The complexity of the Constraint Satisfaction Problem and its variations

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Colloquium of Faculty of Informatics

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Example

Check whether there exists a solution $x_1, x_2, x_3, \ldots \in \{0, 1\}$.

$$\begin{cases} x_1 + x_2 + x_3 = 0 \mod 2 \\ x_1 + x_3 + x_5 = 0 \mod 2 \\ x_2 + x_4 + x_5 = 0 \mod 2 \\ x_2 + x_3 + x_5 = 1 \mod 2 \end{cases}$$

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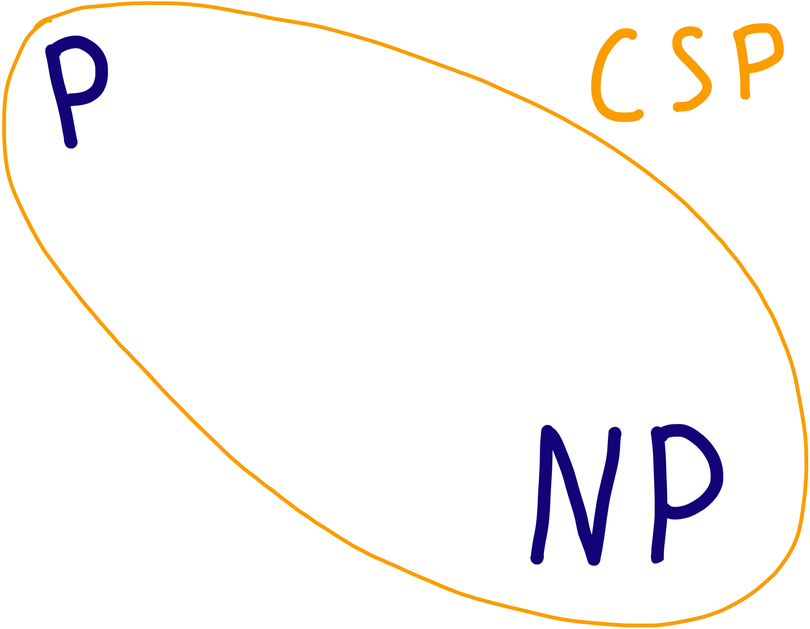
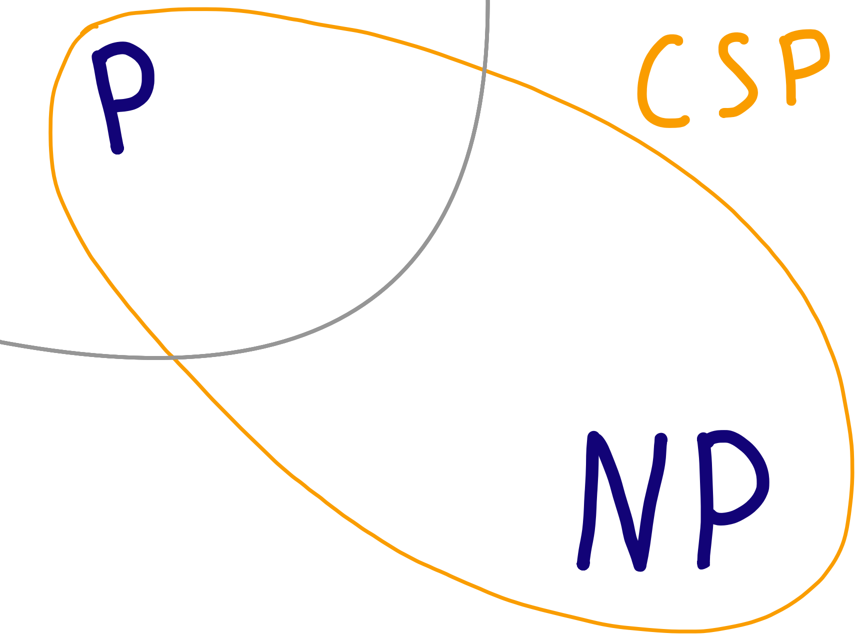
What is the complexity of this problem? Nobody knows!

P

P

NP


CSP
P
NP



P
CSP
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What is CSP?

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Constraint Satisfaction Problem
is a triple $\langle \mathbf{X}, \mathbf{D}, \mathbf{C} \rangle$, where

- $\mathbf{X} = \{x_1, \ldots, x_n\}$ is a set of variables,
- $\mathbf{D} = \{D_1, \ldots, D_n\}$ is a set of the respective domains of values, and
- $\mathbf{C} = \{C_1, \ldots, C_m\}$ is a set of constraints

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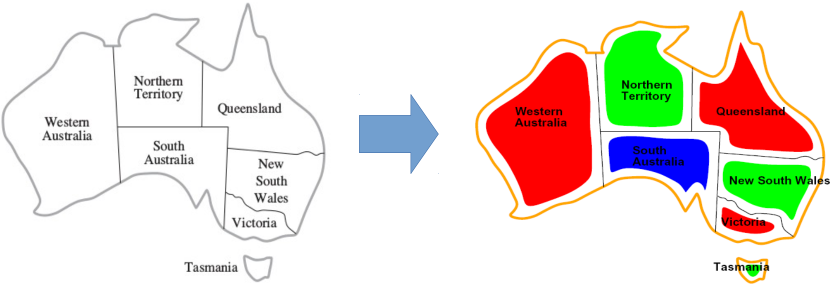
Almost everything is CSP!!!

CSP example: map coloring



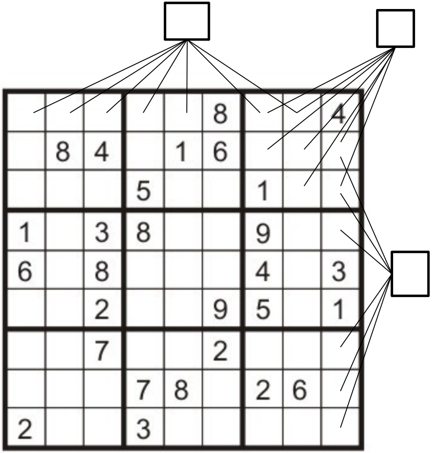
Problem: assign each territory a color such that no two adjacent territories have the same color

Variables: $X = \{WA, NT, Q, NSW, V, SA, T\}$

Domain of variables: $D = \{r, g, b\}$

Constraints: $C = \{SA \neq WA, SA \neq NT, SA \neq Q, \ldots\}$

Another example: sudoku



- Variables:
  - Each (open) square
- Domains:
  - {1,2,…,9}
- Constraints:

9-way alldiff for each column

9-way alldiff for each row

9-way alldiff for each region

(or can have a bunch of pairwise inequality constraints)

Constraint Satisfaction Problem parameterized by a constraint language

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Γ is a set of relations on a finite set A.

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CSP(Γ)

Given: a conjunction of relations, i.e. a formula

$$R_1(x_{i_{1,1}}, \ldots, x_{i_{1,n_1}}) \wedge \cdots \wedge R_s(x_{i_{s,1}}, \ldots, x_{i_{s,n_s}}),$$

where $R_1, \ldots, R_s \in \Gamma$.
Decide: whether the formula is satisfiable.

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$A = \{0, 1, 2\}, \Gamma = \{x < y, x \leq y\}$.

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CSP instances:
$x_1 < x_2 \wedge x_2 < x_3 \wedge x_3 < x_4$,

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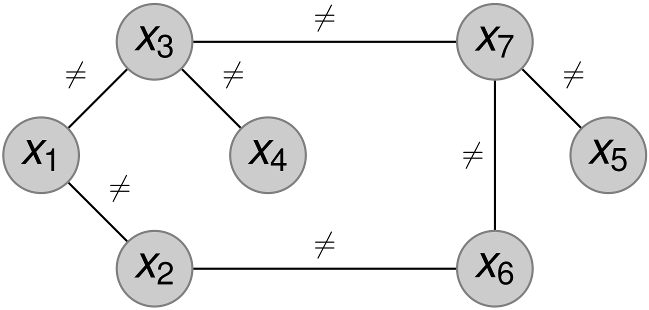
Question

What is the complexity of CSP(Γ) for different Γ?

Graph coloring (two colors)



Domain $D = \{$■, ■$\}$
Constraint language $\Gamma = \{\neq\}$.

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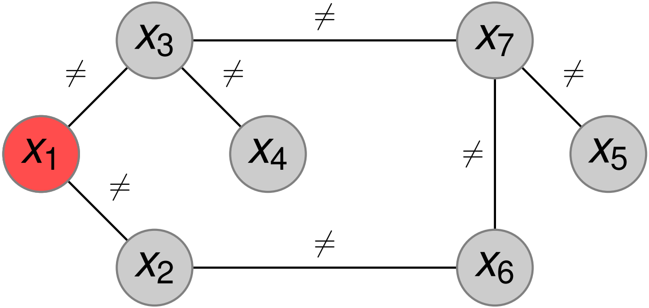

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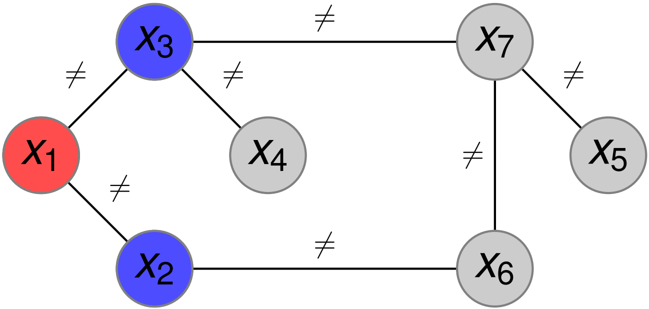

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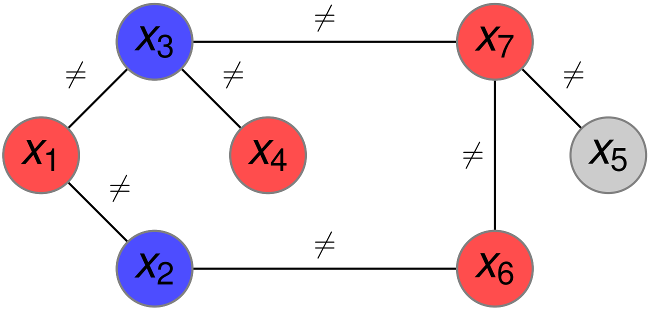

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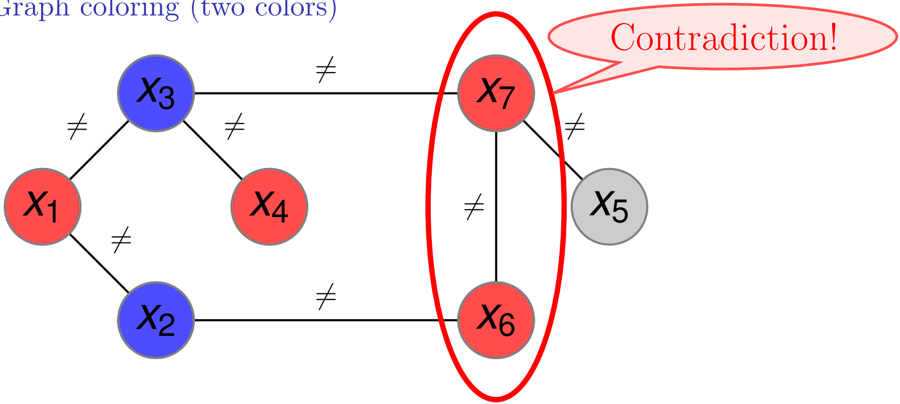

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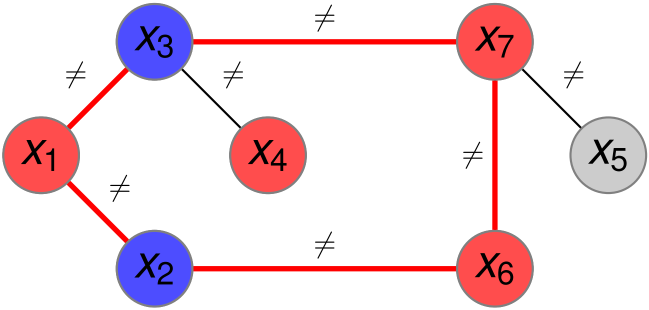

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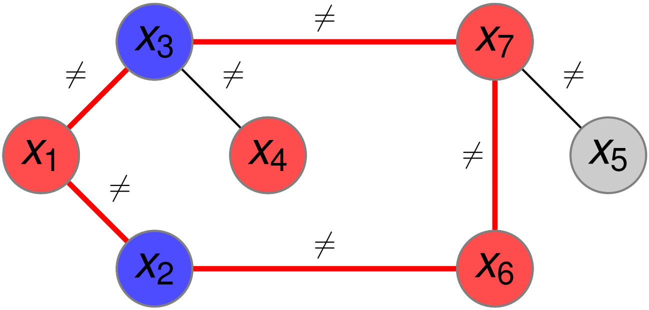

- Either we can color every vertex,
- or we can find an odd cycle.

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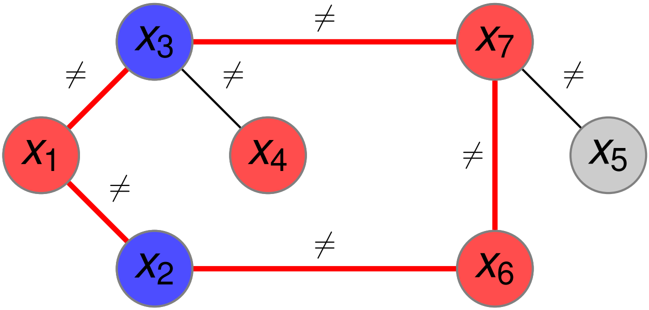

- Either we can color every vertex,
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Local consistency check solves the problem.

Domain $D = \{\blacksquare, \blacksquare\}$

Constraint language $\Gamma = \{\neq\}$.

System of linear equations in a finite field

$$\begin{cases} x_1 + x_2 + 2x_3 = 0 \mod 3 \\ x_1 + 2x_3 + x_5 = 0 \mod 3 \\ 2x_2 + x_4 + x_5 = 0 \mod 3 \\ x_1 + x_3 + 2x_5 = 1 \mod 3 \end{cases}$$

Domain $D = \{0, 1, 2\}$
Constraint language
$\Gamma = \{a_1 x + a_2 y + a_3 z = a_0 \mid a_0, a_1, a_2, a_3 \in D\}$.

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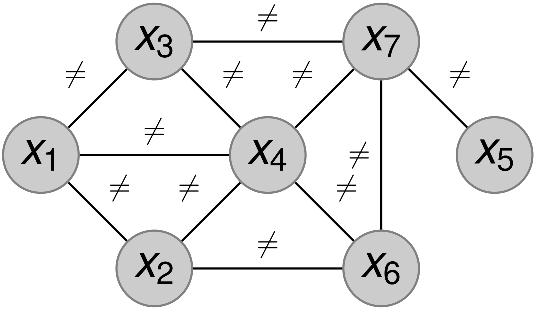

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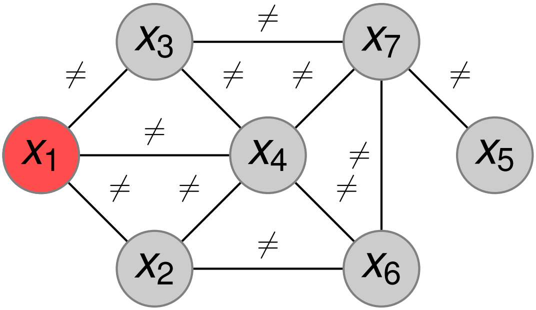

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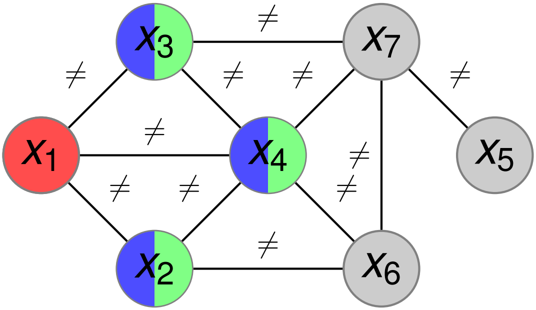

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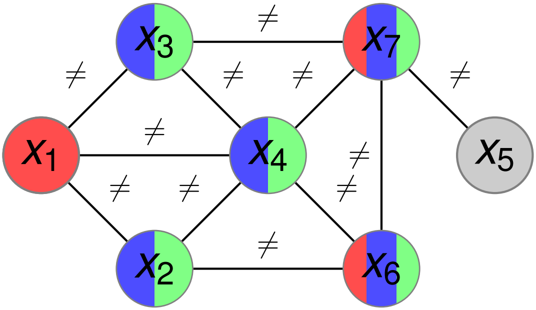

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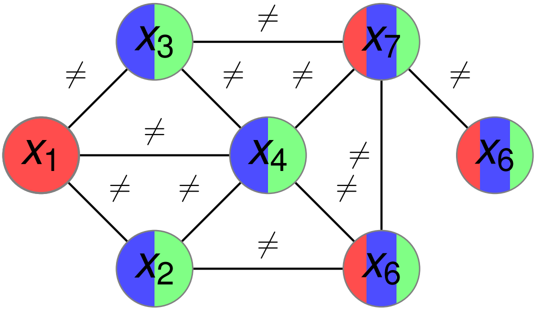

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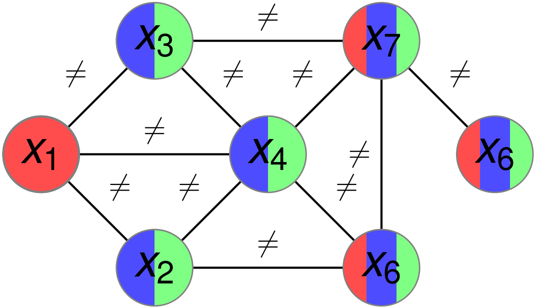

