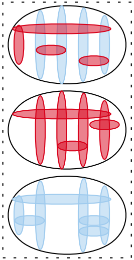

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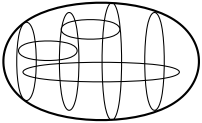

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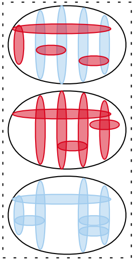

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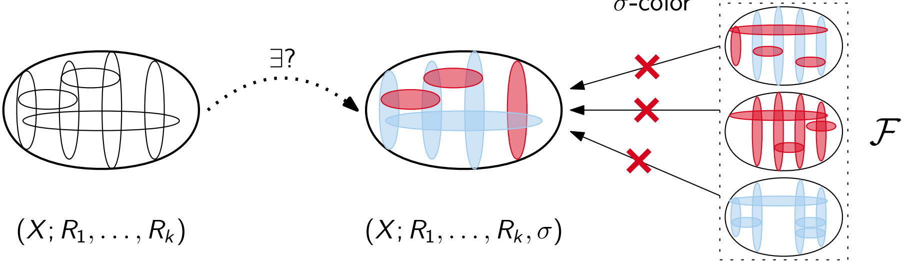

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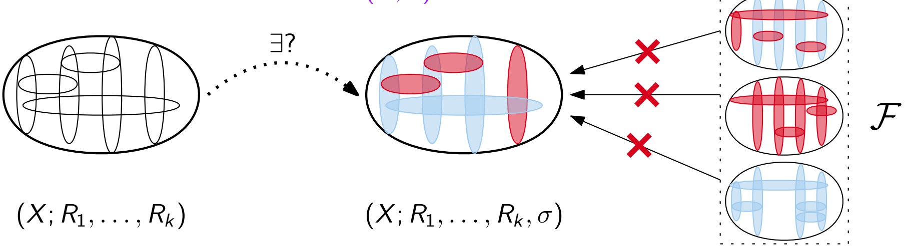

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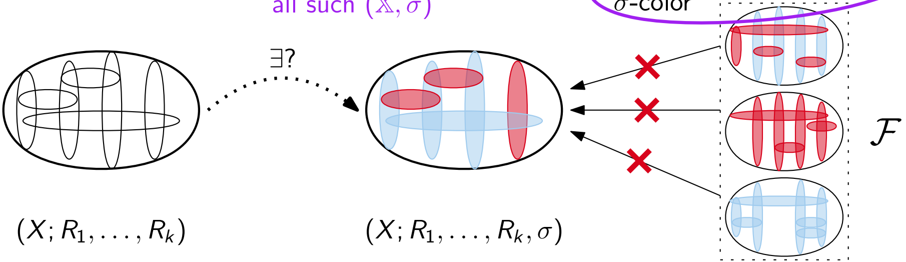

GMSNP: examples

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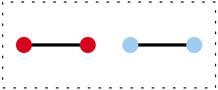

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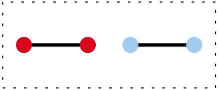

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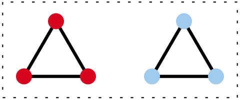

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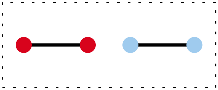

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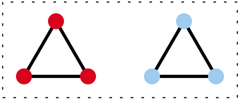
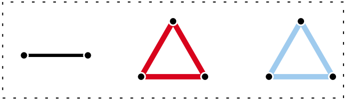


(iii) No-Monochromatic-Edge-Triangle

$\sigma = \{Red(\cdot,\cdot), Blue(\cdot,\cdot)\}$



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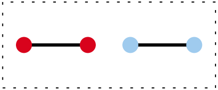

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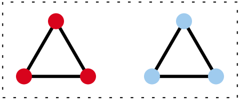

(iii) No-Monochromatic-Edge-Triangle

B., Mottet, Perinti: provably not in MMSNP

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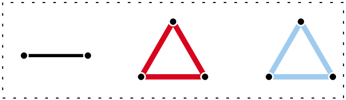

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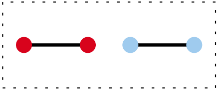

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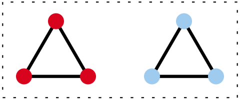

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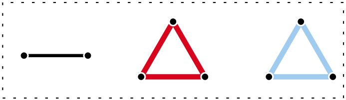

(iv) NOT example: CSP$(\mathbb{Q}, <)$

provably not in GMSNP

GMSNP as infinite-domain CSP

Homogeneity and Amalgamation

Goal: for every $\mathcal{F} \in$ GMSNP, to have “nice” structure $\mathbb{D}_{\mathcal{F}}$ s.t. $\mathbb{X} \in \text{Forb}(\mathcal{F})$ if and only if $\mathbb{X} \to \mathbb{D}_{\mathcal{F}}$

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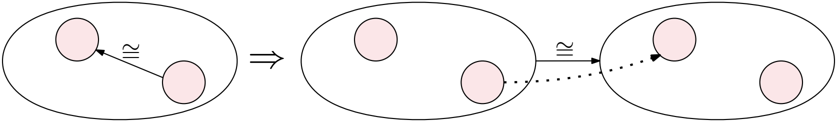

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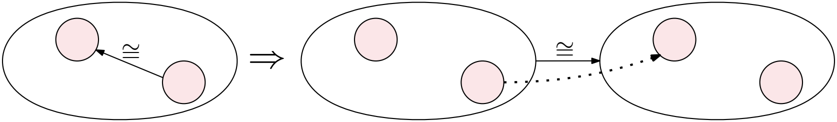

Example: $(0 < 1) \cong (5 < 7) \hookrightarrow (\mathbb{Q}, <)$ take $\alpha \colon \mathbb{Q} \to \mathbb{Q}$, $x \mapsto 2x + 5$

