Π_2^P vs PSpace Dichotomy for the Quantified Constraint Satisfaction Problem

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Main Question

What is the complexity of QCSP(Γ) for different Γ?

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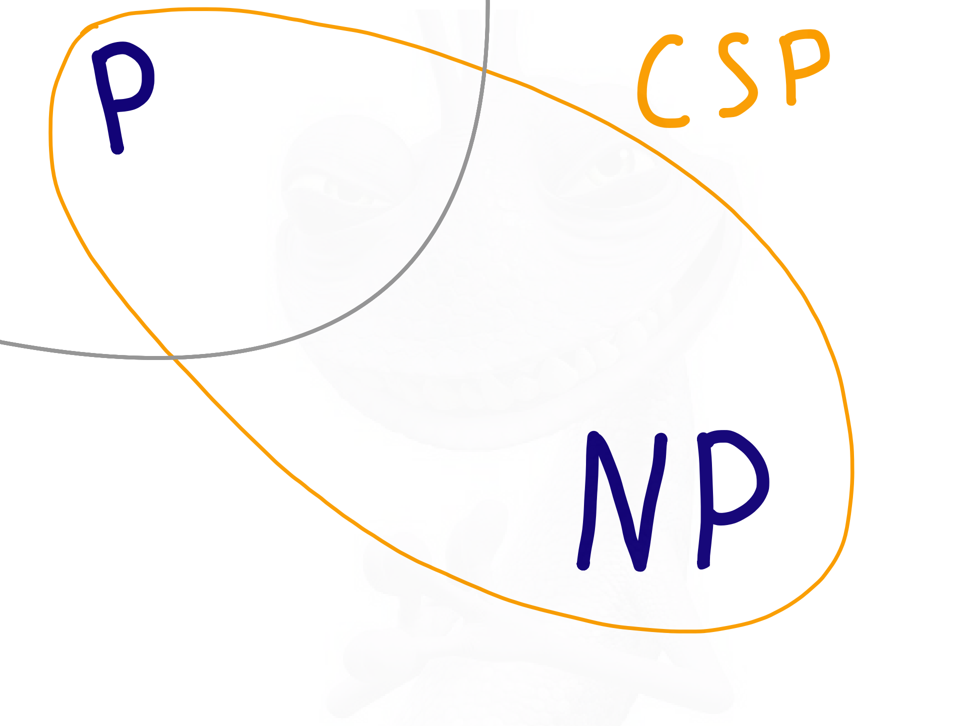
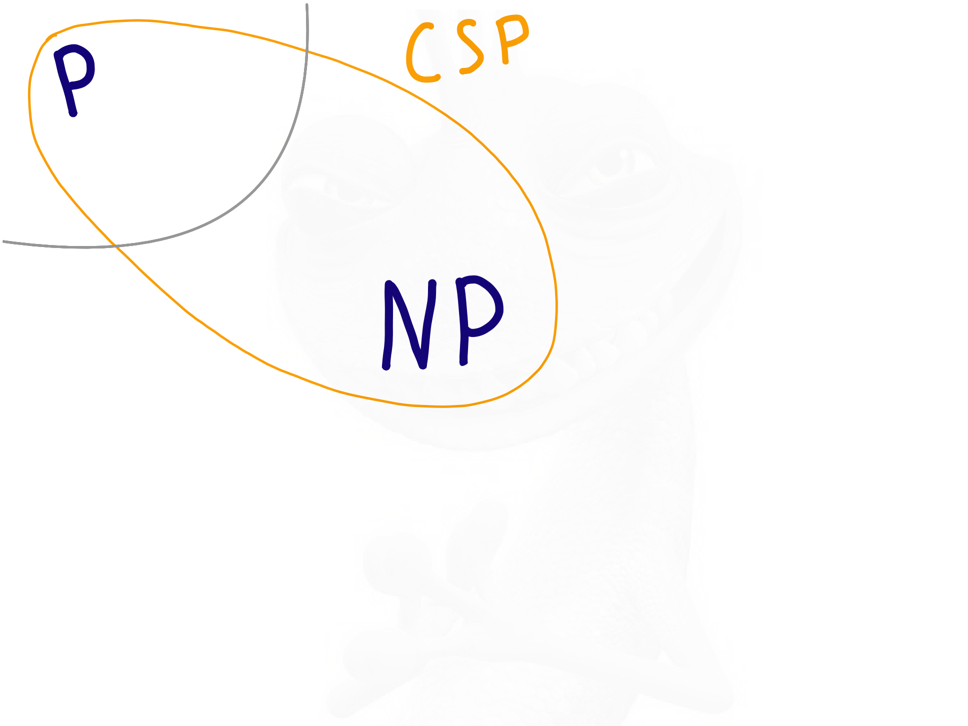
Theorem [Bulatov, Zhuk, 2017]

CSP(Γ) is

- either solvable in polynomial time,
- or NP-complete.