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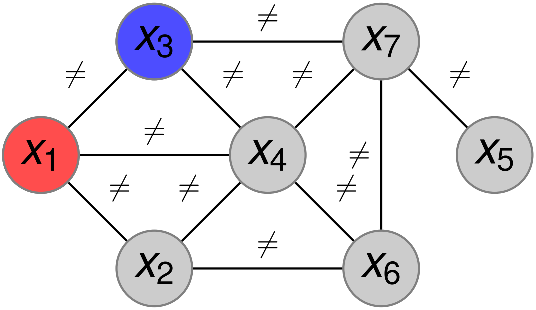

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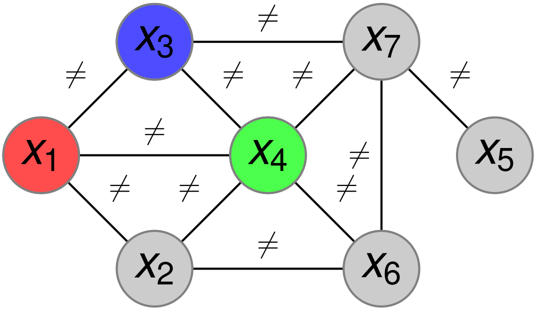

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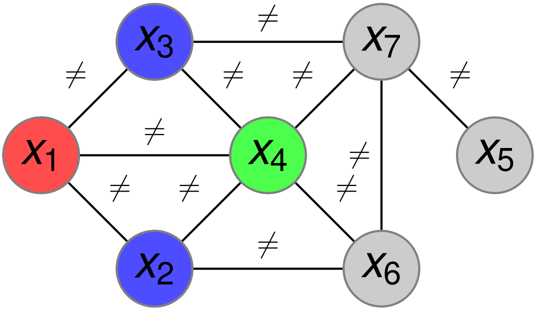

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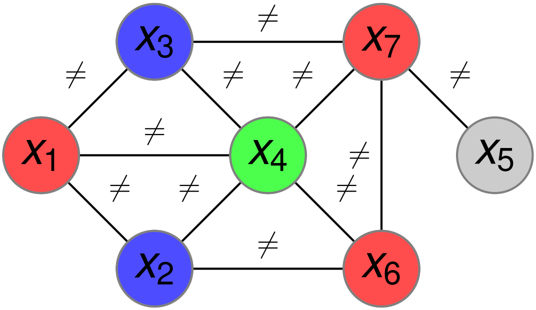

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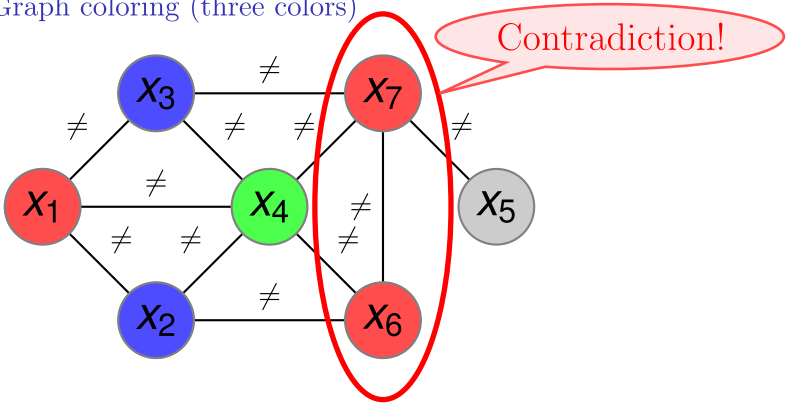

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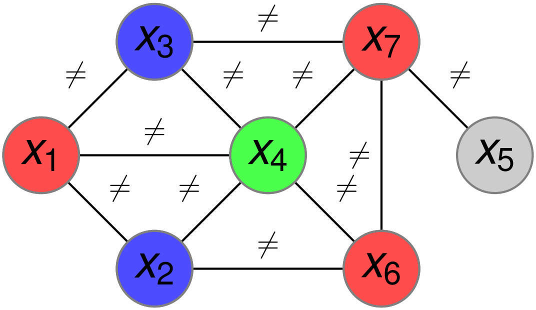

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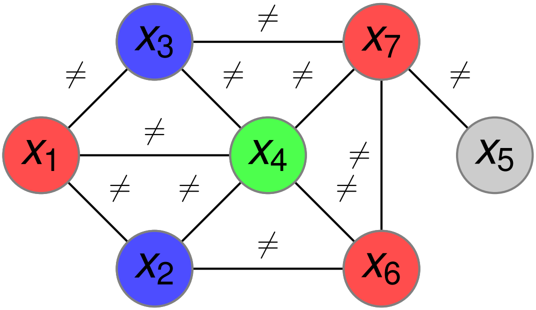

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The problem is NP-hard.

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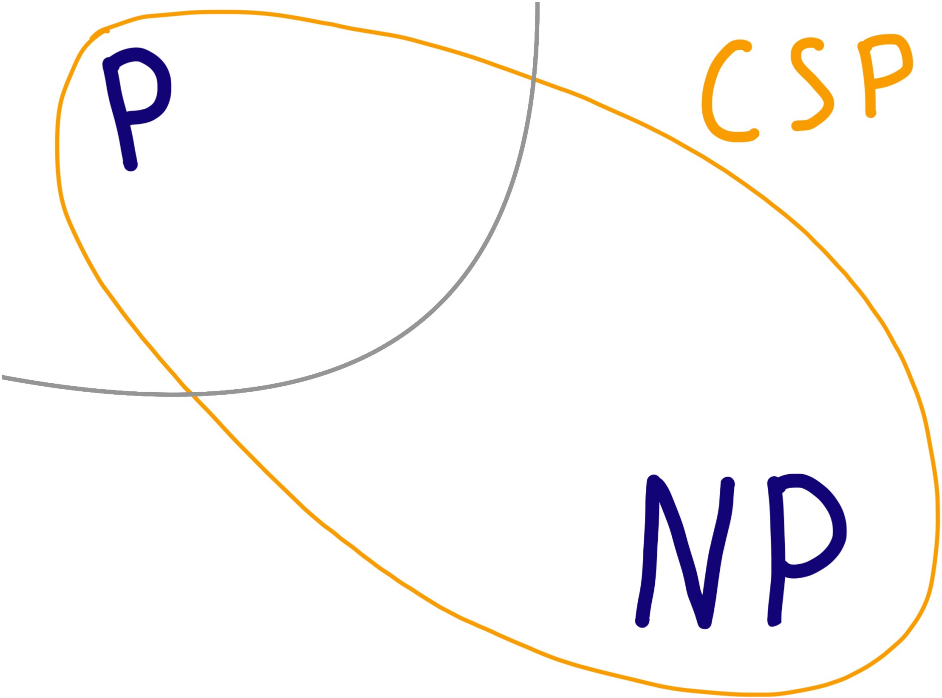
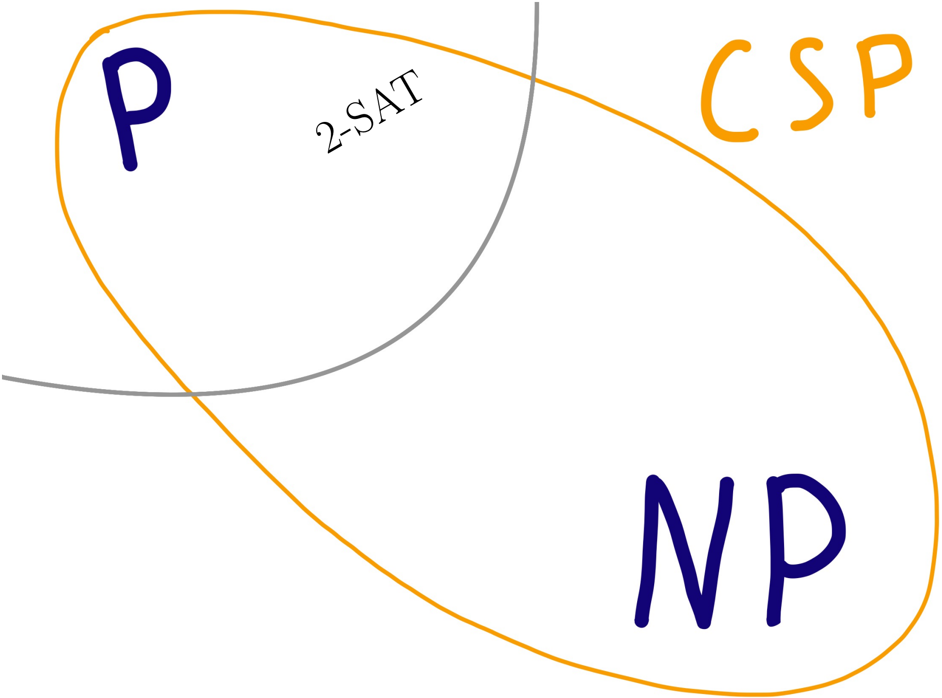
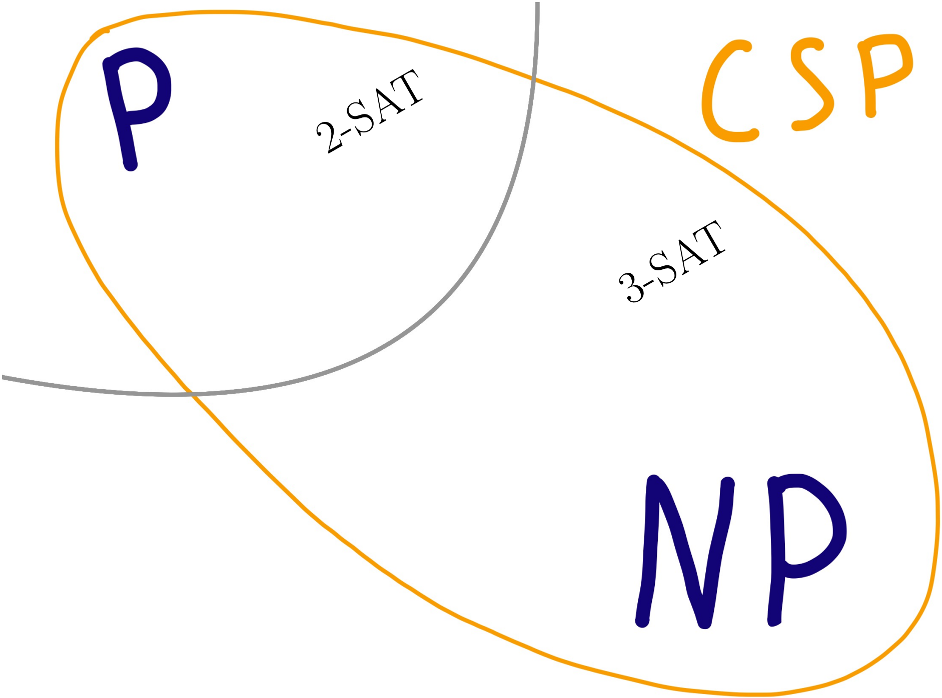
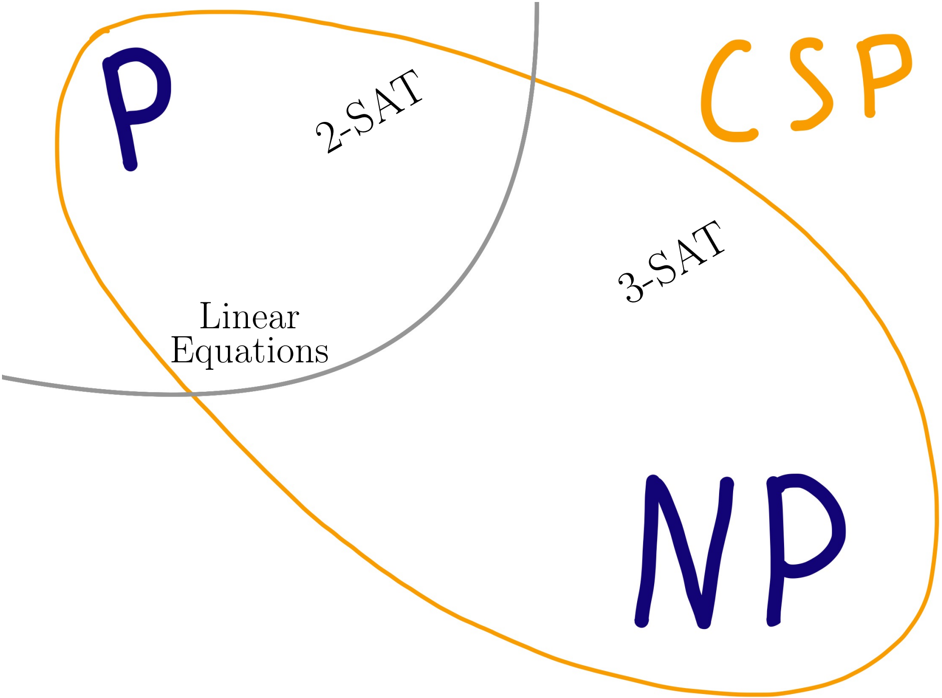
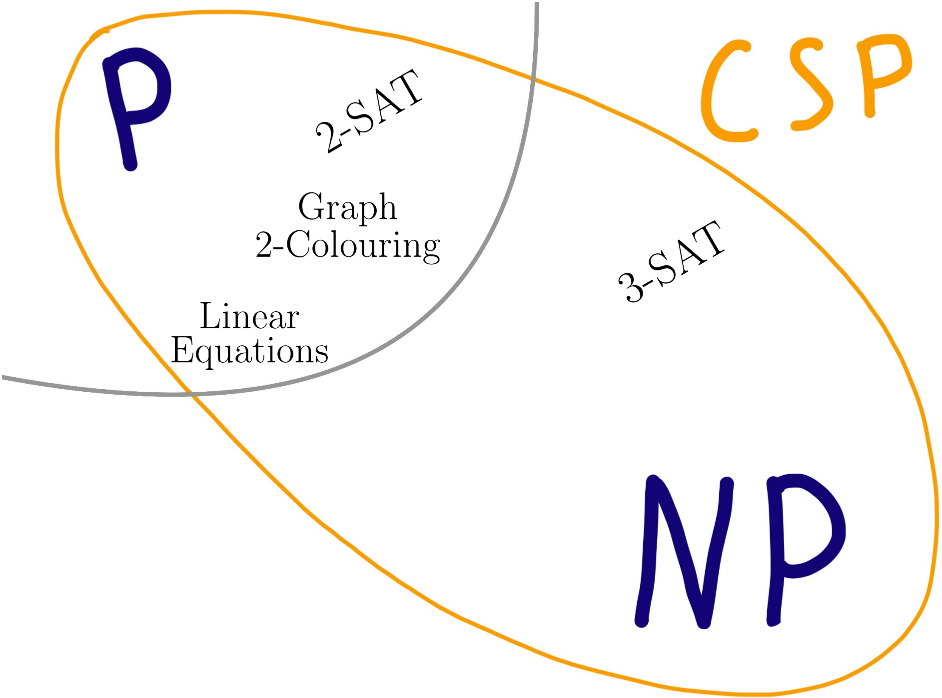
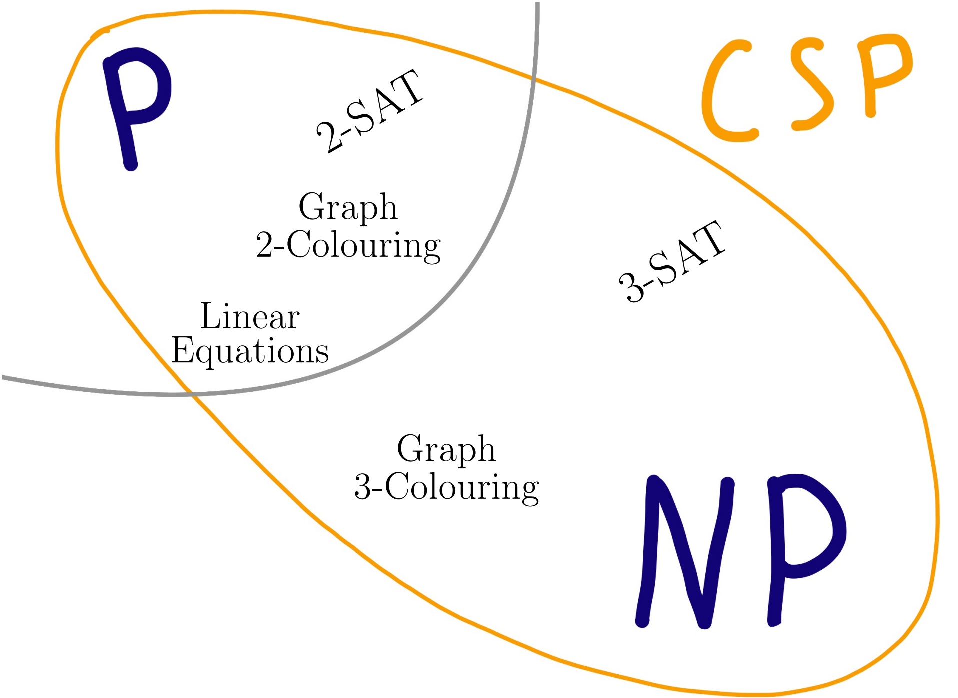

P
CSP
NP


P
2-SAT
CSP
NP


P
CSP
2-SAT
3-SAT
NP


P
2-SAT
CSP
3-SAT
Linear
Equations
NP


P
CSP
2-SAT
Graph
2-Colouring
3-SAT
Linear
Equations
NP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Equations
Graph
3-Colouring
NP

Reduction from one language to another

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CSP(Γ)

Given: a sentence

$$\exists x_1 \dots \exists x_n \, R_1(x_{i_{1,1}}, \dots, x_{i_{1,n_1}}) \wedge \dots \wedge R_s(x_{i_{s,1}}, \dots, x_{i_{s,n_s}}),$$

where $R_1, \dots, R_s \in \Gamma$.
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Fact [Schaefer, 1978]

Suppose $|\Gamma_1| < \infty$, $|\Gamma_2| < \infty$, Γ_2 pp-defines Γ_1. Then CSP(Γ_1) is log-space reducible to CSP(Γ_2).