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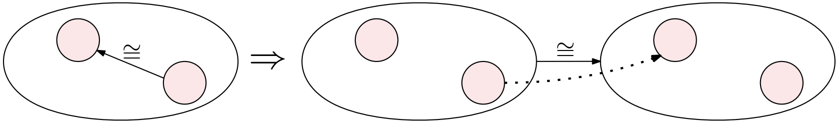

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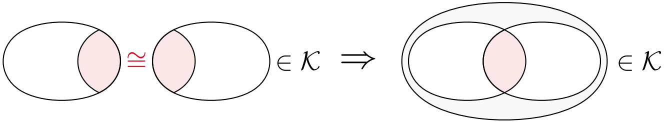

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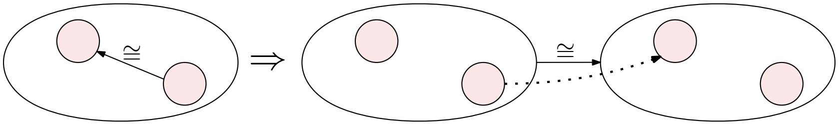

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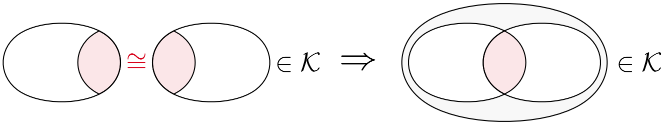

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Amalgamation for GMSNP

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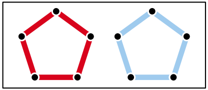

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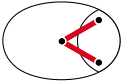
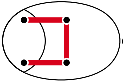
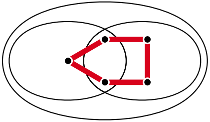
 $\cong$  $\in \text{Forb}(\mathcal{F})$ but  $\notin \text{Forb}(\mathcal{F})$

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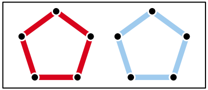

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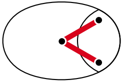
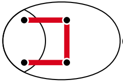
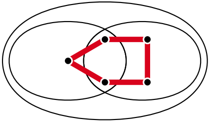
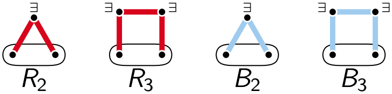
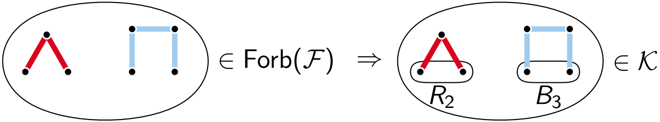
 $\cong$  $\in \text{Forb}(\mathcal{F})$ but  $\notin \text{Forb}(\mathcal{F})$

Solution: construct a class $\mathcal{K}$ from $\text{Forb}(\mathcal{F})$ by adding new relations





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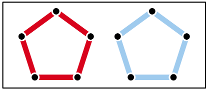

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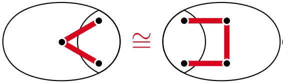

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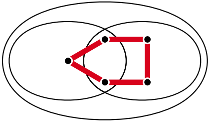

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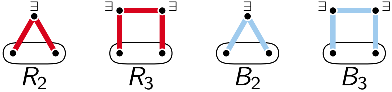

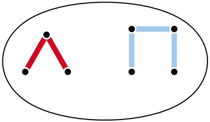

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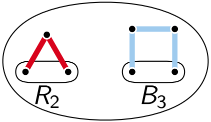

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Infinite-domain CSP for GMSNP

Thm (Bodirsky, Knäuer, Starke): For every template $\mathcal{F}$ there is countably infinite structure $\mathbb{B}$ s.t. input $\mathbb{X}$ has a solution iff $\mathbb{X} \to \mathbb{B}$

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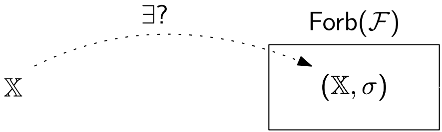

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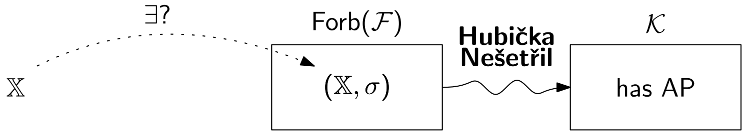

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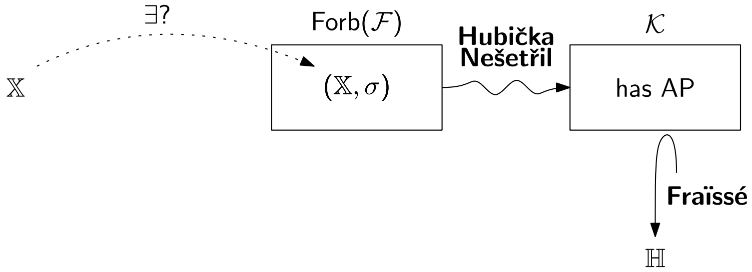

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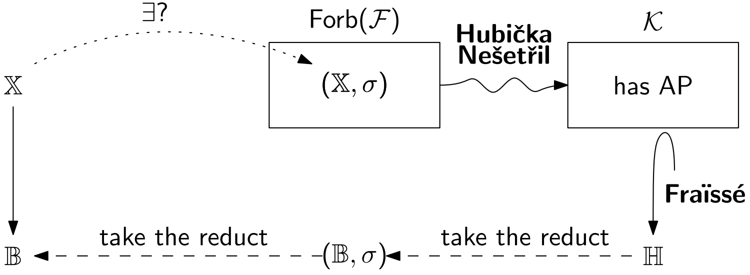

Cor: every GMSNP is a CSP of a reduct of finitely-bounded homogeneous structure, i.e., it falls into the scope of Bodirsky-Pinsker conjecture