CSP
P
NP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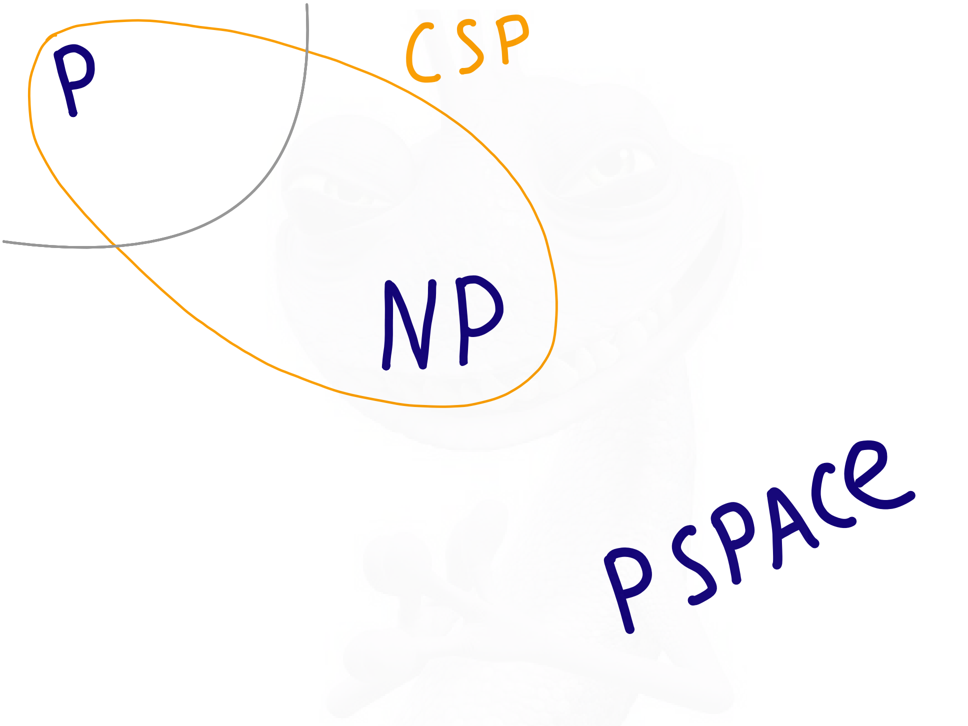
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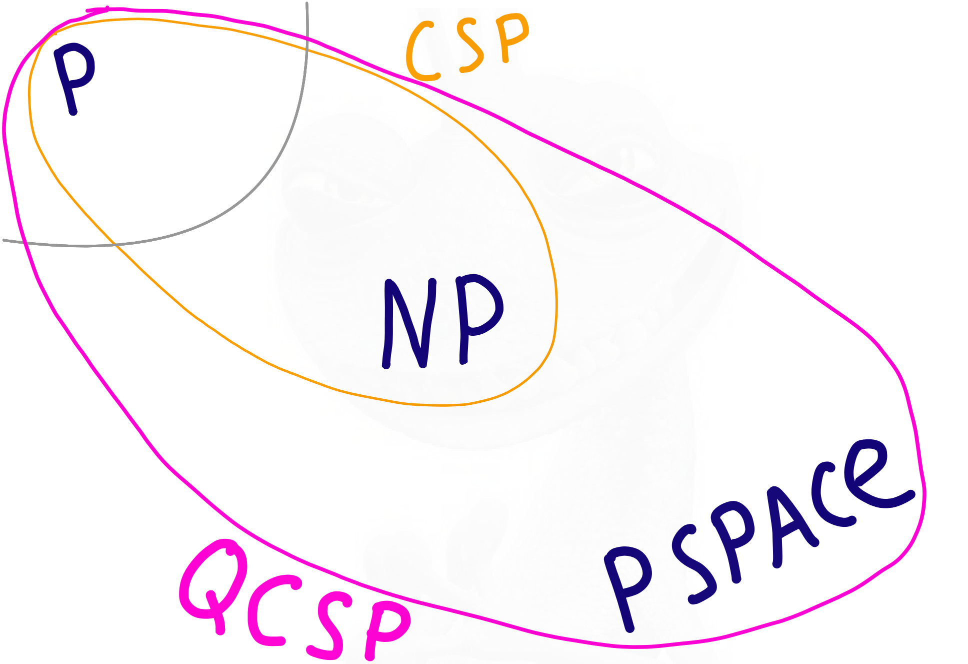
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Theorem [Schaefer 1978 + Creignou et al. 2001/ Dalmau 1997.]