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Theorem [Bodnarchuk, Kaluzhnin, Kotov, Romov, Geiger, 1969]

Γ_2 pp-defines Γ_1 IFF every operation preserving Γ_2 preserves Γ_1

Polymorphisms

Polymorphisms

An operation f preserves a relation R,
(equivalently, f is a polymorphism of R)

if for all $\begin{pmatrix} a_1^1 \\ \vdots \\ a_1^s \end{pmatrix}, \dots, \begin{pmatrix} a_n^1 \\ \vdots \\ a_n^s \end{pmatrix} \in R$,

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The relation $\leq$ on $\{0, 1, 2\}$

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An operation f preserves $\leq$ iff $f \begin{pmatrix} a_1^1 & \dots & a_n^1 \\ \wedge\!\!\backslash & \ddots & \wedge\!\!\backslash \\ a_1^2 & \dots & a_n^2 \end{pmatrix} = \begin{pmatrix} b^1 \\ \wedge\!\!\backslash \\ b^2 \end{pmatrix}$

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or, equivalently, f is monotone.

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CSP Dichotomy Theorem [Bulatov, Zhuk, 2017]

CSP(Γ) is solvable in polynomial time if there is a WNU operation preserving Γ; it is NP-complete otherwise.

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A weak near unanimity operation (WNU) is an operation f satisfying $f(x, \dots, x, y) = f(x, \dots, x, y, x) = \dots = f(y, x, \dots, x)$.

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Examples: $x \vee y, x \wedge y, xy \vee xz \vee yz, x + y + z, 0, \min(x, y), \ldots$

Hardness

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Suppose Γ is not preserved by a WNU.

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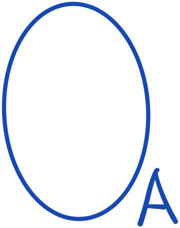

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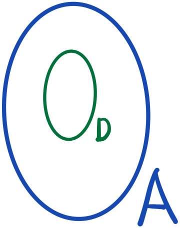

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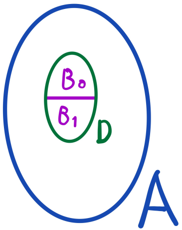

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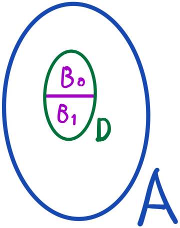
Suppose Γ is not preserved by a WNU. Then there exists $D \subseteq A$, $D = B_0 \sqcup B_1$ s.t. $D^3 \setminus (B_0^3 \cup B_1^3)$ is pp-definable from Γ.



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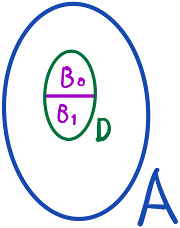

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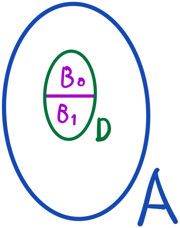

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Suppose Γ is not preserved by a WNU. Then

- CSP(NAE3) is log-space reducible to CSP(Γ)

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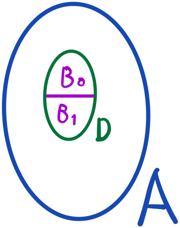

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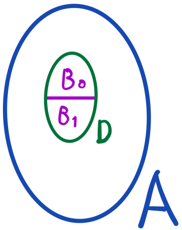

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Suppose Γ is not preserved by a WNU. Then

- CSP(NAE3) is log-space reducible to CSP(Γ)
- CSP(Γ) is NP-complete.
- CSP(Δ) is log-space reducible to CSP(Γ) for any finite constraint language Δ.

Polynomial algorithm

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Tractable cases on $\{0, 1\}$

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$\{0\}$ is a strong subset.

- 2-SAT: $\Gamma = \{R \mid R \subseteq \{0,1\}^2\}$

How to solve: force local consistency.

How to find a solution: set $x_i := 0$ or $x_i := 1$ and force consistency again.

Both $\{0\}$ and $\{1\}$ are strong subsets.

- System of linear equations: $\Gamma = \{a_1 x_1 + \cdots + a_n x_n = a_0 \mid a_0, \ldots, a_n \in \{0,1\}\}$.

How to find a solution: Gaussian elimination.

$B \subseteq D_i$ is called strong if reducing $x_i \in B$ does not kill all the solution for any consistent enough instance of CSP.

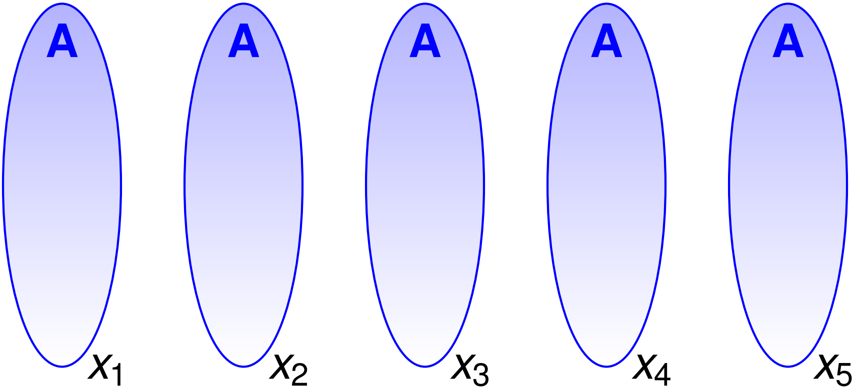
Polynomial algorithm

Polynomial algorithm

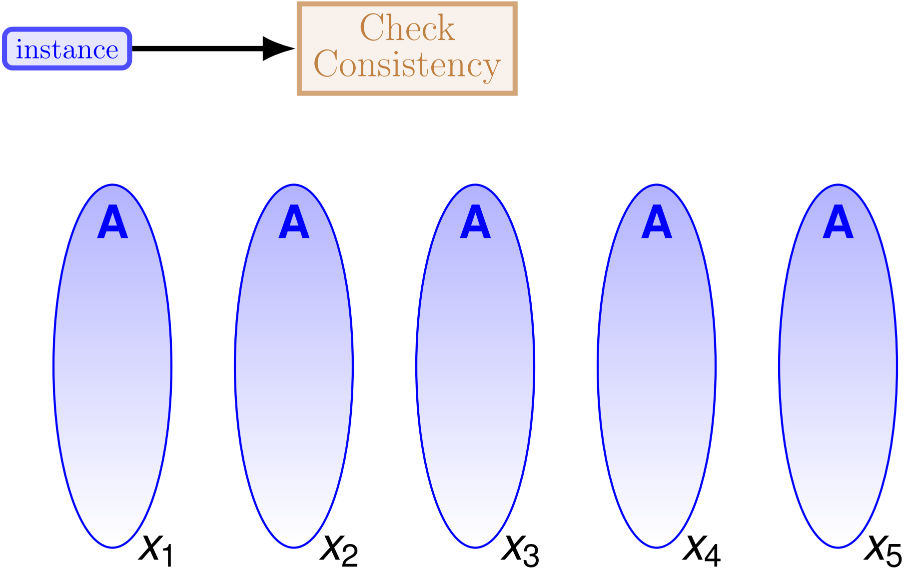
$\boxed{\text{instance}}$

Polynomial algorithm

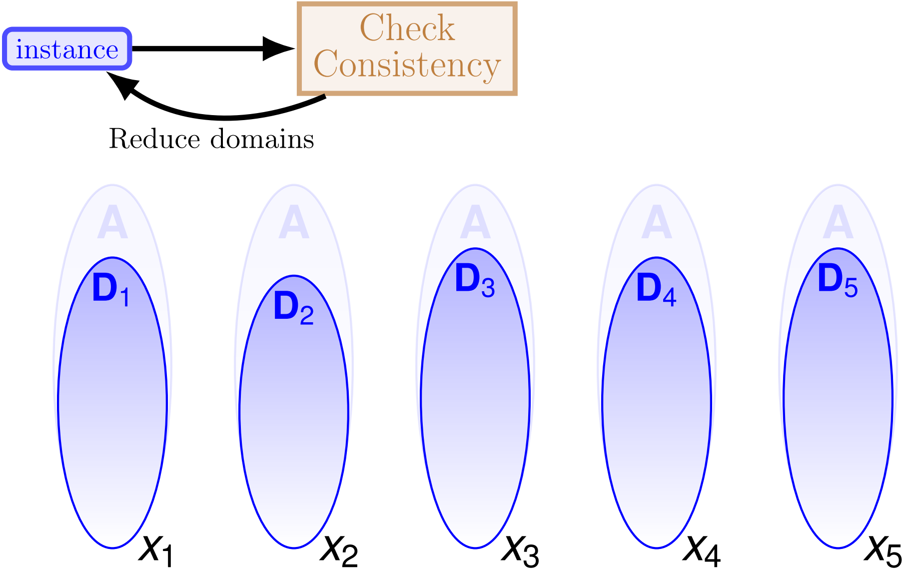
instance



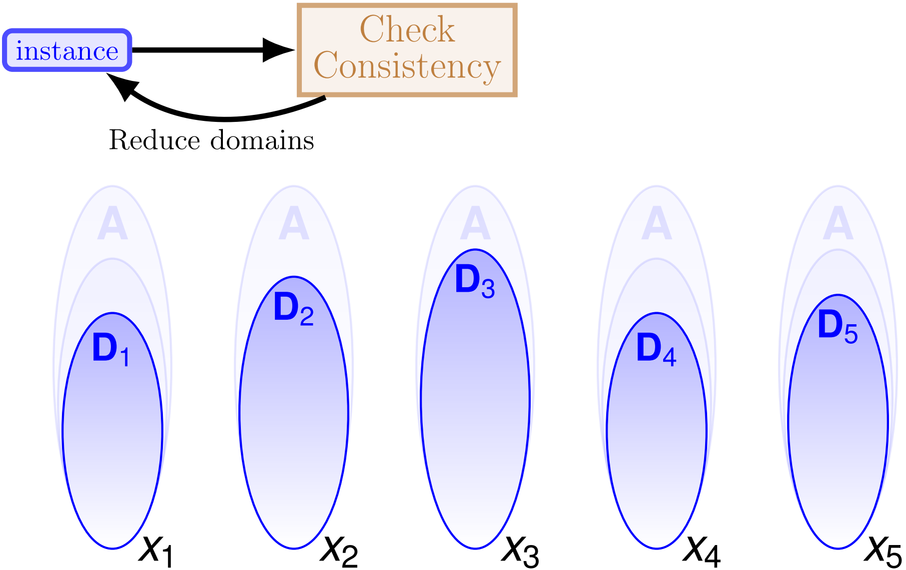
Polynomial algorithm



Polynomial algorithm



Polynomial algorithm



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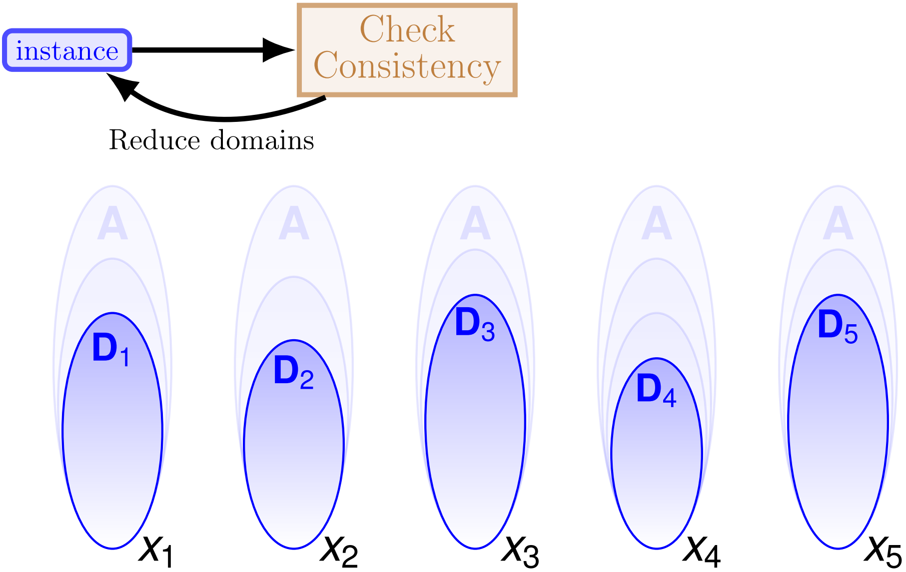
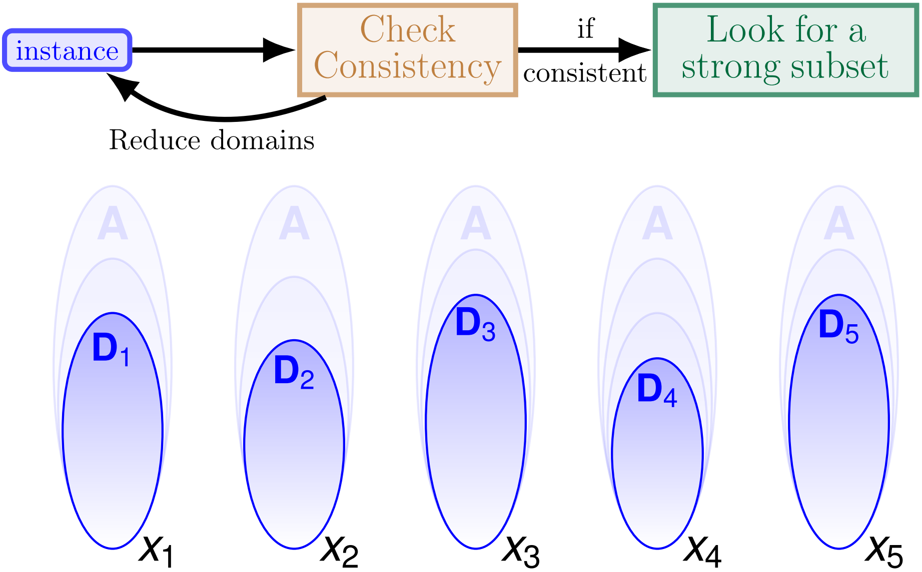
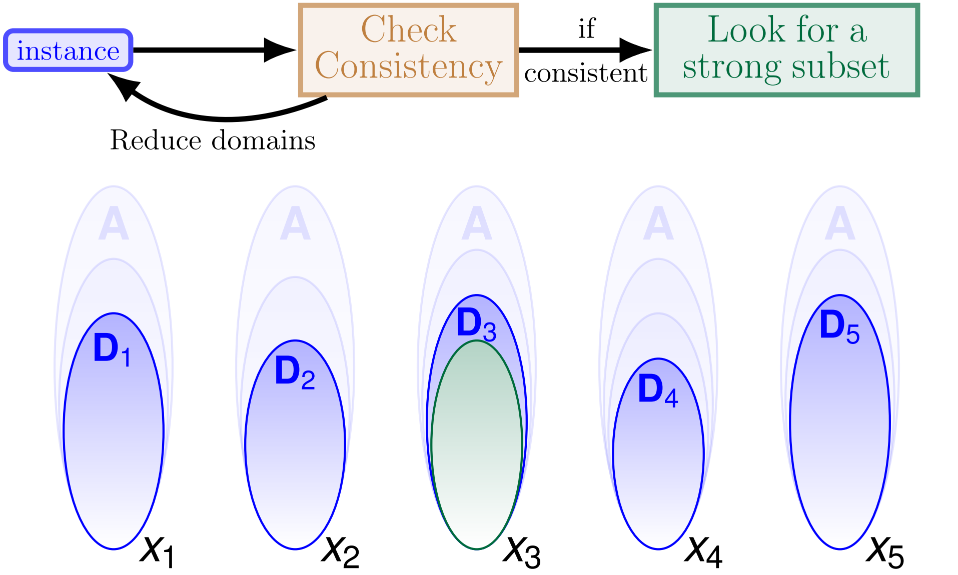
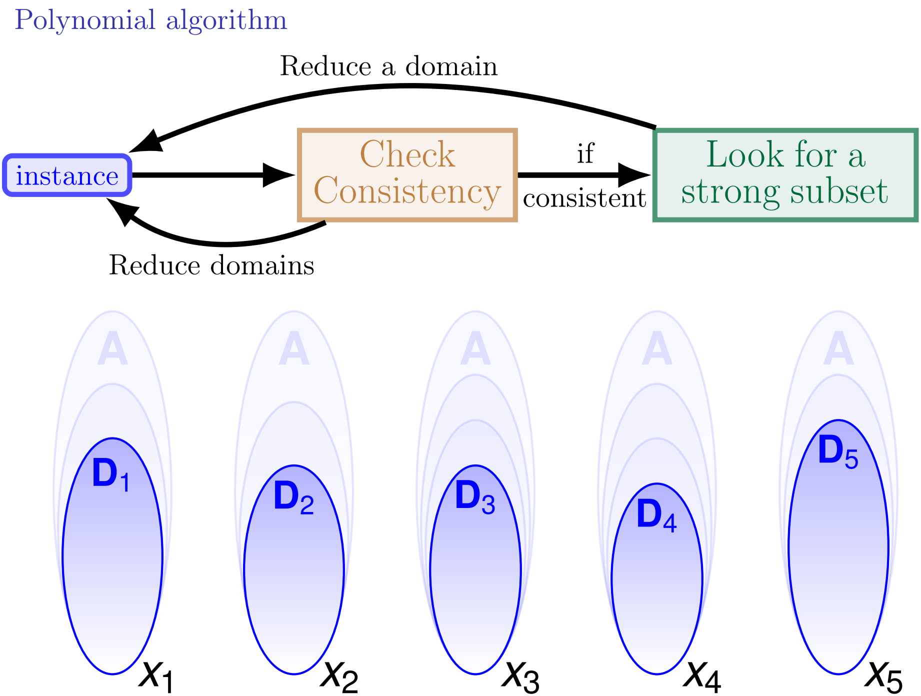