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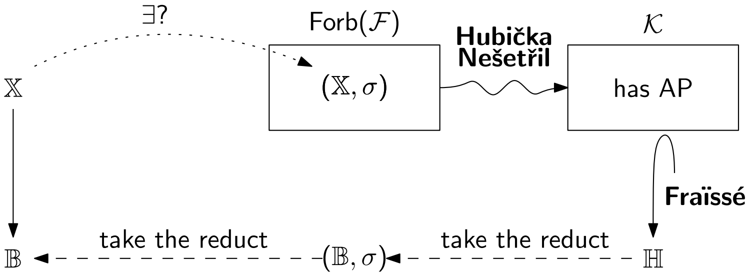

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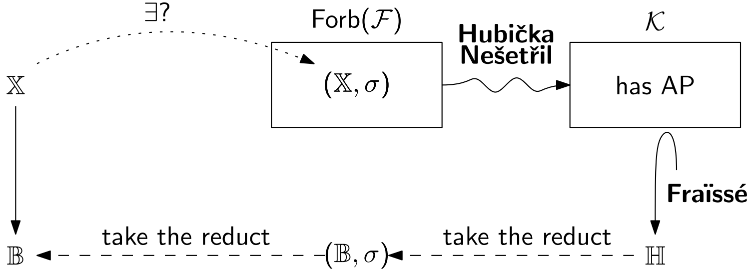

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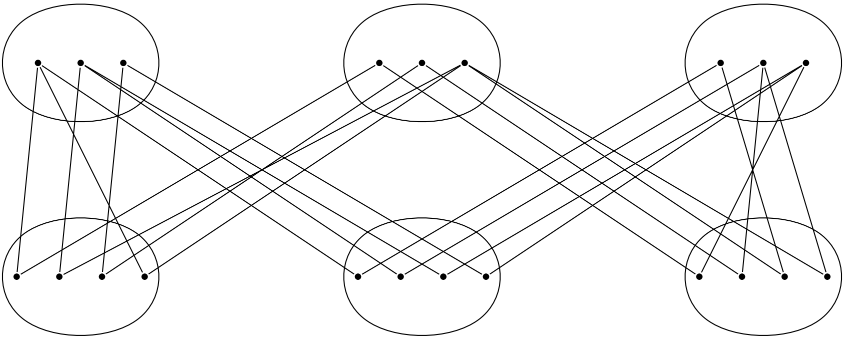
GMSNP as finite-domain CSP

Label Cover

Input: family of finite sets $X_1, \ldots, X_n$ and family of maps $f_1, \ldots, f_m$

Q: find $s\colon [n] \to \bigsqcup_i X_i$ s.t. $s(i) \in X_i$ and that $f(s(i)) = s(j)$, for every map $f\colon X_i \to X_j$



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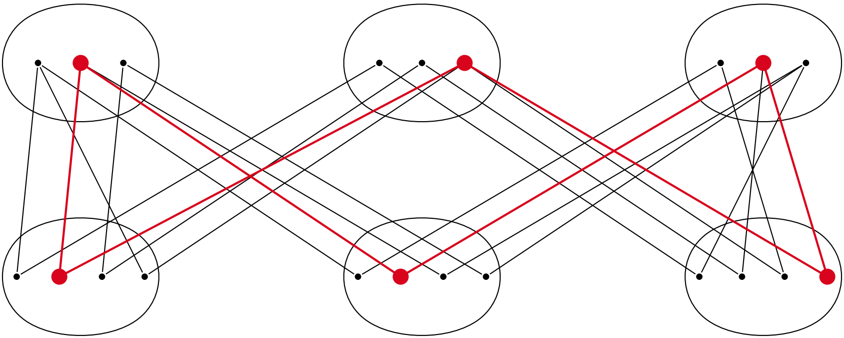
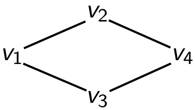


Label Cover for finite-domain CSPs

Input: 

Template: CSP()

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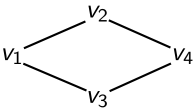
Input: 

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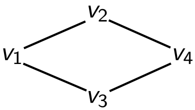
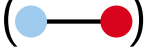
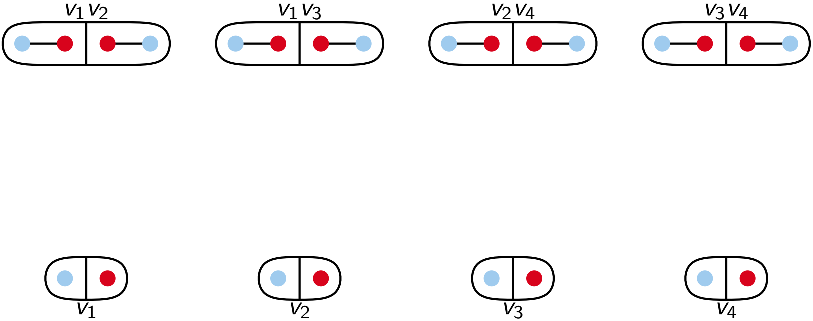


Template: CSP()



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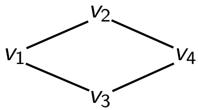
Input: 

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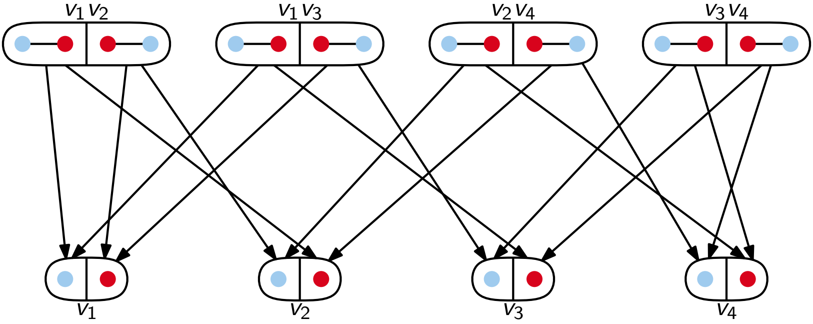
Maps are projections of "constraint" tuples on their subtuples



Label Cover for infinite-domain CSPs: MMSNP

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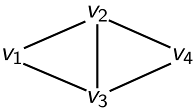

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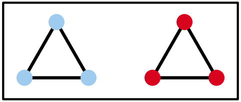