Suppose Γ is a constraint language on $\{0, 1\}$. Then

- QCSP(Γ) is in P if Γ is preserved by an idempotent WNU operation,
- QCSP(Γ) is PSPACE-complete otherwise.

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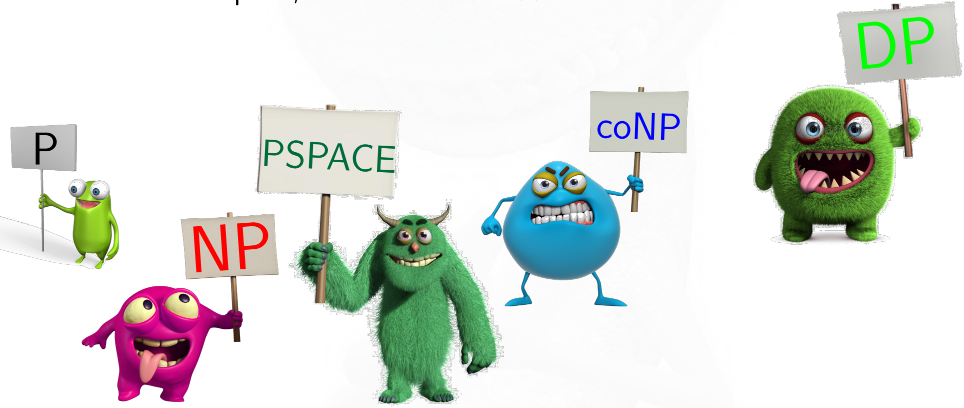

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- There exists Γ on a 10-element domain such that QCSP(Γ) is Θ_2^P-complete.

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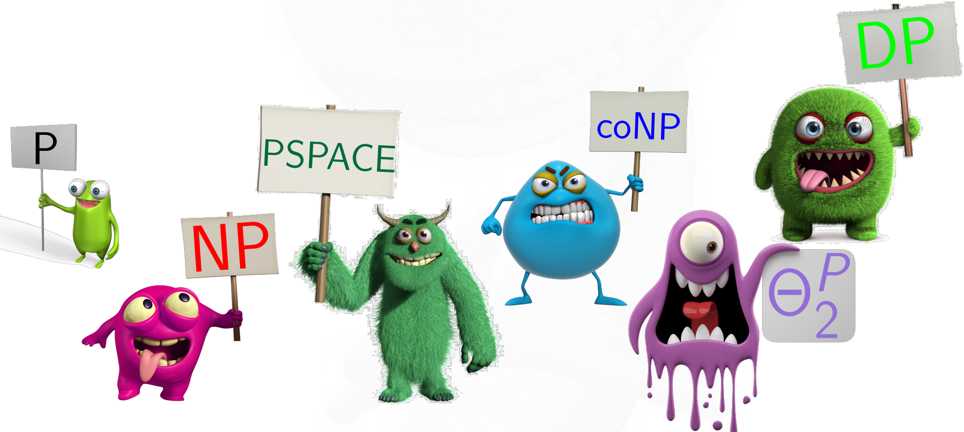

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Theorem [Zhuk, Martin, 2019]

Suppose Γ is a constraint language on $\{0, 1, 2\}$ containing $\{x = a \mid a \in \{0, 1, 2\}\}$. Then QCSP($\Gamma$) is

- in P, or
- NP-complete, or
- coNP-complete, or
- PSPACE-complete.