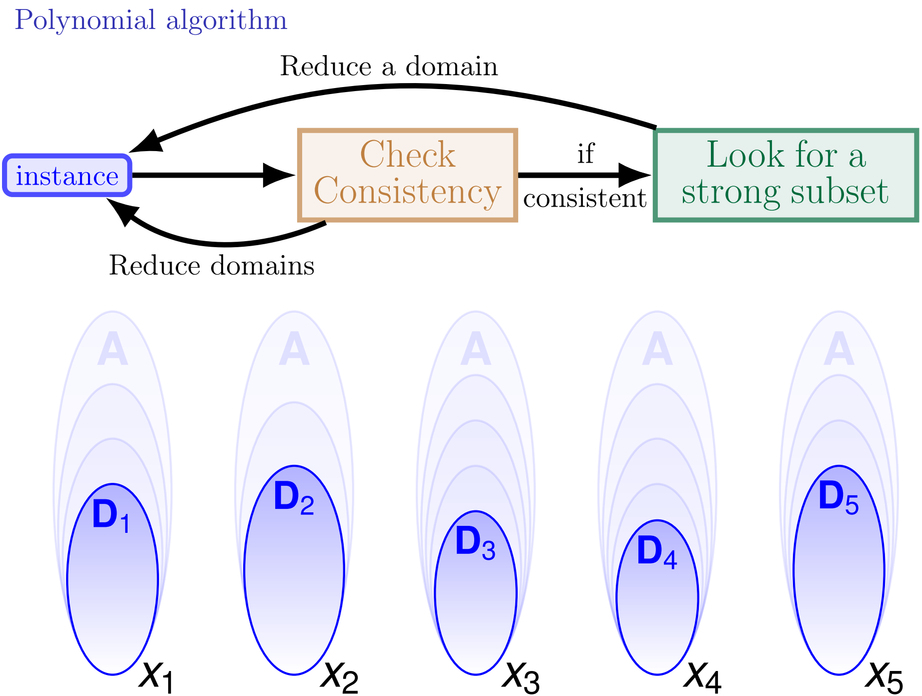
Polynomial algorithm



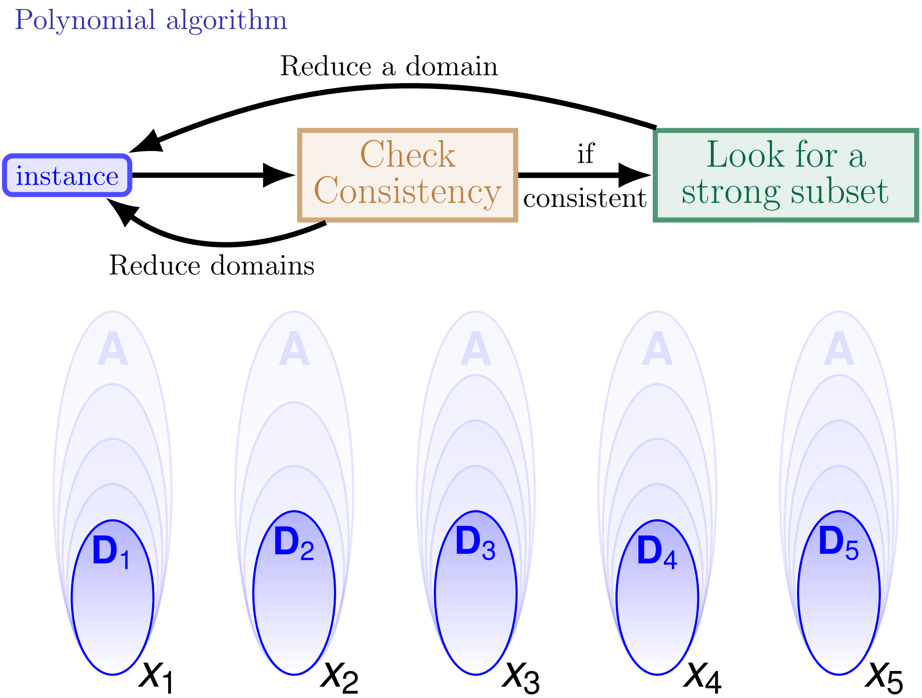
Polynomial algorithm




Polynomial algorithm
Reduce a domain
instance
Check Consistency
if consistent
Look for a strong subset
Reduce domains
A
D1
D2
D3
D4
D5
x1
x2
x3
x4
x5



Polynomial algorithm
Reduce a domain
instance
Check Consistency
if consistent
Look for a strong subset
Reduce domains
A
A
A
A
A
D_1
D_2
D_3
D_4
D_5
x_1
x_2
x_3
x_4
x_5


Polynomial algorithm
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Check Consistency
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A
A
A
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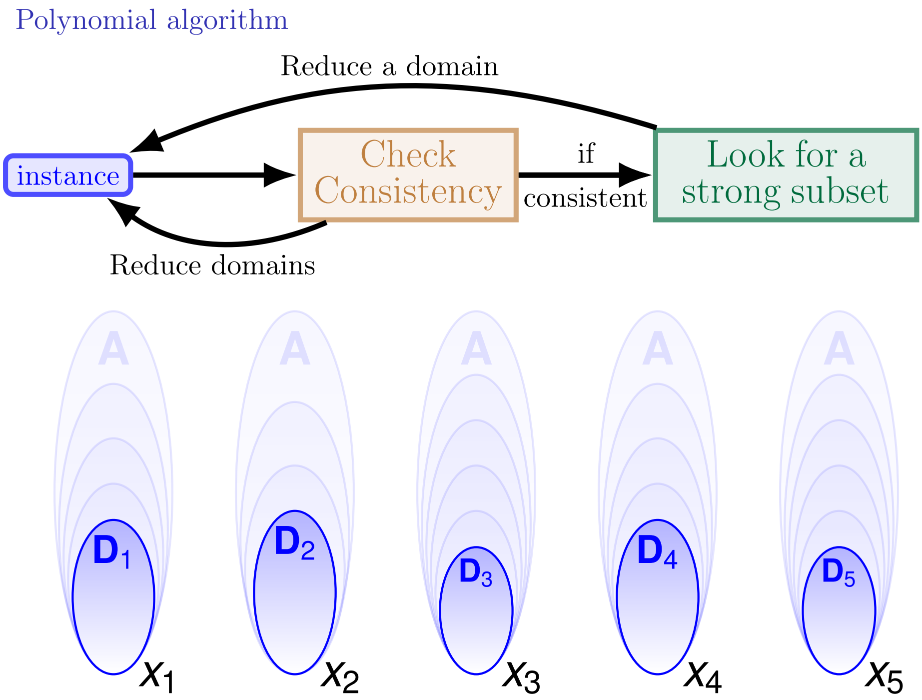
D_1
D_2
D_3
D_4
D_5
x_1
x_2
x_3
x_4
x_5


Polynomial algorithm
Reduce a domain
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Check Consistency
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Look for a strong subset
Reduce domains
A
A
A
A
A
D1
D2
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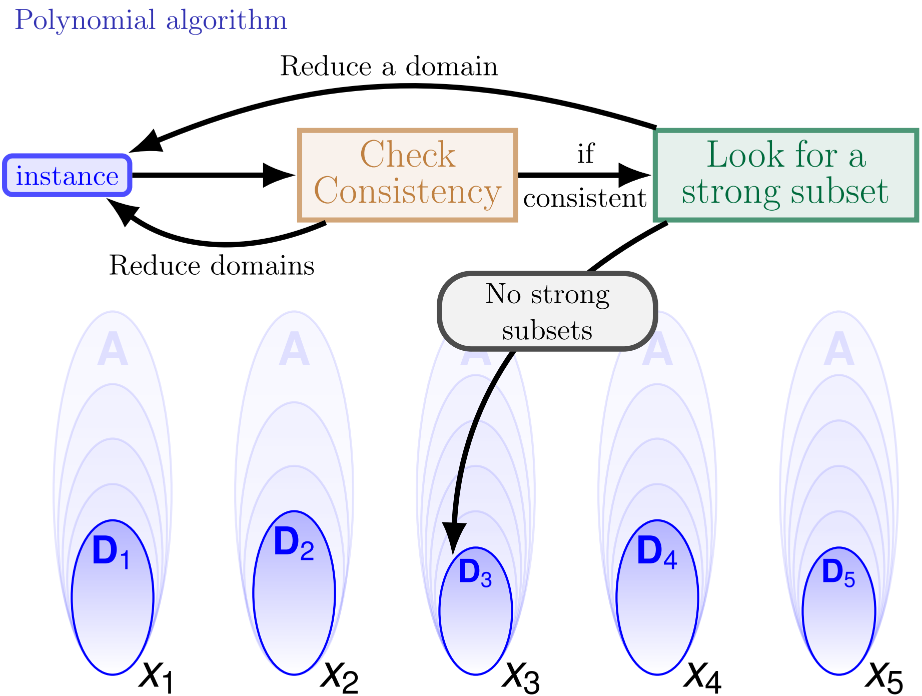
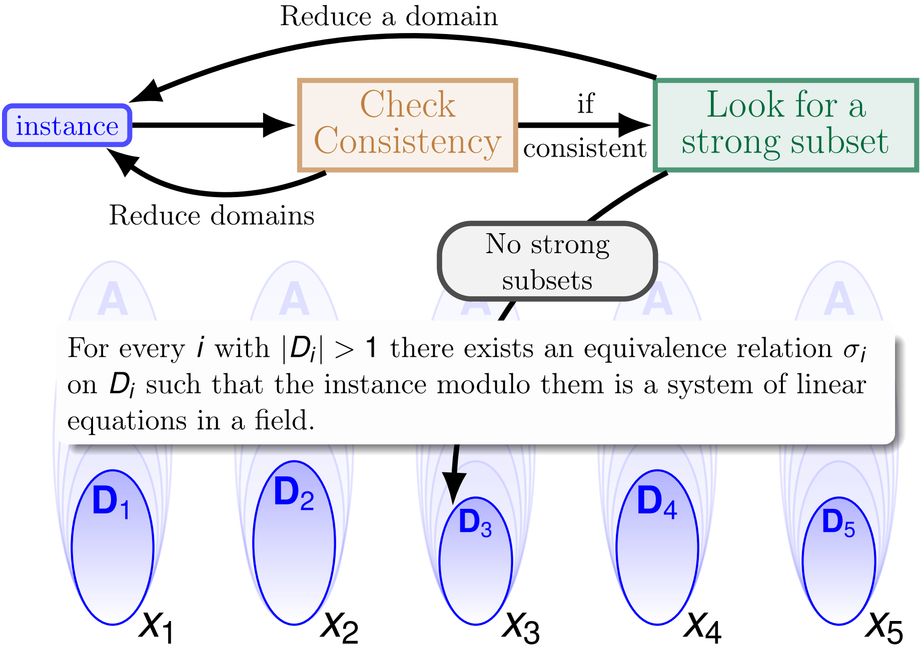
x1
x2
x3
x4
x5

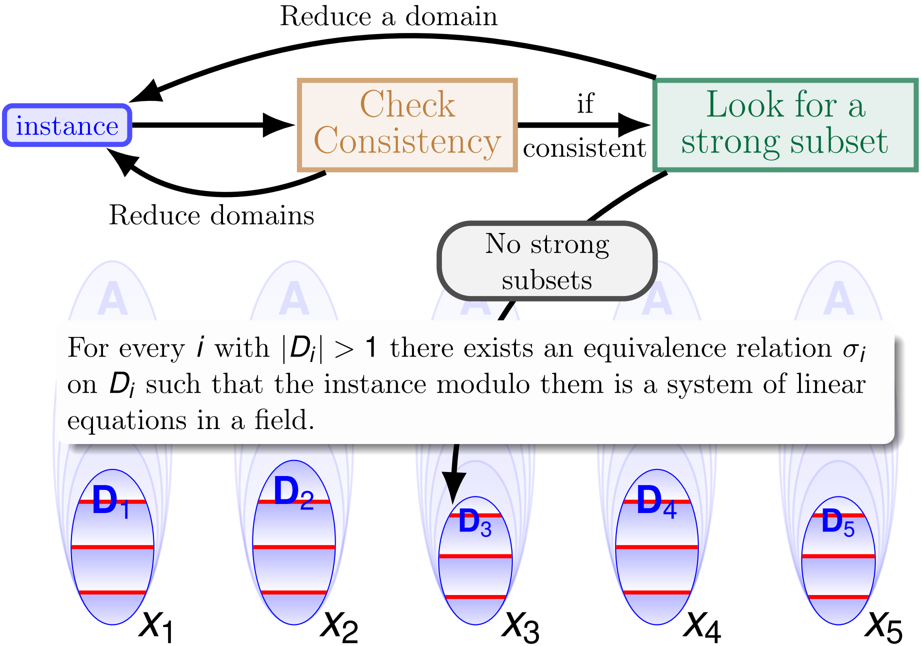

Polynomial algorithm
Reduce a domain
instance
Check Consistency
if consistent
Look for a strong subset
Reduce domains
No strong subsets
A
A
A
A
A
D_1
D_2
D_3
D_4
D_5
x_1
x_2
x_3
x_4
x_5

Polynomial algorithm



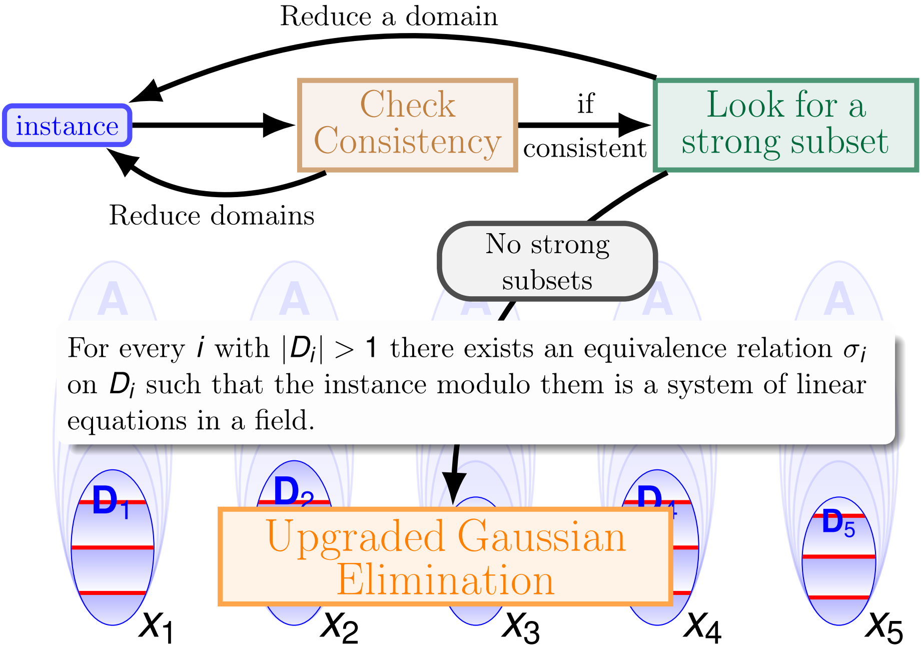
For every i with $|D_i| > 1$ there exists an equivalence relation σ_i on D_i such that the instance modulo them is a system of linear equations in a field.

Polynomial algorithm



For every i with $|D_i| > 1$ there exists an equivalence relation σ_i on D_i such that the instance modulo them is a system of linear equations in a field.