$\mathcal{F}$

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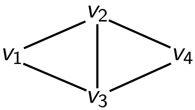
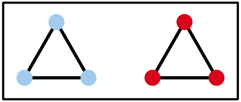
Input: 

Template:  $\mathcal{F}$

What matters:

Tuples to which we assign values (vertices)

Tuples that are hom-images of reducts of forbidden structures (triangles)

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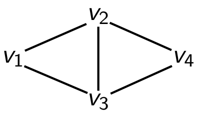
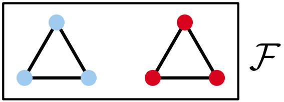
Input: 

Template: 

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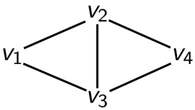
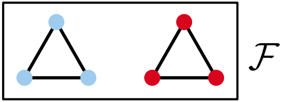
Input: 

Template: 

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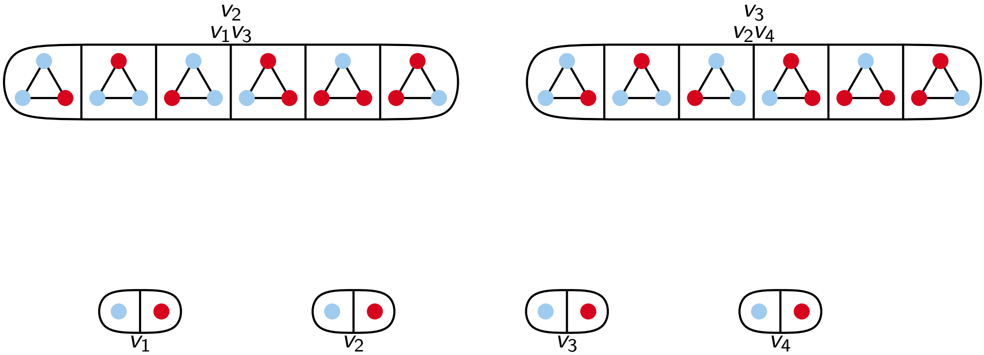

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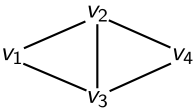

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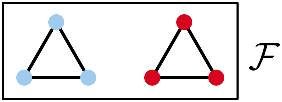

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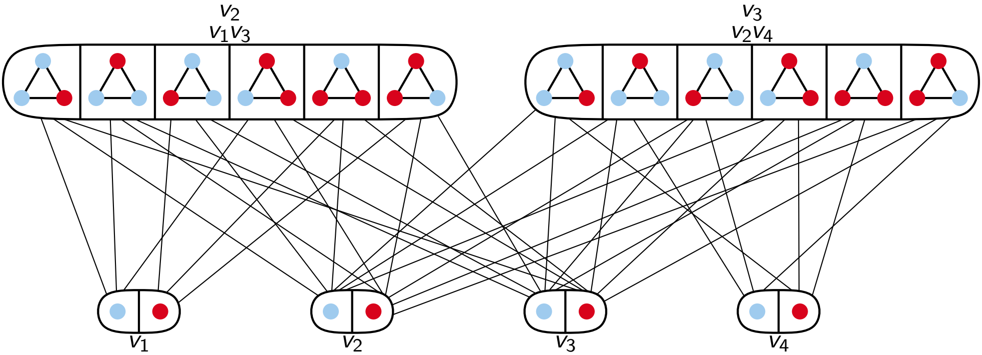

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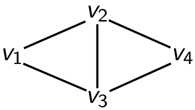

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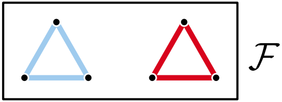

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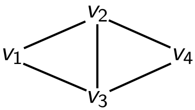

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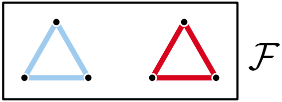

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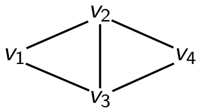
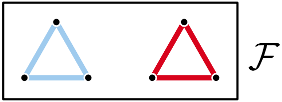


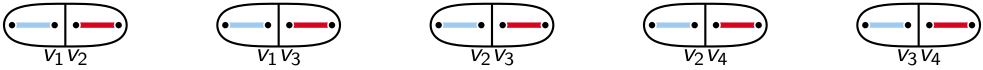
Template:



What matters:

Tuples to which we assign values (edges)

Tuples that are hom-images of reducts of forbidden structures (triangles)



Label Cover for infinite-domain CSPs: GMSNP

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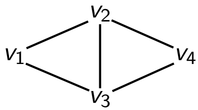

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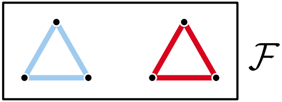

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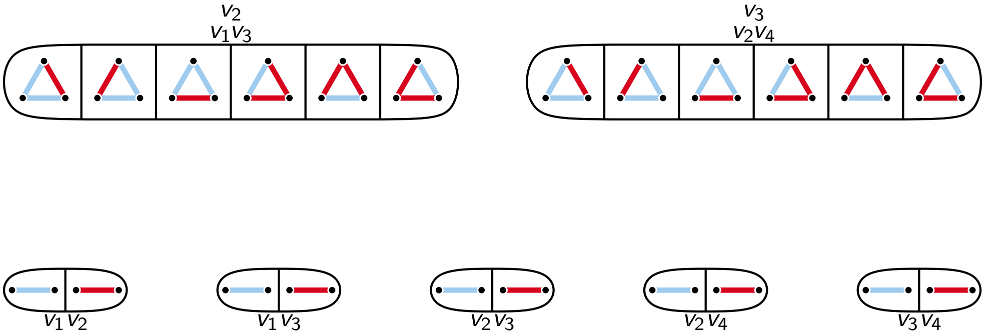

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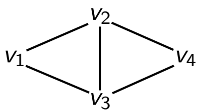

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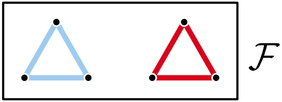