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Proof:

$$\Pi_2^P = \{\forall X^{|X|<p(|Z|)} \exists Y^{|Y|<q(|Z|)} \mathcal{V}(X, Y, Z) : \mathcal{V} \in \mathrm{P}, p, q \text{ - polynomials}\}$$

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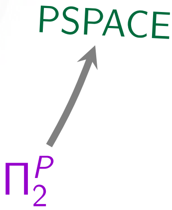

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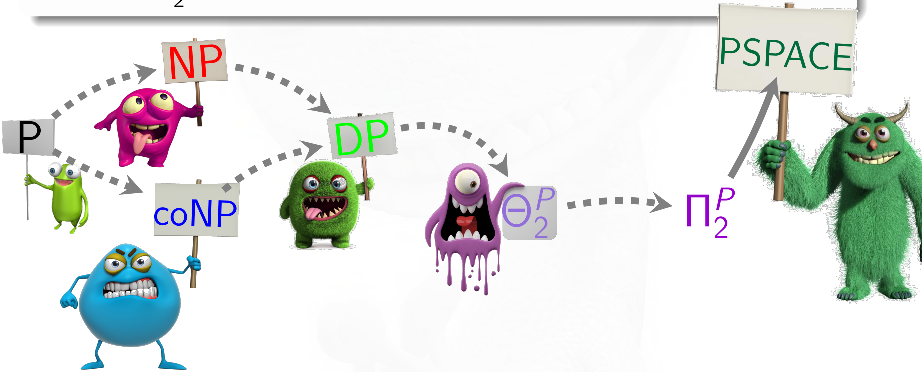

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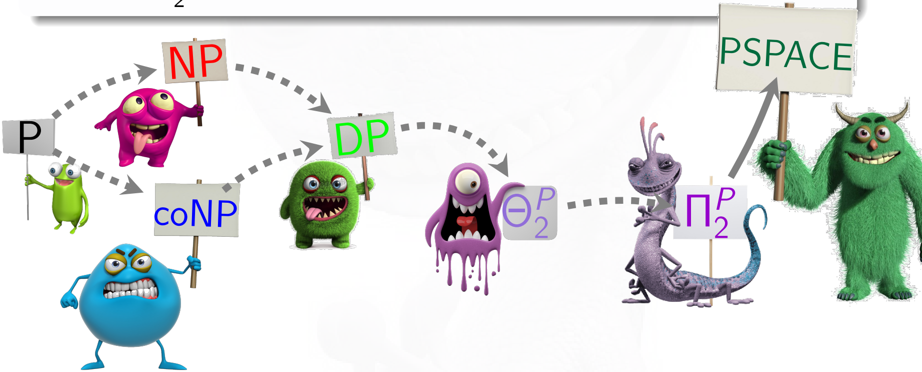

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There exists Γ on a 6-element set such that QCSP(Γ) is Π_2^P-complete.

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Are there any other complexity classes?

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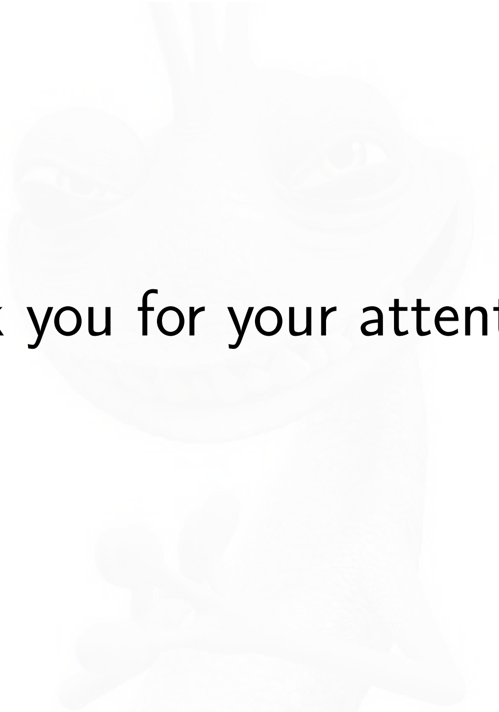
- If the tree-instance is small then there exists a polynomial-size subinstance of $\mathcal{I}$ without a solution.
- The polynomial-size subinstance gives $S \subseteq A^n$ s. .t. $\exists y_0 \forall x_1 \exists y_1 \dots \forall x_n \exists y_n ((x_1, \dots, x_n) \in S \rightarrow \Psi)$ is not satisfiable.



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