Polynomial algorithm

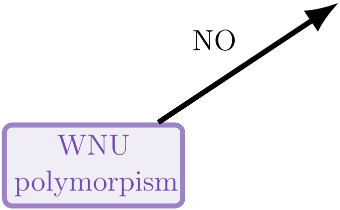


To sum up

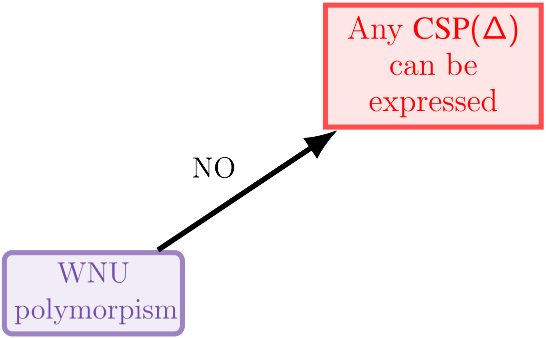
To sum up

WNU
polymorpism

To sum up



To sum up



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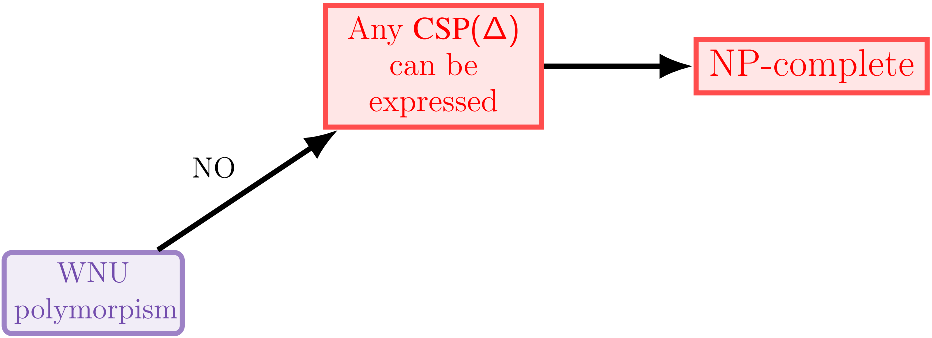
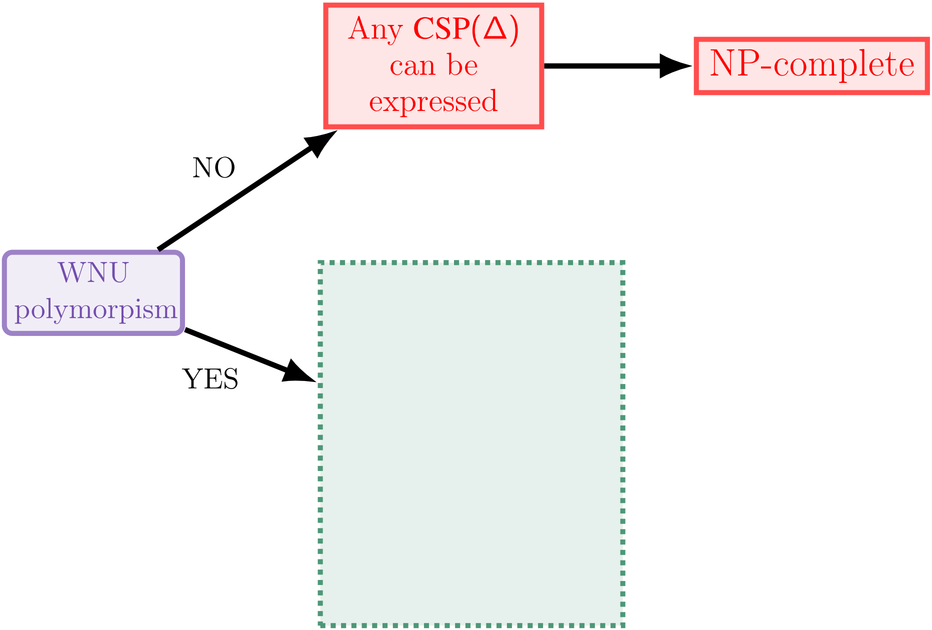
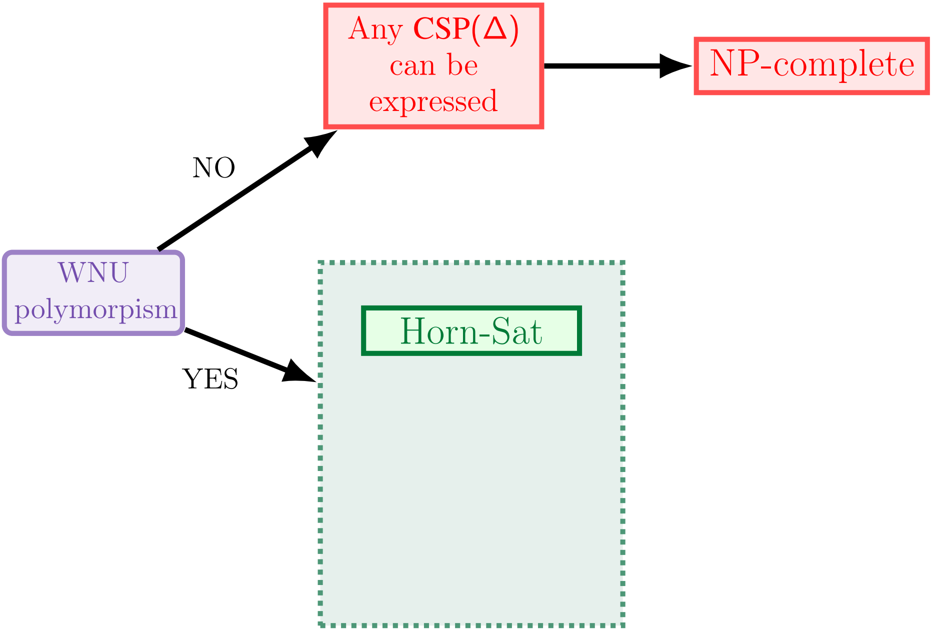


To sum up

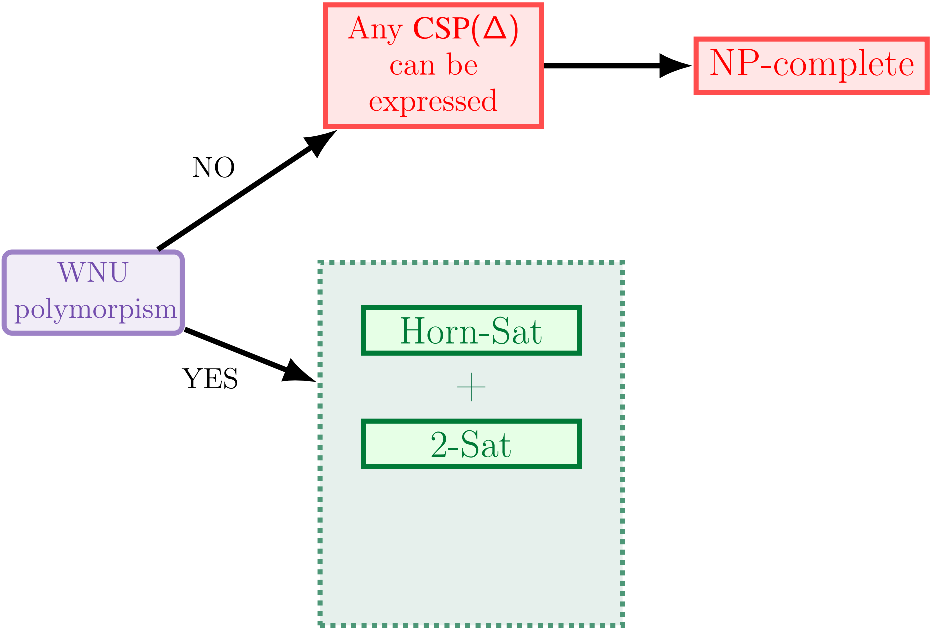

Any $\mathbf{CSP}(\Delta)$ can be expressed
NP-complete
NO
WNU polymorpism
YES

To sum up



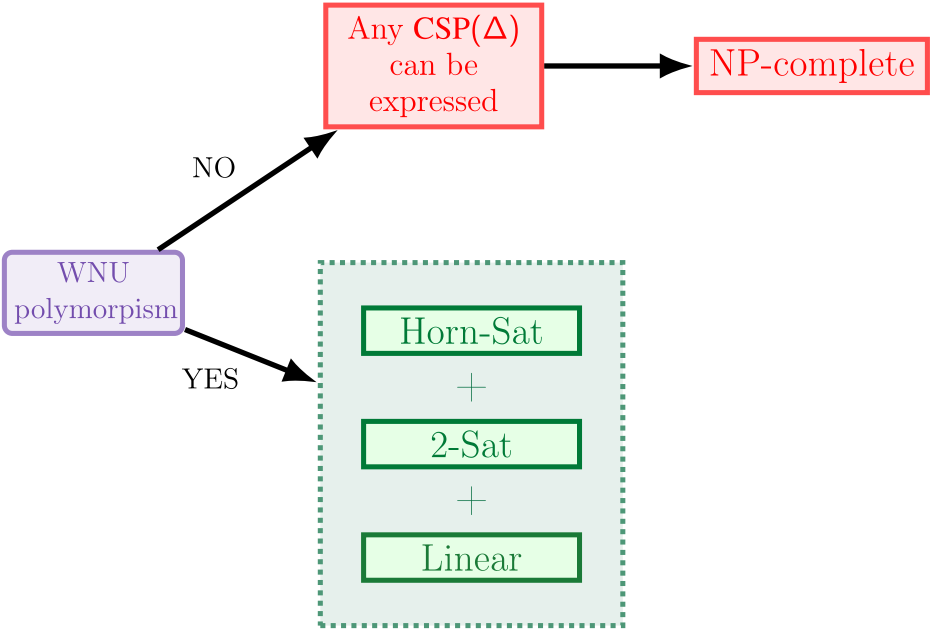
Any **CSP**(Δ) can be expressed
NP-complete
NO
WNU polymorpism
YES
Horn-Sat

To sum up



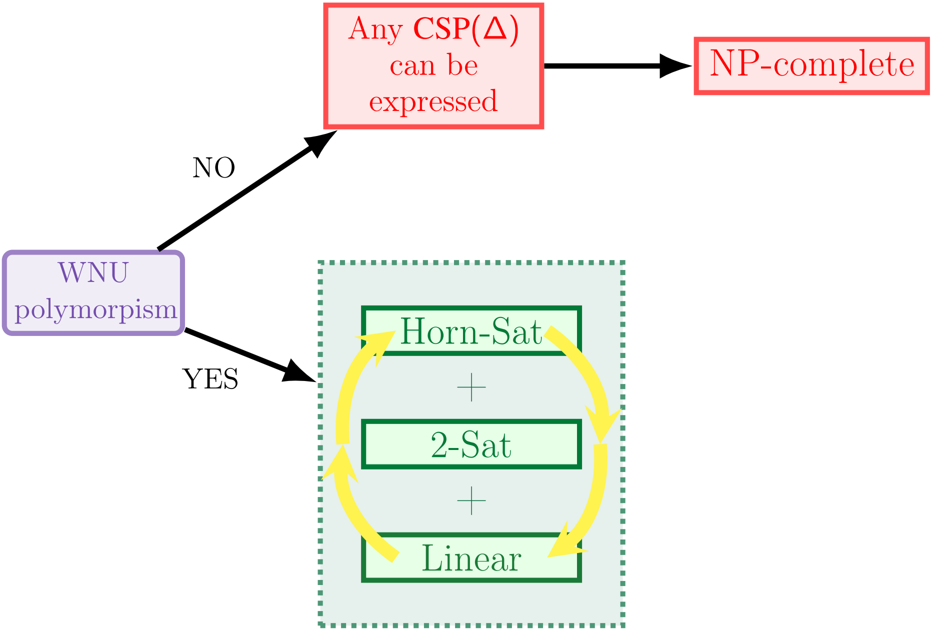
Any $\mathbf{CSP}(\Delta)$ can be expressed
NP-complete
NO
WNU polymorpism
YES
Horn-Sat
+
2-Sat

To sum up



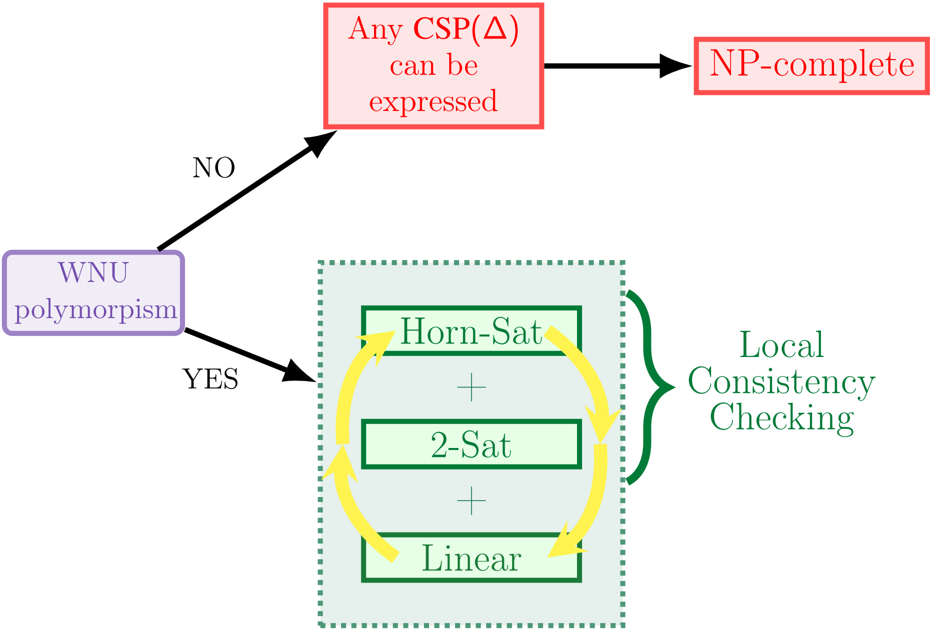
Any **CSP**(Δ) can be expressed
NP-complete
NO
WNU polymorpism
YES
Horn-Sat
+
2-Sat
+
Linear

To sum up

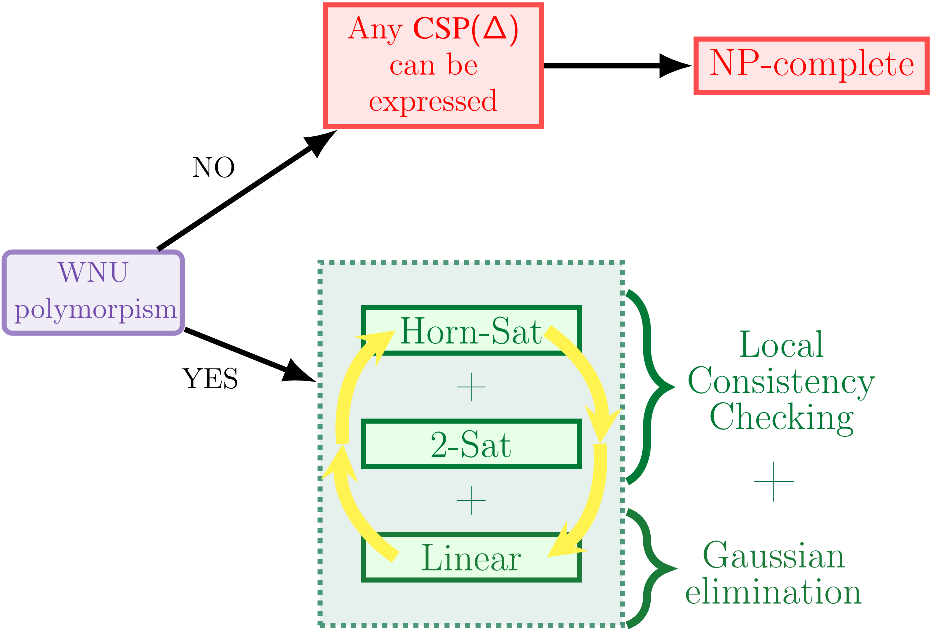


Any $\mathbf{CSP}(\Delta)$
can be
expressed
NP-complete
NO
WNU
polymorpism
YES
Horn-Sat
+
2-Sat
+
Linear

To sum up


Any **CSP(Δ)** can be expressed
NP-complete
NO
WNU polymorpism
YES
Horn-Sat
+
2-Sat
+
Linear
Local Consistency Checking

To sum up



Any CSP(Δ) can be expressed
NP-complete
NO
WNU polymorpism
YES
Horn-Sat
+
2-Sat
+
Linear
Local Consistency Checking
+
Gaussian elimination

		CSP					
Domain	finite						
	infinite						

		CSP					
Domain	finite						
	infinite						

		CSP					
Domain	finite						
	infinite						

Full classification

		CSP					
Domain	finite						
	infinite						

Full classification

Infinite Domain CSP

Infinite Domain CSP

Γ is a set of relations on $\mathbb{Q}$.

Infinite Domain CSP

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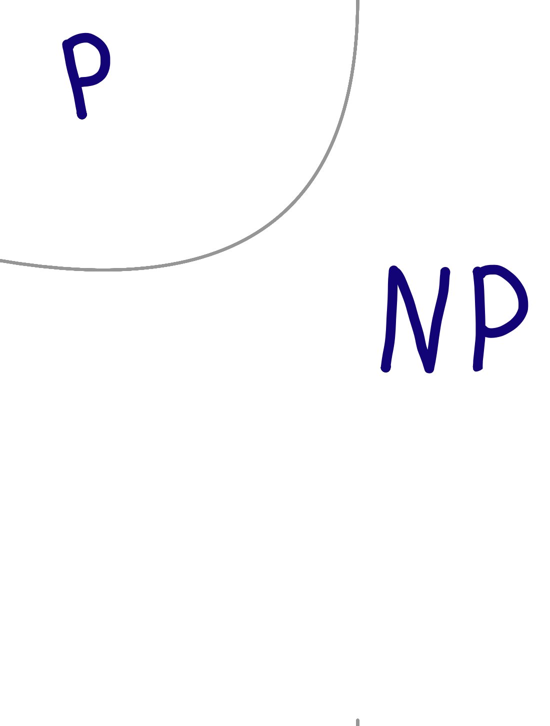
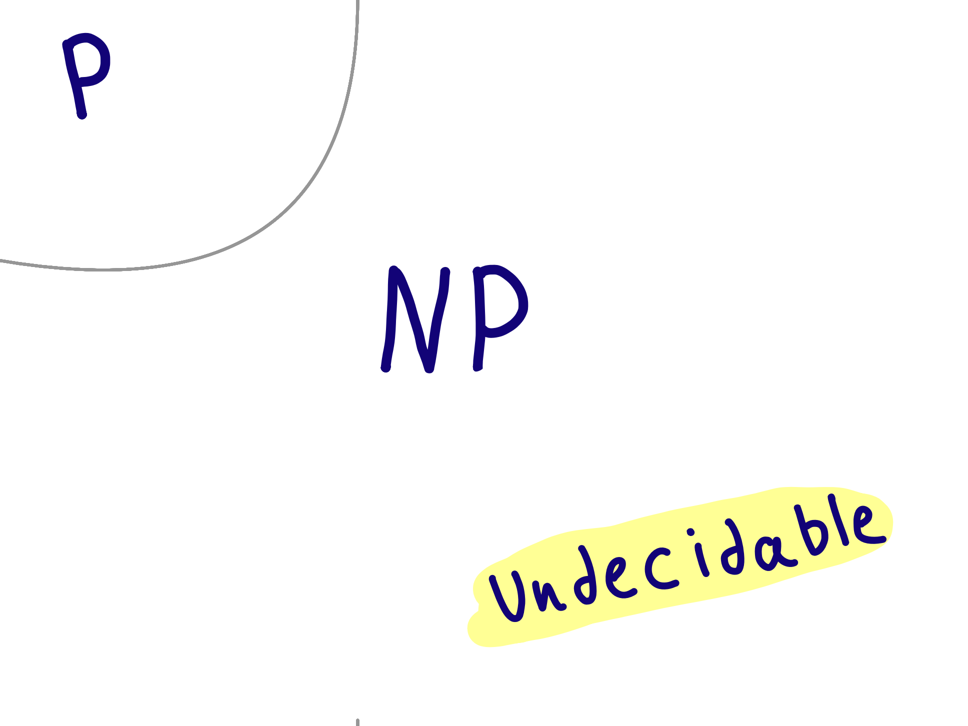
CSP(Γ)