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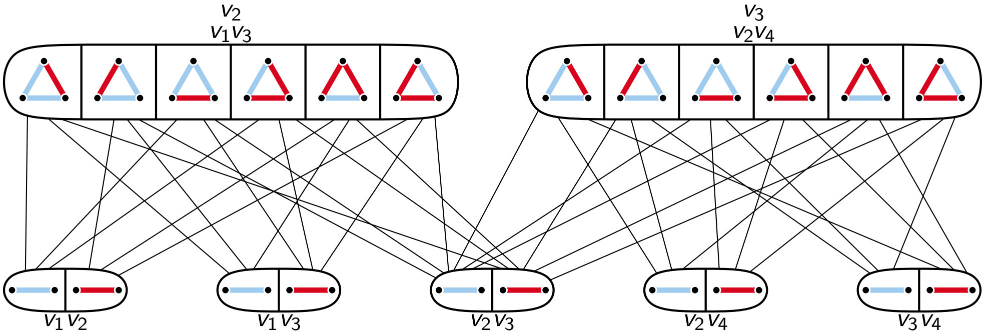

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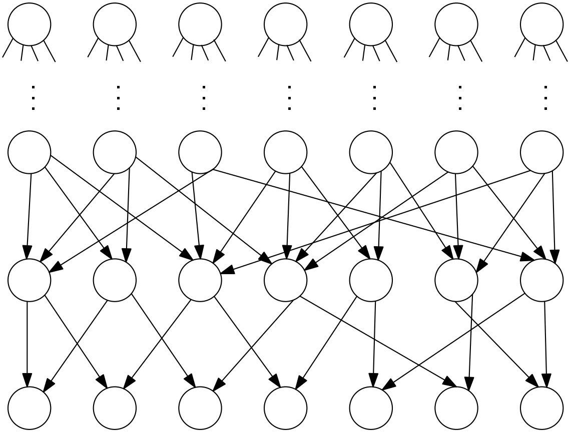

m-many levels, where $m = \max |\mathbb{F}|$

values on level k are k-element orbits

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and by the bounds from the template, for example, "no-monochromatic-triangle"

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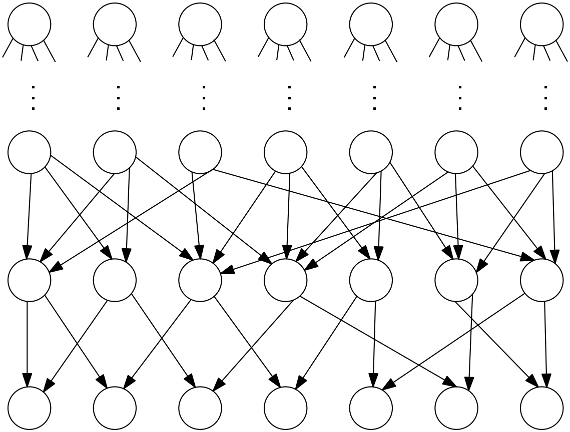

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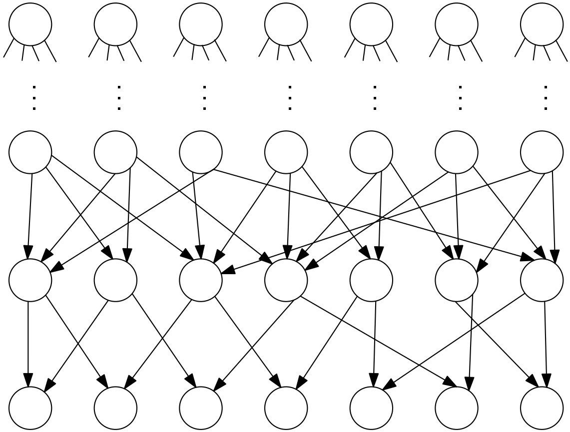

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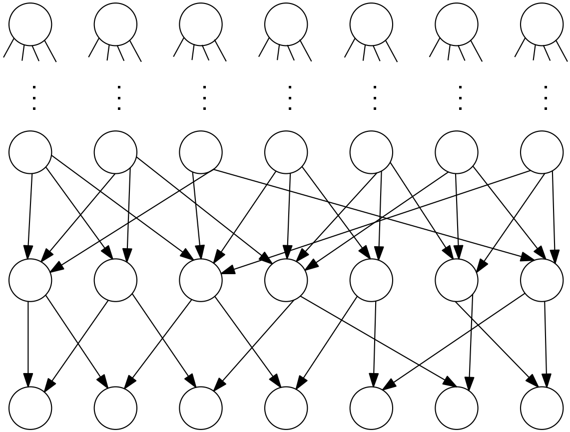

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Decidability of containment

Containment: definition

Def: Given problems Φ and Ψ with the same input, to decide whether $\mathbb{I} \in \Phi \Rightarrow \mathbb{I} \in \Psi$, for every input $\mathbb{I}$, denoted $\Phi \subseteq \Psi$

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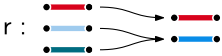
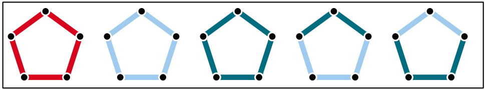
Q (Bienvenu, ten Cate, Lutz, Wolter) (Bourhis, Lutz): Is it still decidable for GMSNP?

Recoloring

Def: $r : \{\text{colors of } \Phi\} \to \{\text{colors of } \Psi\}$ is a *recoloring* from Φ to Ψ



if the preimage $r^{-1}(\mathcal{F}_\Psi)$ has no $\mathcal{F}_\Phi$-free structures



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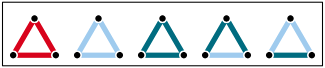

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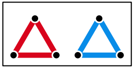

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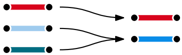

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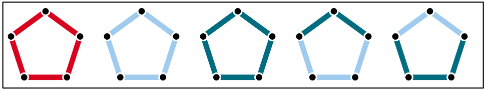

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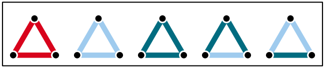

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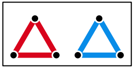
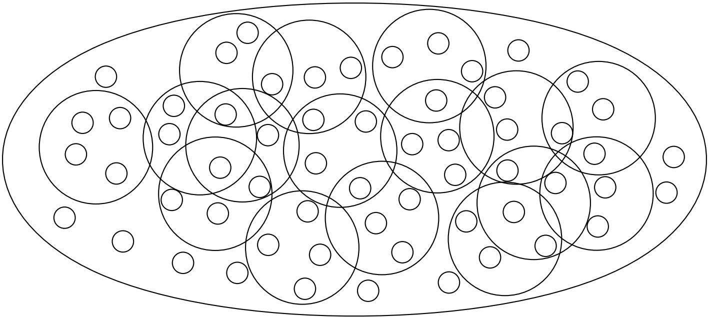


$\mathcal{F}_\Psi$

$\exists$ recoloring from Φ to $\Psi \Rightarrow \Phi \subseteq \Psi$

Ramsey Property

Class $\mathcal{K}$ is **Ramsey** if for all $\mathbb{A}, \mathbb{B} \in \mathcal{K}$ and all $n \in \mathbb{N}$ there is $\mathbb{C} \in \mathcal{K}$ s.t. for all $\chi\colon \binom{\mathbb{C}}{\mathbb{A}} \to [n]$ there is $\mathbb{B}_0 \in \binom{\mathbb{C}}{\mathbb{B}}$ s.t. χ is constant on $\binom{\mathbb{B}_0}{\mathbb{A}}$



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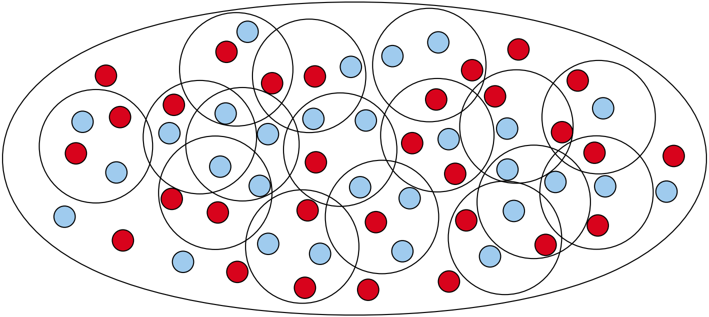

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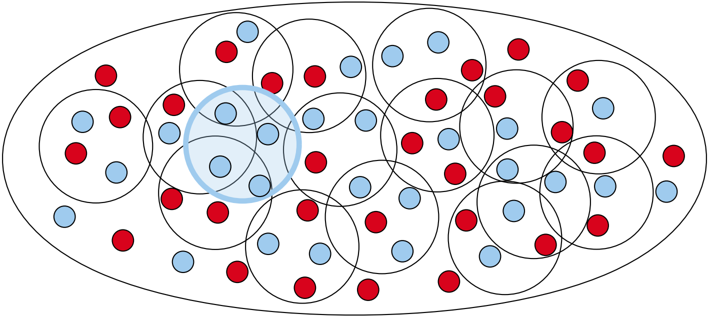
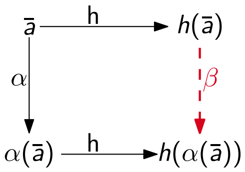


Canonical mappings

Def: mapping $h\colon A \to B$ is *canonical* w.r.t. groups G and H acting on sets A and B, if for every n, every $\bar{a} \in A^n$, and every $\alpha \in G$ there is $\beta \in H$ s.t.



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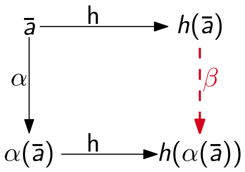

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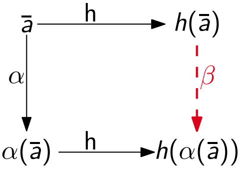

Thm (Bodirsky, Pinsker, Tsankov): If $h\colon \mathbb{A} \to \mathbb{B}$ is hom between ω-categorical structures and if $\mathbb{A}$ has homogeneous Ramsey expansion $\mathbb{A}^*$, then there is a hom $g\colon \mathbb{A} \to \mathbb{B}$ which is canonical w.r.t. $Aut(\mathbb{A}^*)$ and $Aut(\mathbb{B})$

Containment implies recoloring

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containment $\Rightarrow$ recoloring

Searching for recoloring is 2NEXPTIME-complete problem

Conclusion on containment

Decidability of containment for class $\mathcal{C}$ can motivate to study the complexity of *promise* problems on $\mathcal{C}$:

Input: X	**Q:** Y, if X is accepted by Φ N, if X is rejected by Ψ	**Template:** $\Phi, \Psi \in \mathcal{C}$ s.t. $\Phi \subseteq \Psi$

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MMSNP and now even GMSNP

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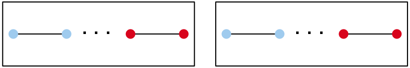
Approximate graph coloring: for all $m < n$, the following is NP-hard

PromiseMMSNP()

m-many colors n-many colors

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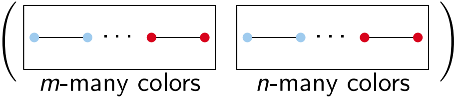
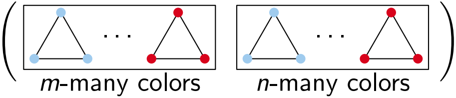


Obs: for all $m < n$, the following is p-time equiv to PCSP(NAE_m, NAE_n)

PromiseMMSNP



Conclusion on containment

MMSNP and now even GMSNP

Decidability of containment for class $\mathcal{C}$ can motivate to study the complexity of *promise* problems on $\mathcal{C}$:

Input: X **Q:** Y, if X is accepted by Φ
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Approximate graph coloring: for all $m < n$, the following is NP-hard

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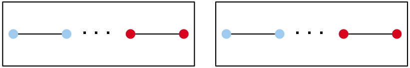
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Obs: for all $m < n$, the following is p-time equiv to PCSP(NAE_m, NAE_n)

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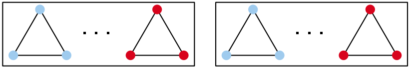
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Q: what is the complexity of the following PromiseMMSNP?