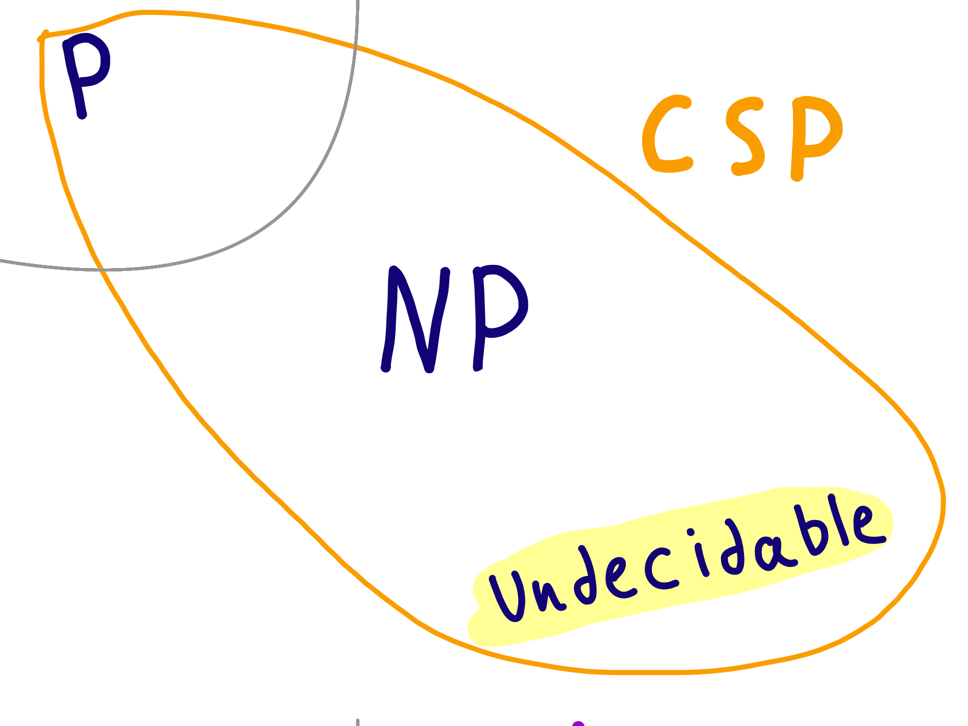
Given: a conjunction of relations, i.e. a formula

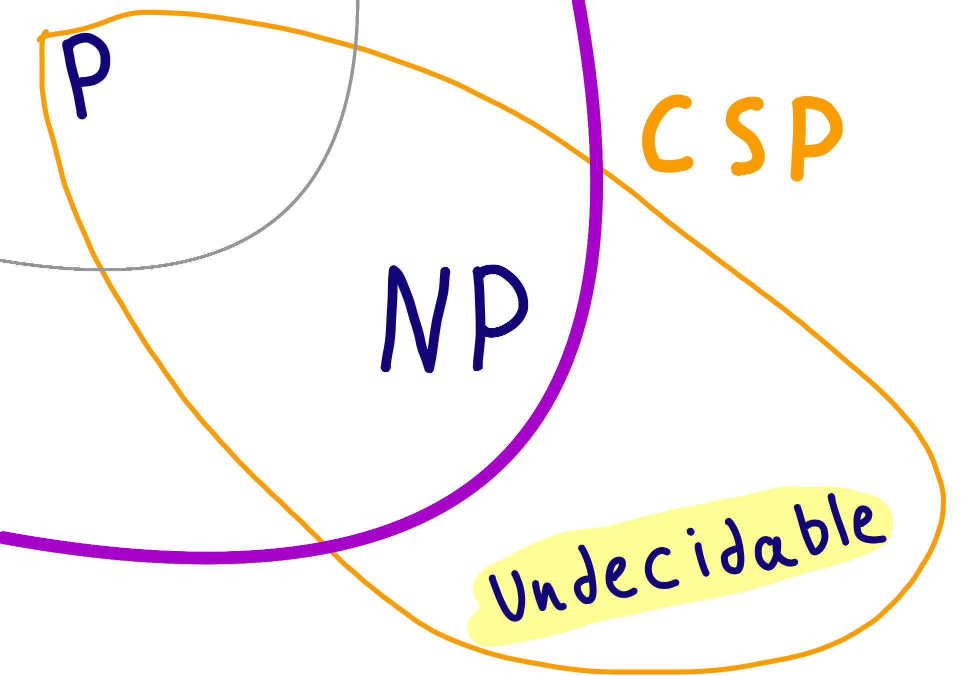
$$R_1(x_{i_{1,1}}, \ldots, x_{i_{1,n_1}}) \wedge \cdots \wedge R_s(x_{i_{s,1}}, \ldots, x_{i_{s,n_s}}),$$

where $R_1, \ldots, R_s \in \Gamma$.
Decide: whether the formula is satisfiable.


P
NP


P
NP
Undecidable


P
CSP
NP
Undecidable


P
NP
CSP
Undecidable

Infinite Domain CSP

Γ is a set of relations on $\mathbb{Q}$.

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Examples

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Examples

1. CSP($\{x < y\}$)

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Examples

1. CSP($\{x < y\}$)

CSP instances:

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where $R_1, \ldots, R_s \in \Gamma$.
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Examples

1. CSP($\{x < y\}$)

CSP instances:
$x_1 < x_2 \wedge x_2 < x_3 \wedge x_1 < x_3$,

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where $R_1, \ldots, R_s \in \Gamma$.
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Examples

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$x_1 < x_2 \wedge x_2 < x_3 \wedge x_1 < x_3$, has a solution

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Examples

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CSP instances:
$x_1 < x_2 \wedge x_2 < x_3 \wedge x_1 < x_3$, has a solution
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Examples

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CSP instances:
$x_1 < x_2 \wedge x_2 < x_3 \wedge x_1 < x_3$, has a solution
$x_1 < x_2 \wedge x_2 < x_3 \wedge x_3 < x_1$, has no solutions

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where $R_1, \ldots, R_s \in \Gamma$.
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Examples

1. CSP($\{x < y\}$)

CSP instances:
$x_1 < x_2 \wedge x_2 < x_3 \wedge x_1 < x_3$, has a solution
$x_1 < x_2 \wedge x_2 < x_3 \wedge x_3 < x_1$, has no solutions
The instance has a solution IFF there is no oriented cycle.

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CSP(Γ)

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$$R_1(x_{i_{1,1}}, \ldots, x_{i_{1,n_1}}) \wedge \cdots \wedge R_s(x_{i_{s,1}}, \ldots, x_{i_{s,n_s}}),$$

where $R_1, \ldots, R_s \in \Gamma$.
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Examples

1. CSP($\{x < y\}$) is in P.

CSP instances:
$x_1 < x_2 \wedge x_2 < x_3 \wedge x_1 < x_3$, has a solution
$x_1 < x_2 \wedge x_2 < x_3 \wedge x_3 < x_1$, has no solutions
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where $R_1, \ldots, R_s \in \Gamma$.
Decide: whether the formula is satisfiable.

Examples

1. CSP($\{x < y\}$) is in P.
2. CSP($\{x < y < z \vee z < y < x\}$)

Infinite Domain CSP

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CSP(Γ)

Given: a conjunction of relations, i.e. a formula

$$R_1(x_{i_{1,1}}, \ldots, x_{i_{1,n_1}}) \wedge \cdots \wedge R_s(x_{i_{s,1}}, \ldots, x_{i_{s,n_s}}),$$

where $R_1, \ldots, R_s \in \Gamma$.
Decide: whether the formula is satisfiable.

Examples

1. CSP($\{x < y\}$) is in P.
2. CSP($\{x < y < z \vee z < y < x\}$) is NP-complete.

Infinite Domain CSP

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CSP(Γ)

Given: a conjunction of relations, i.e. a formula

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where $R_1, \ldots, R_s \in \Gamma$.
Decide: whether the formula is satisfiable.

Examples

1. CSP($\{x < y\}$) is in P.
2. CSP($\{x < y < z \vee z < y < x\}$) is NP-complete.
3. CSP($\{x = y < z \vee x = z < y \vee y = z < x\}$)

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CSP(Γ)

Given: a conjunction of relations, i.e. a formula

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where $R_1, \ldots, R_s \in \Gamma$.
Decide: whether the formula is satisfiable.

Examples

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2. CSP($\{x < y < z \vee z < y < x\}$) is NP-complete.
3. CSP($\{x = y < z \vee x = z < y \vee y = z < x\}$) is in P.

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where $R_1, \ldots, R_s \in \Gamma$.
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Examples

1. CSP($\{x < y\}$) is in P.
2. CSP($\{x < y < z \vee z < y < x\}$) is NP-complete.
3. CSP($\{x = y < z \vee x = z < y \vee y = z < x\}$) is in P.
4. CSP($\{x = y < z \vee x = z < y\}$)

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where $R_1, \ldots, R_s \in \Gamma$.
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Examples

1. CSP($\{x < y\}$) is in P.
2. CSP($\{x < y < z \vee z < y < x\}$) is NP-complete.
3. CSP($\{x = y < z \vee x = z < y \vee y = z < x\}$) is in P.
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where $R_1, \dots, R_s \in \Gamma$.
Decide: whether the formula is satisfiable.

Examples

1. CSP($\{x < y\}$) is in P.
2. CSP($\{x < y < z \vee z < y < x\}$) is NP-complete.
3. CSP($\{x = y < z \vee x = z < y \vee y = z < x\}$) is in P.
4. CSP($\{x = y < z \vee x = z < y\}$) is NP-complete.
5. CSP($\{x = y < z \vee x = z < y \vee y = z < x, x = y + 1\}$)

Infinite Domain CSP

Γ is a set of relations on $\mathbb{Q}$.

CSP(Γ)

Given: a conjunction of relations, i.e. a formula

$$R_1(x_{i_{1,1}}, \ldots, x_{i_{1,n_1}}) \wedge \cdots \wedge R_s(x_{i_{s,1}}, \ldots, x_{i_{s,n_s}}),$$

where $R_1, \ldots, R_s \in \Gamma$.
Decide: whether the formula is satisfiable.

Examples

1. CSP($\{x < y\}$) is in P.
2. CSP($\{x < y < z \vee z < y < x\}$) is NP-complete.
3. CSP($\{x = y < z \vee x = z < y \vee y = z < x\}$) is in P.
4. CSP($\{x = y < z \vee x = z < y\}$) is NP-complete.
5. CSP($\{x = y < z \vee x = z < y \vee y = z < x, x = y + 1\}$) is NP-complete.

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Classification for temporal constraint languages [Bodirsky, Kára, 2008]

A full classification of the complexity for constraint languages admitting a first-order definition in$(\mathbb{Q}; <)$ (P vs NP-complete).

		CSP					
Domain	finite						
	infinite						

Full classification

		CSP					
Domain	finite						
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Full classification

Some classifications

		CSP	Quantified CSP				
Domain	finite						
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Full classification

Some classifications

Quantified Constraint Satisfaction Problem

Γ is a set of relations on a finite set A.

CSP(Γ)

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Decide: whether it holds.

Quantified Constraint Satisfaction Problem

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Given: a sentence

$$\exists y_1 \forall x_1 \ldots \exists y_t \forall x_t (R_1(\ldots) \wedge \cdots \wedge R_s(\ldots)),$$

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Decide: whether it holds.

Examples:
$A = \{0, 1, 2\}, \Gamma = \{x \neq y\}$. QCSP instances:

$\forall x \exists y_1 \exists y_2 (x \neq y_1 \wedge x \neq y_2 \wedge y_1 \neq y_2),$

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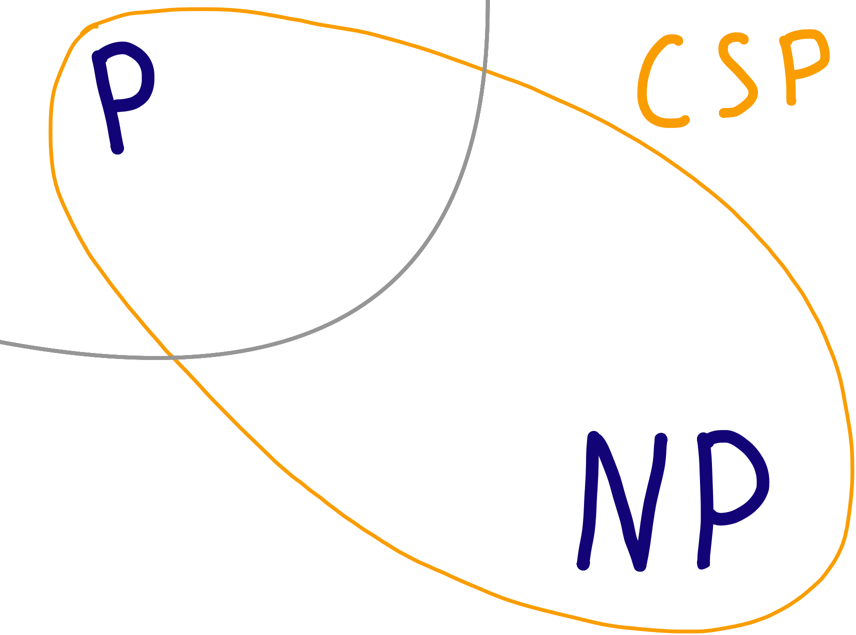
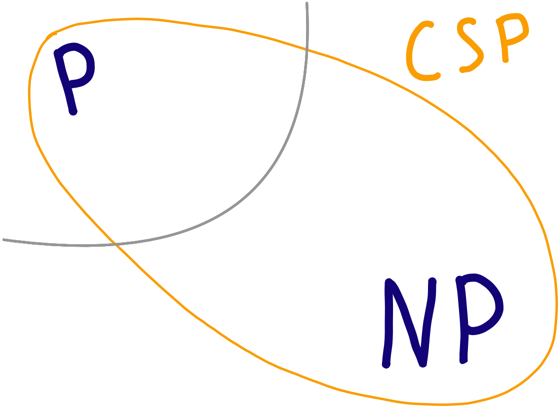
Question

What is the complexity of QCSP(Γ) for different Γ?


CSP
P
NP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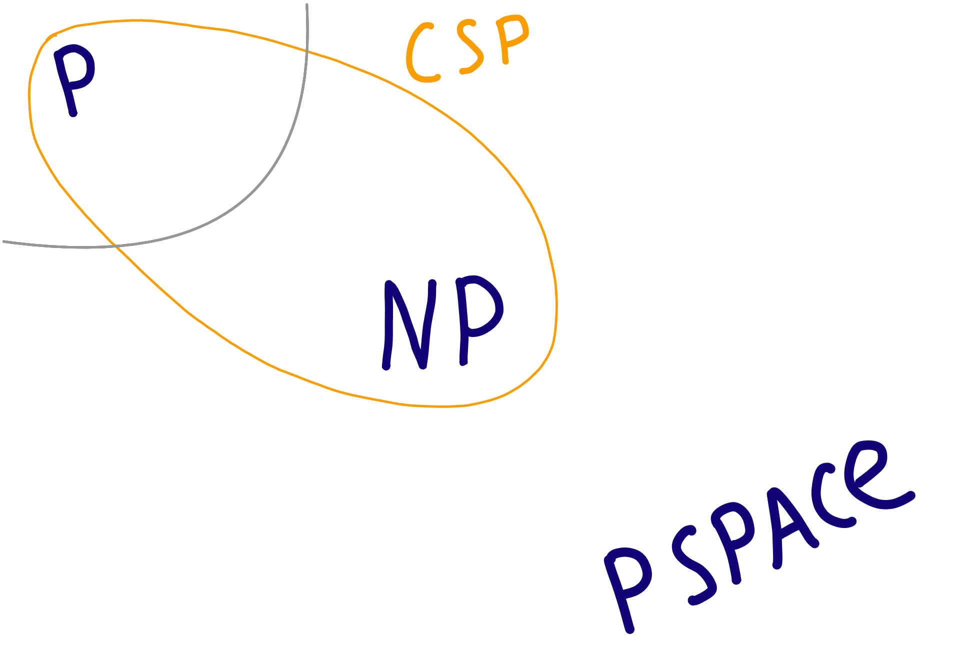
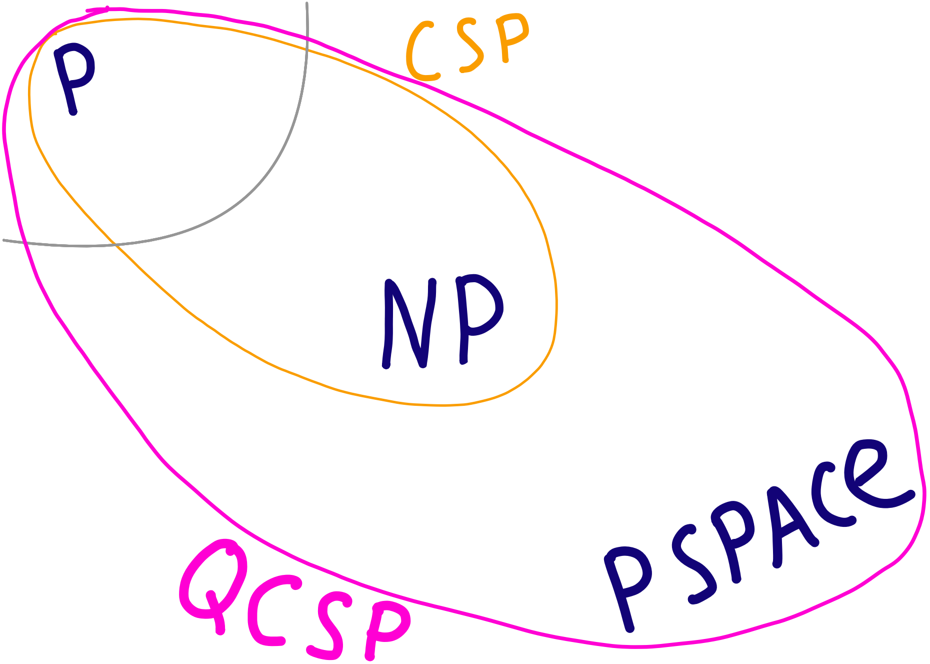


CSP
P
NP
PSPACE



P
CSP
NP
PSPACE
QCSP

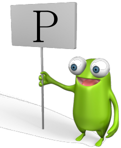
QCSP Complexity Classes

QCSP Complexity Classes

- If Γ contains all predicates then $\mathsf{QCSP}(\Gamma)$ is PSPACE-complete.