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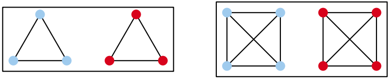
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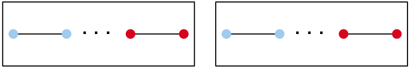
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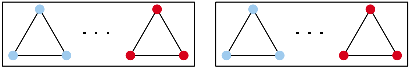
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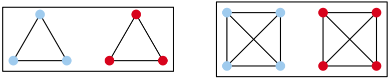
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each problem is equiv to Bool CSP: 3-ary and 4-ary NAE

Dichotomy for GMSNP

whose complexity classification belongs to the most prominent open problems in infinite-domain constraint satisfaction

Feller, Pinsker

Forbidden tournaments

Def: *$\mathcal{F}$-free orientation problems* have the form:

Input: undirected graph $\mathbb{G}$

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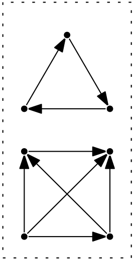

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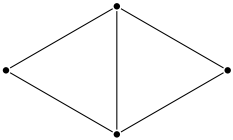

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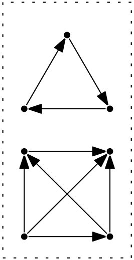

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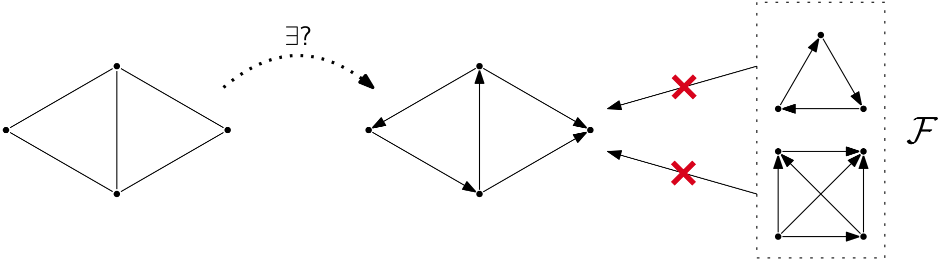

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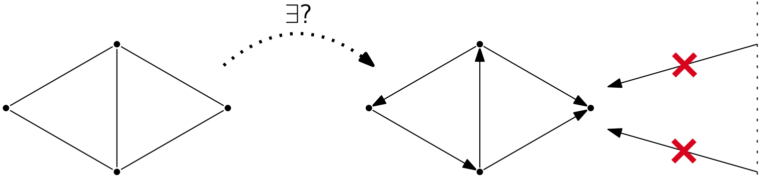

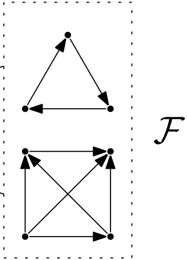

Let $\mathbb{D}_{\mathcal{F}}$ be the Fraïssé-limit of Forb($\mathcal{F}$) and $\mathbb{H}_{\mathcal{F}}$ be its *graph reduct* (forget orientation of edges)

Thm (Bodirsky, Guzmán-Pro) (Bitter, Mottet) (Feller, Pinsker): Pol($\mathbb{H}_{\mathcal{F}}$) $\to$ Proj and CSP($\mathbb{H}_{\mathcal{F}}$) is NP-complete or Pol($\mathbb{H}_{\mathcal{F}}$) $\not\to$ Proj and CSP($\mathbb{H}_{\mathcal{F}}$) is in P

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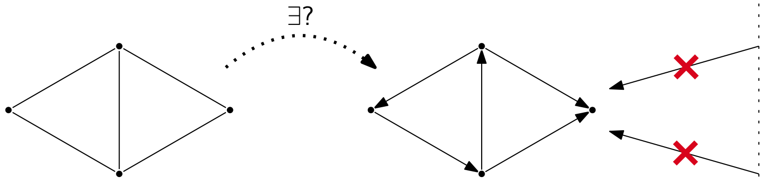

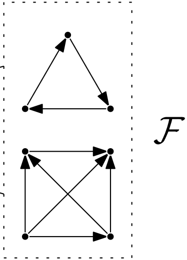

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Polymorphisms of that Boolean CSP are canonical polymorphisms of $\mathbb{D}_{\mathcal{F}}$

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Motivation: explore step **(i)** for more general classes of GMSNP: such like edge-colored graphs

Precolored GMSNP: definition

Def: *Precolored GMSNP* consists of the problems of the form:

Input: relational τ-structure $\mathbb{X}$ with some τ-tuples colored with σ

Q: to complete coloring of τ-tuples of $\mathbb{X}$ s.t. for NO $\mathbb{F} \in \mathcal{F}$, there is hom $\mathbb{F} \to (\mathbb{X}, \sigma)$

Template: finite set σ of new colors for τ-tuples and finite family $\mathcal{F}$ of finite structures s.t. every τ-tuple is colored with some σ-color

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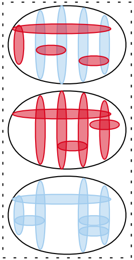

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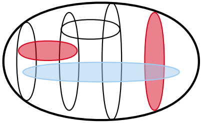
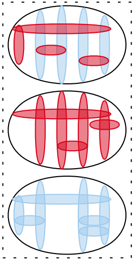