QCSP Complexity Classes

- If Γ contains all predicates then **QCSP(Γ)** is PSPACE-complete.
- If Γ consists of linear equations in a finite field then **QCSP(Γ)** is in P.



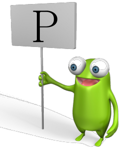
QCSP Complexity Classes

- If Γ contains all predicates then $\mathsf{QCSP}(\Gamma)$ is PSPACE-complete.
- If Γ consists of linear equations in a finite field then $\mathsf{QCSP}(\Gamma)$ is in P.

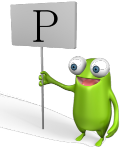
Theorem [Schaefer 1978 + Creignou et al. 2001/ Dalmau 1997.]

Suppose Γ is a constraint language on $\{0, 1\}$. Then

- $\mathsf{QCSP}(\Gamma)$ is in P if Γ is preserved by an idempotent WNU operation,
- $\mathsf{QCSP}(\Gamma)$ is PSPACE-complete otherwise.

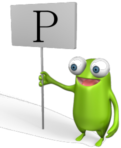


QCSP Complexity Classes



QCSP Complexity Classes

- Put $A' = A \cup \{*\}$, Γ' is Γ extended to A'. Then $\mathbf{QCSP}(\Gamma')$ is equivalent to $\mathbf{CSP}(\Gamma)$.



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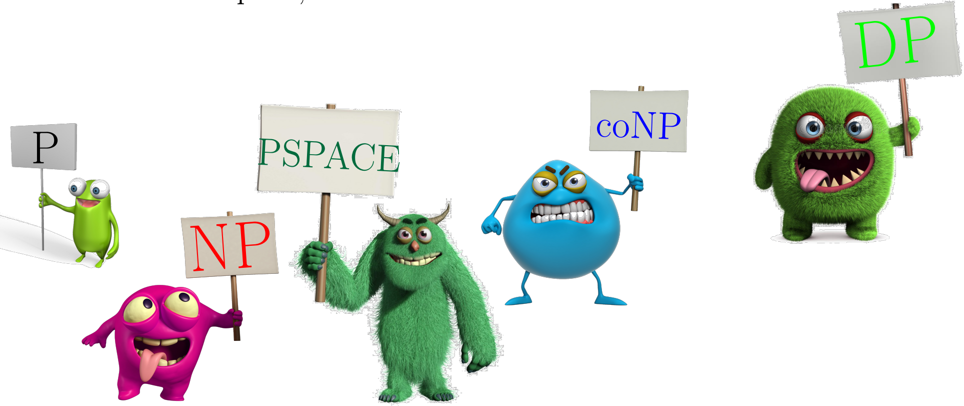

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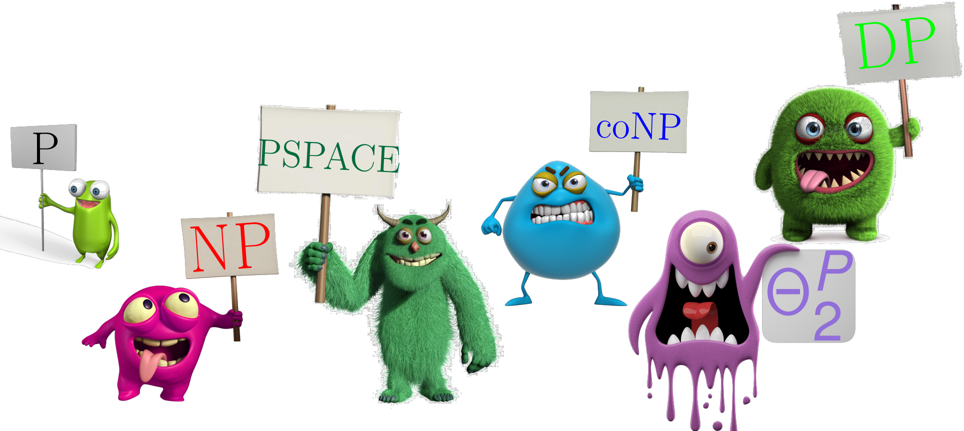
- there exists Γ on a 10-element domain such that $\mathrm{QCSP}(\Gamma)$ is Θ_2^P-complete.

QCSP Complexity Classes



QCSP Complexity Classes

Theorem [Zhuk, Martin, 2019]

Suppose Γ is a constraint language on $\{0, 1, 2\}$ containing $\{x = a \mid a \in \{0, 1, 2\}\}$. Then QCSP($\Gamma$) is

- in P, or
- NP-complete, or
- coNP-complete, or
- PSPACE-complete.

QCSP Complexity Classes

Theorem [Zhuk, 2024]

QCSP(Γ)

- is either PSpace-complete,
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PSPACE

Π_2^P

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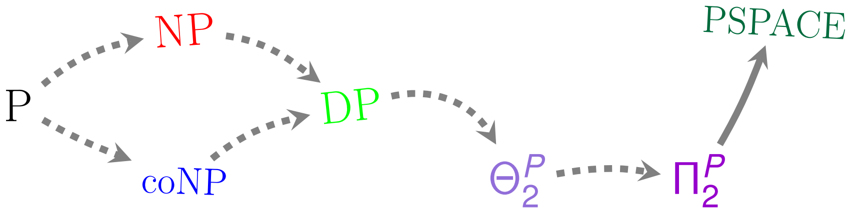

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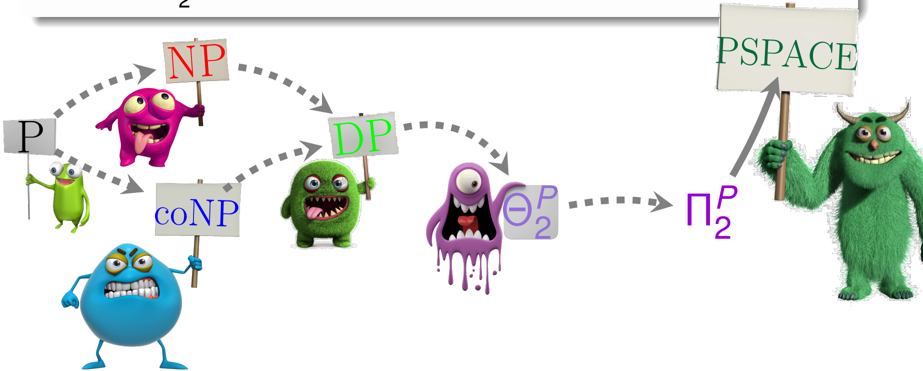

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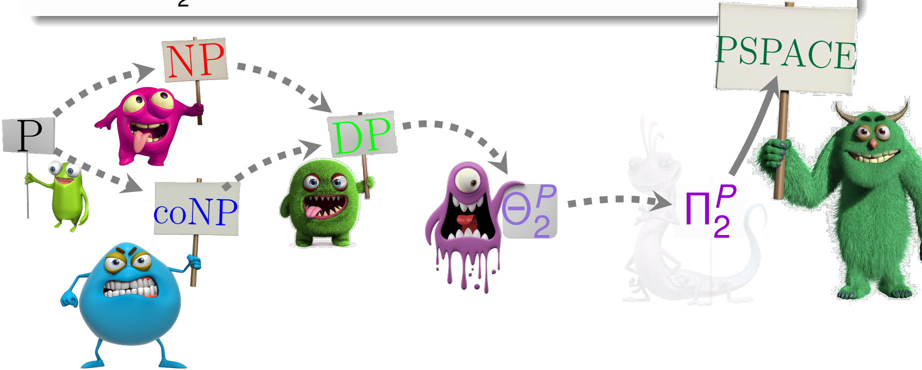

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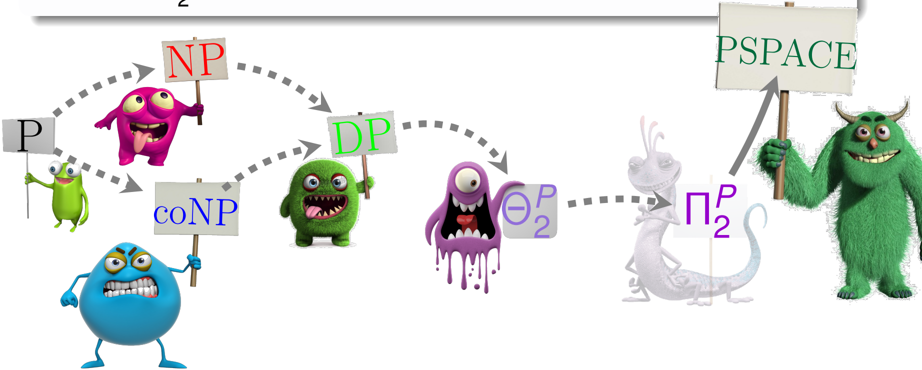

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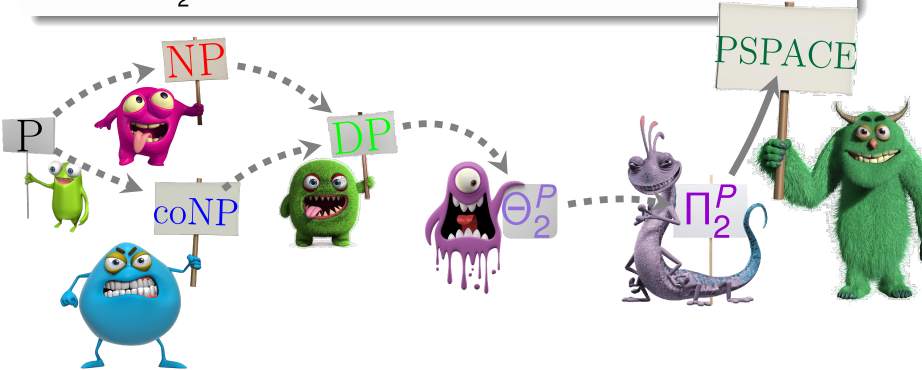

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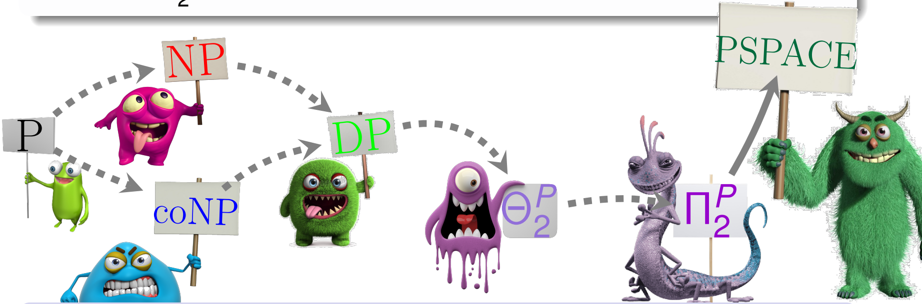

Lemma [Zhuk, 2024]

There exists Γ on a 6-element set such that $\mathrm{QCSP}(\Gamma)$ is Π_2^P-complete.

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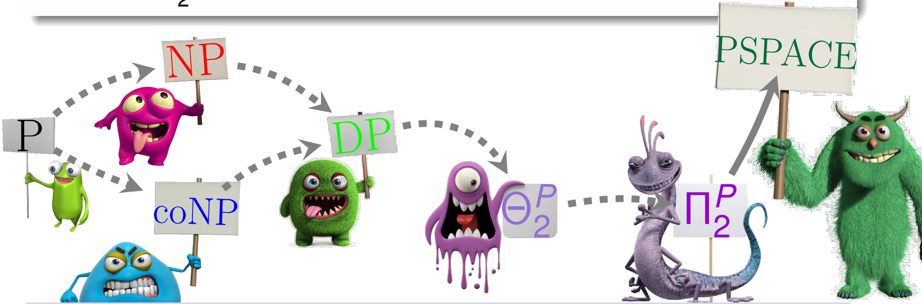

Lemma [Zhuk, 2024]

There exists Γ on a 6-element set such that QCSP(Γ) is Π_2^P-complete.

Are there any other complexity classes?

		CSP	Quantified CSP				
Domain	finite						
	infinite						

Full classification

Some classifications

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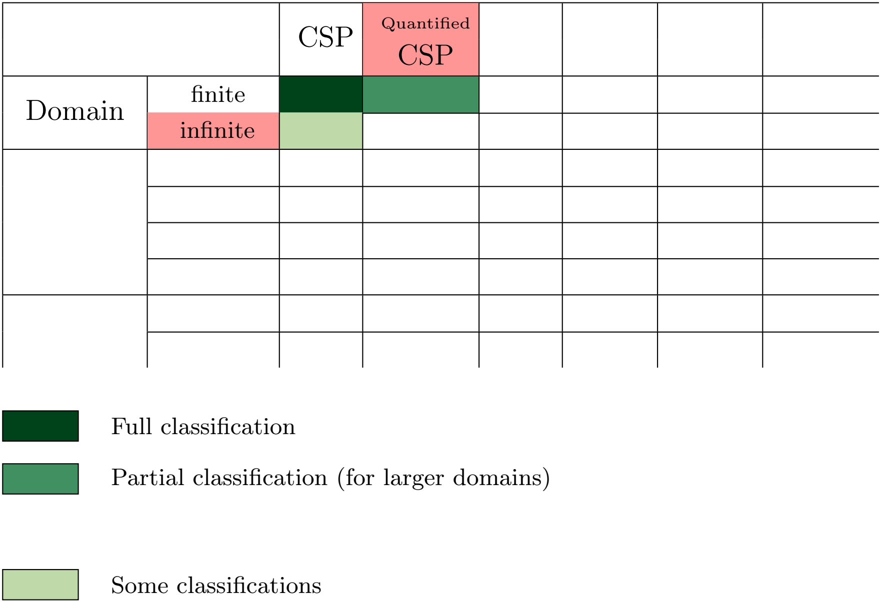
Full classification

Partial classification (for larger domains)

Some classifications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Infinite Domain QCSP

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Given: a sentence $\exists y_1 \forall x_1 \ldots \exists y_t \forall x_t (R_1(\ldots) \wedge \cdots \wedge R_s(\ldots))$, where $R_1, \ldots, R_s \in \Gamma$.
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QCSP instances:

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Examples

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2. QCSP($\{x = y \vee y = z\}$) is NP-complete.

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Examples

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2. QCSP($\{x = y \vee y = z\}$) is NP-complete.
3. QCSP($\{x = y \rightarrow z = t\}$) is PSPACE-complete [Bodirsky, Chen, 2010].

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QCSP(Γ)

Given: a sentence $\exists y_1 \forall x_1 \ldots \exists y_t \forall x_t (R_1(\ldots) \wedge \cdots \wedge R_s(\ldots))$, where $R_1, \ldots, R_s \in \Gamma$.
Decide: whether it holds.