$(X; R_1, \ldots, R_k)$



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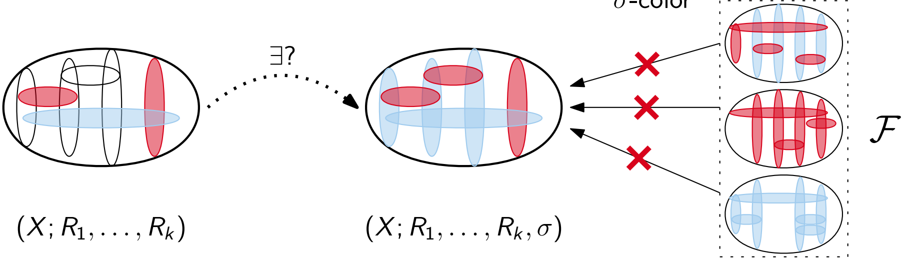

Precolored GMSNP: definition

Obs: GMSNP($\mathcal{F}$) always reduces to precolored version

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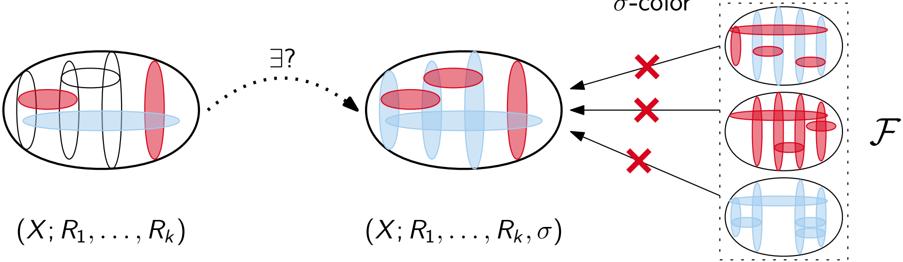

Colored determiners: definition

Motivation: Prove that original and precolored versions always have the same complexity and to work only with precolored GMSNP

Def: for $\mathbb{G}$ – graph, ξ – partial edge-coloring, e – edge of G, i – color, tuple $(\mathbb{G}, \xi, e)$ is a *colored determiner* for color i if

- **(i)** one of the two vertices incident to e is not incident to any edge of $\mathrm{dom}(\xi)$
- **(ii)** there is an $\mathcal{F}$-free extension of ξ
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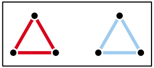
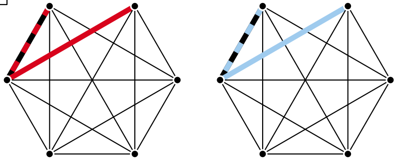
Ex: for $\mathcal{F} =$  colored red- and blue-determiners are



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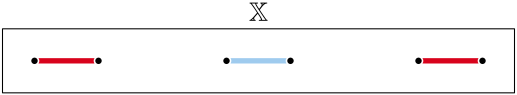

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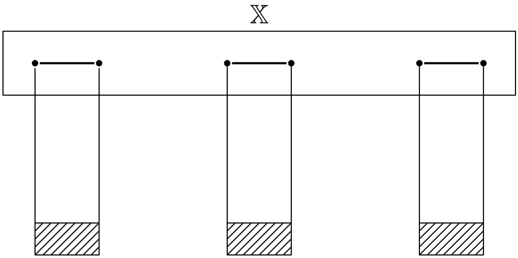

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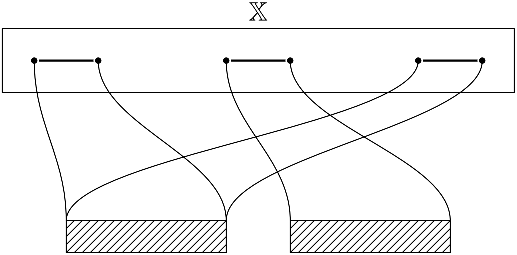

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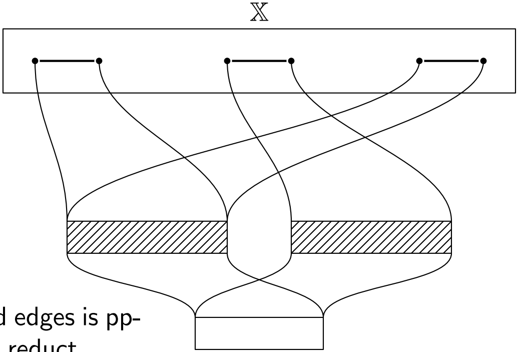

the orbit of colored edges is pp-definable in graph reduct

Colored determiners: results

B., Mottet, Perinti: Using colored determiners can prove $\mathsf{P} \cup \mathsf{NP}$-complete dichotomy for various classes $\mathcal{F}$ such like

all families of monochromatic odd cycles
all families of monochromatic cliques

even if the input is restricted on $\mathbb{K}_m$-free graphs for $m >$ max size of obstruction

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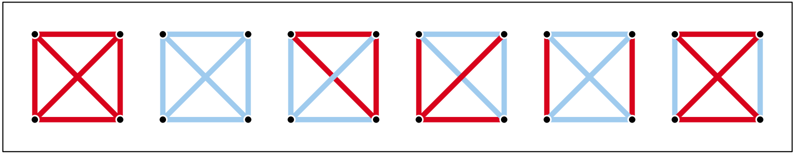
A: No. Take $\mathcal{F}$ to be



Colored determiners: results

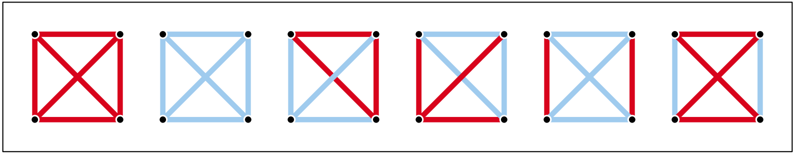
B., Mottet, Perinti: Using colored determiners can prove $\mathrm{P} \cup \mathrm{NP}$-complete dichotomy for various classes $\mathcal{F}$ such like

all families of monochromatic odd cycles
all families of monochromatic cliques

even if the input is restricted on $\mathbb{K}_m$-free graphs for $m >$ max size of obstruction

Q: Do colored determiners always exist? at least for edge-colored cliques? at least for 2-edge-colored cliques?

A: No. Take $\mathcal{F}$ to be colored determiners provably do not exist for $\mathcal{F}$



Thank You!

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