Examples

1. QCSP($\{x = y\}$) is in P.
2. QCSP($\{x = y \vee y = z\}$) is NP-complete.
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Classification for equality constraints [Bodirsky, Chen, 2010 + Zhuk, Martin, 2021]

A full classification of the complexity for constraint languages whose relations are boolean combinations of equalities. (P, NP-complete, PSPACE-complete)

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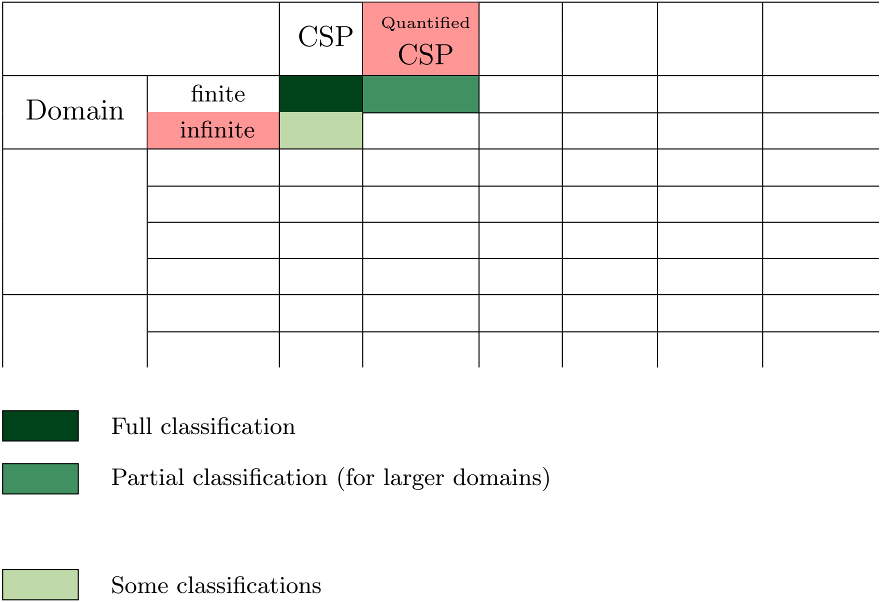
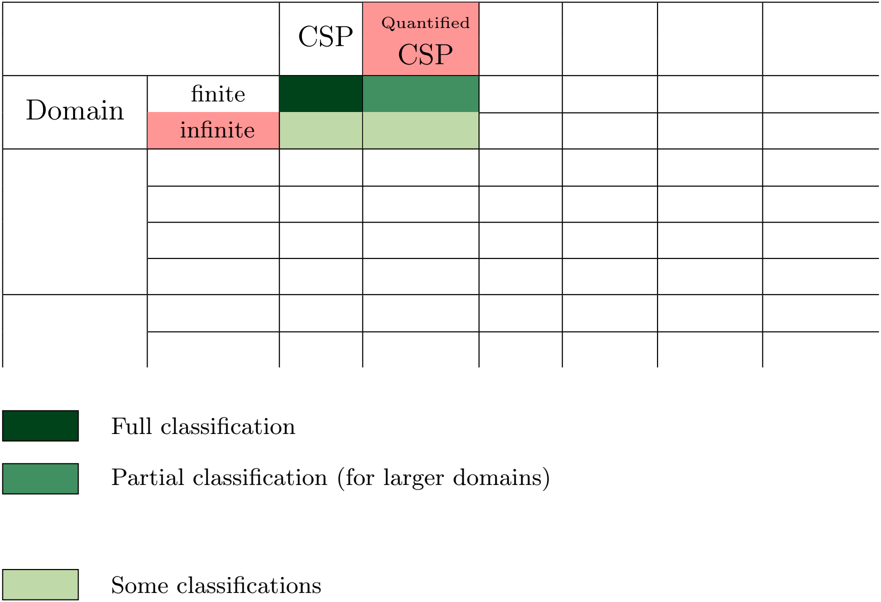
What is the complexity of QCSP($\{x < y \vee y < z\}$)?


CSP
Quantified CSP
Domain
finite
infinite
Full classification
Partial classification (for larger domains)
Some classifications


CSP
Quantified CSP
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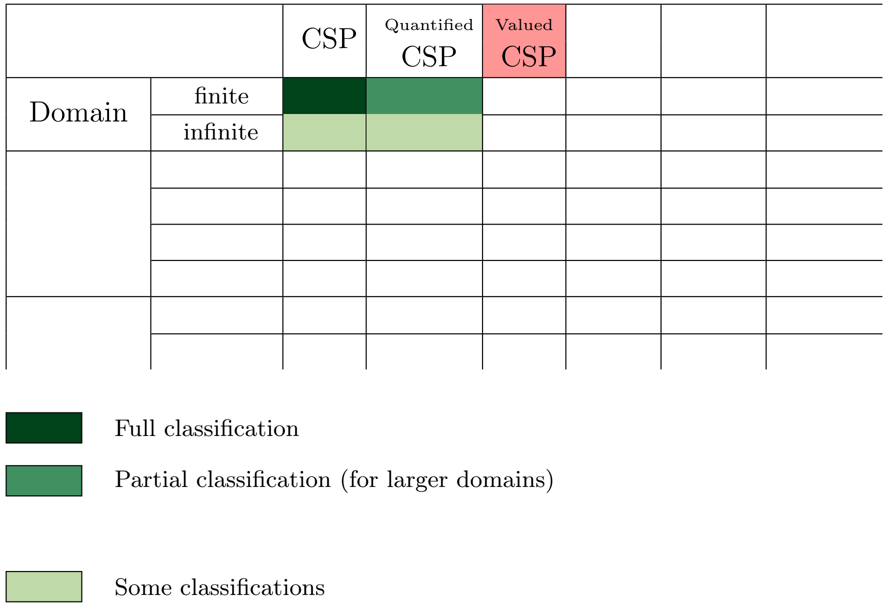
CSP
Quantified CSP
Valued CSP
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Valued Constraint Satisfaction Problem

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Γ is a set of cost functions on a finite set A, i.e. mappings $A^n \to \mathbb{Q} \cup \{\infty\}$.

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VCSP(Γ)

Given: a threshold T and a sum $f_1(\dots) + f_2(\dots) + \cdots + f_s(\dots)$, where $f_1, \dots, f_s \in \Gamma$.
Decide: whether $f_1(\dots) + f_2(\dots) + \cdots + f_s(\dots) < T$ is satisfiable.

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Example

$$A = \{0, 1\}, f(x, y) = \begin{cases} 1, & \text{if } x = y \\ 0, & \text{otherwise} \end{cases}.$$

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- VCSP($\{f\}$) is equivalent to MAX-CUT problem.

Valued Constraint Satisfaction Problem

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- VCSP($\{f\}$) is NP-complete.

Complexity classification [Kolmogorov, Krokhin, Rolínek, 2015+Bulatov, Zhuk, 2017]

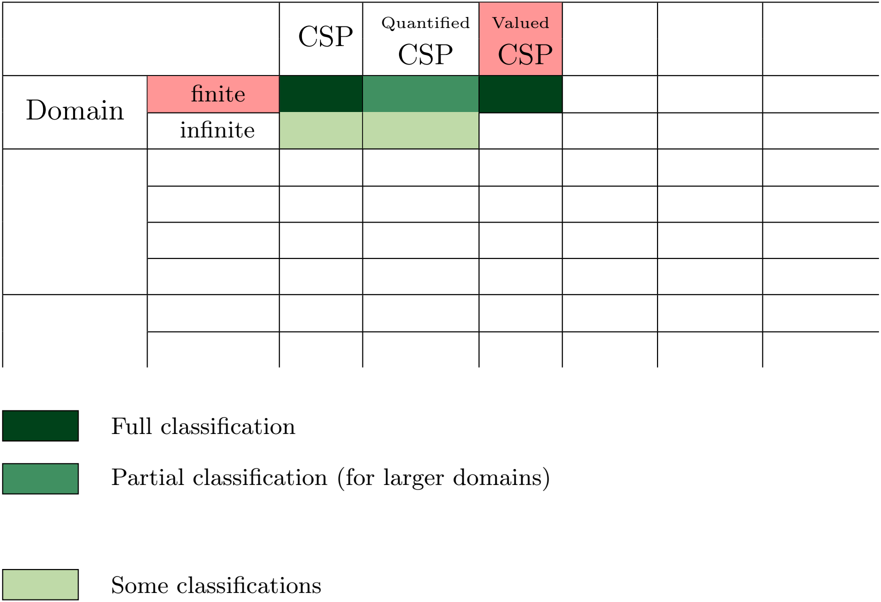
A full classification of the complexity for any finite set of cost functions Γ (P vs NP-complete).

		CSP	Quantified CSP	Valued CSP			
Domain	finite						
	infinite						

Full classification

Partial classification (for larger domains)

Some classifications


CSP
Quantified CSP
Valued CSP
Domain
finite
infinite
Full classification
Partial classification (for larger domains)
Some classifications

		CSP	Quantified CSP	Valued CSP	Promise CSP		
Domain	finite						
	infinite						

Full classification

Partial classification (for larger domains)

Some classifications

Promise Constraint Satisfaction Problem

There are two versions of each relation (weak and strong) in Γ

Promise Constraint Satisfaction Problem

There are two versions of each relation (weak and strong) in Γ

PCSP(Γ)

Given a formula $R_1(x_{i_{1,1}}, \ldots, x_{i_{1,n_1}}) \wedge \cdots \wedge R_s(x_{i_{s,1}}, \ldots, x_{i_{s,n_s}})$,
Distinguish between two cases:

- Strong version is satisfied
- Weak version is not satisfied

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Given a formula $R_1(x_{i_{1,1}}, \dots, x_{i_{1,n_1}}) \wedge \cdots \wedge R_s(x_{i_{s,1}}, \dots, x_{i_{s,n_s}})$,
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Example 1

- Strong version is 1IN3 $= \{(1,0,0),(0,1,0),(0,0,1)\}$

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Example 1

- Strong version is $1\mathrm{IN}3 = \{(1,0,0),(0,1,0),(0,0,1)\}$
- Weak version is $\mathrm{NAE}3 = \{0,1\}^3 \setminus \{(0,0,0),(1,1,1)\}$

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- Weak version is $\mathrm{NAE}3 = \{0,1\}^3 \setminus \{(0,0,0),(1,1,1)\}$
- $\mathrm{CSP}(\{\mathrm{NAE}3\})$ and $\mathrm{CSP}(\{1\mathrm{IN}3\})$ are NP-hard, but $\mathrm{PCSP}(\{1\mathrm{IN}3, \mathrm{NAE}3\})$ is in P

Promise Constraint Satisfaction Problem

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Example 1

- Strong version is 1IN3 $= \{(1,0,0),(0,1,0),(0,0,1)\}$
- Weak version is NAE3 $= \{0,1\}^3 \setminus \{(0,0,0),(1,1,1)\}$
- CSP($\{$NAE3$\}$) and CSP($\{$1IN3$\}$) are NP-hard, but PCSP($\{$1IN3, NAE3$\}$) is in P (promise helps).

Promise Constraint Satisfaction Problem

There are two versions of each relation (weak and strong) in Γ

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Given a formula $R_1(x_{i_{1,1}}, \ldots, x_{i_{1,n_1}}) \wedge \cdots \wedge R_s(x_{i_{s,1}}, \ldots, x_{i_{s,n_s}})$,
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Example 1

- Strong version is 1IN3 $= \{(1,0,0),(0,1,0),(0,0,1)\}$
- Weak version is NAE3 $= \{0,1\}^3 \setminus \{(0,0,0),(1,1,1)\}$
- CSP({NAE3}) and CSP({1IN3}) are NP-hard, but PCSP({1IN3, NAE3}) is in P (promise helps).

Theorem [Ficak, Kozik, Olsák, Stankiewicz, 2019])

A classification of the complexity of PCSP(Γ) for Γ consising of symmetric relations on $\{0,1\}$.

Promise Constraint Satisfaction Problem

There are two versions of each relation (weak and strong) in Γ

PCSP(Γ)

Given a formula $R_1(x_{i_{1,1}}, \ldots, x_{i_{1,n_1}}) \wedge \cdots \wedge R_s(x_{i_{s,1}}, \ldots, x_{i_{s,n_s}})$,
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Example 2 ((K, L)-colorability)

Given a graph G.

Promise Constraint Satisfaction Problem

There are two versions of each relation (weak and strong) in Γ

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Given a formula $R_1(x_{i_{1,1}}, \ldots, x_{i_{1,n_1}}) \wedge \cdots \wedge R_s(x_{i_{s,1}}, \ldots, x_{i_{s,n_s}})$,
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Example 2 ((K, L)-colorability)

Given a graph G. Distinguish between two cases

- the graph is K-colorable;
- the graph is not even L-colorable;

Promise Constraint Satisfaction Problem

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Open questions

- What is the complexity of $(3, 6)$-colorability?

Promise Constraint Satisfaction Problem

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Given a formula $R_1(x_{i_{1,1}}, \dots, x_{i_{1,n_1}}) \wedge \dots \wedge R_s(x_{i_{s,1}}, \dots, x_{i_{s,n_s}})$,
Distinguish between two cases:

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- Weak version is not satisfied

Example 2 ((K, L)-colorability)

Given a graph G. Distinguish between two cases

- the graph is K-colorable;
- the graph is not even L-colorable;

Open questions

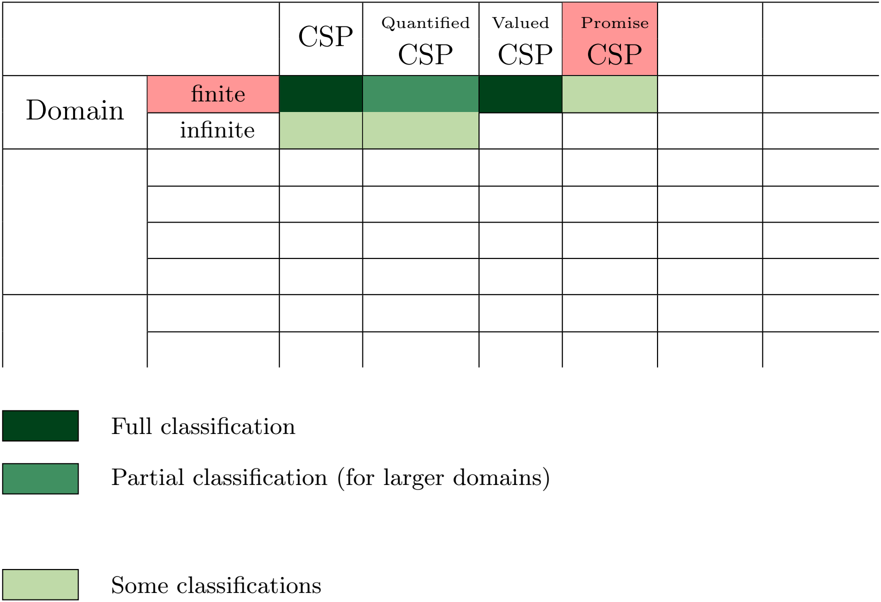
- What is the complexity of $(3, 6)$-colorability?
- What is the complexity of $(3, 1000000000)$-colorability?

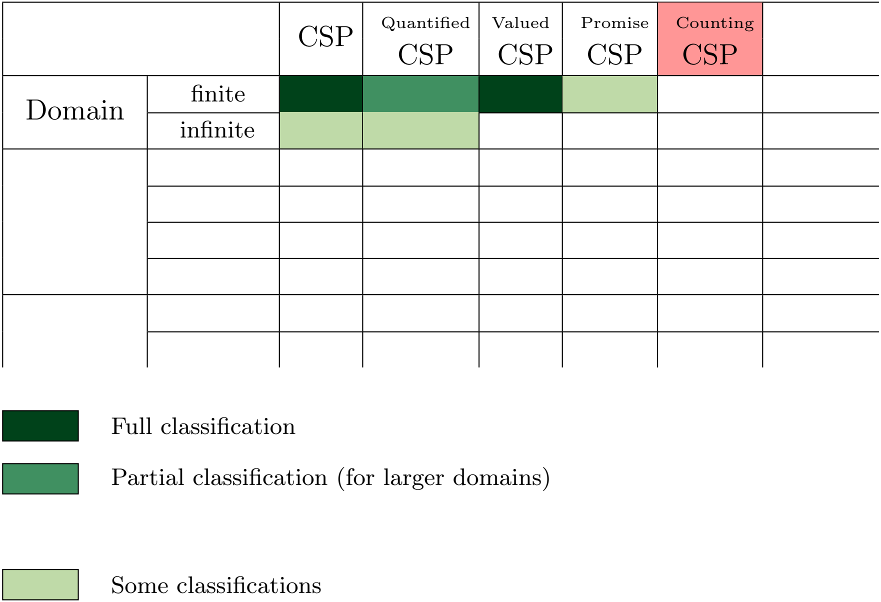
		CSP	Quantified CSP	Valued CSP	Promise CSP		
Domain	finite						
	infinite						

Full classification

Partial classification (for larger domains)

Some classifications


CSP
Quantified CSP
Valued CSP
Promise CSP
Domain
finite
infinite
Full classification
Partial classification (for larger domains)
Some classifications


CSP
Quantified CSP
Valued CSP
Promise CSP
Counting CSP
Domain
finite
infinite
Full classification
Partial classification (for larger domains)
Some classifications

Counting Constraint Satisfaction Problem

Γ is a set of relations on a finite set A.

Counting-CSP(Γ)

Given: a formula $R_1(x_{i_{1,1}}, \ldots, x_{i_{1,n_1}}) \wedge \cdots \wedge R_s(x_{i_{s,1}}, \ldots, x_{i_{s,n_s}})$, where $R_1, \ldots, R_s \in \Gamma$.
Find the number of solutions.

Counting Constraint Satisfaction Problem

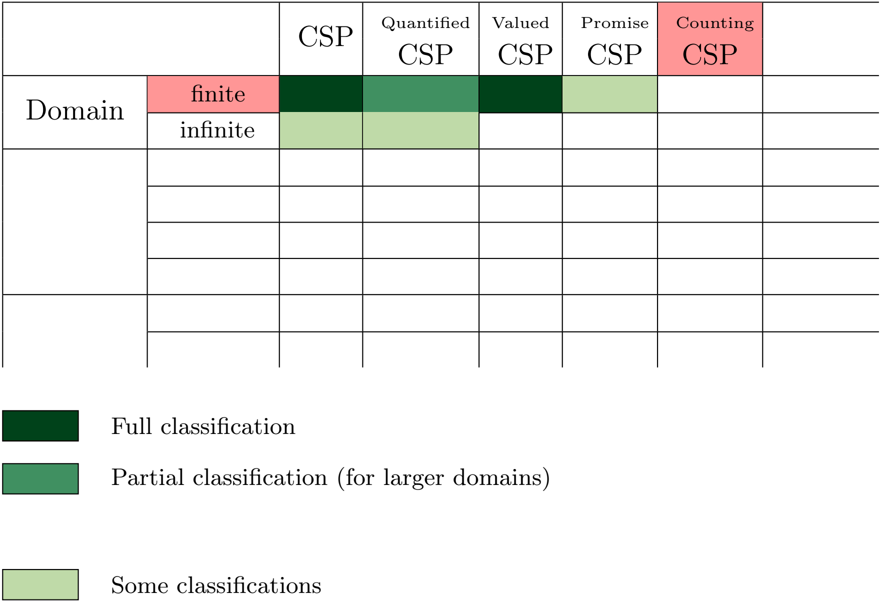
Γ is a set of relations on a finite set A.

Counting-CSP(Γ)

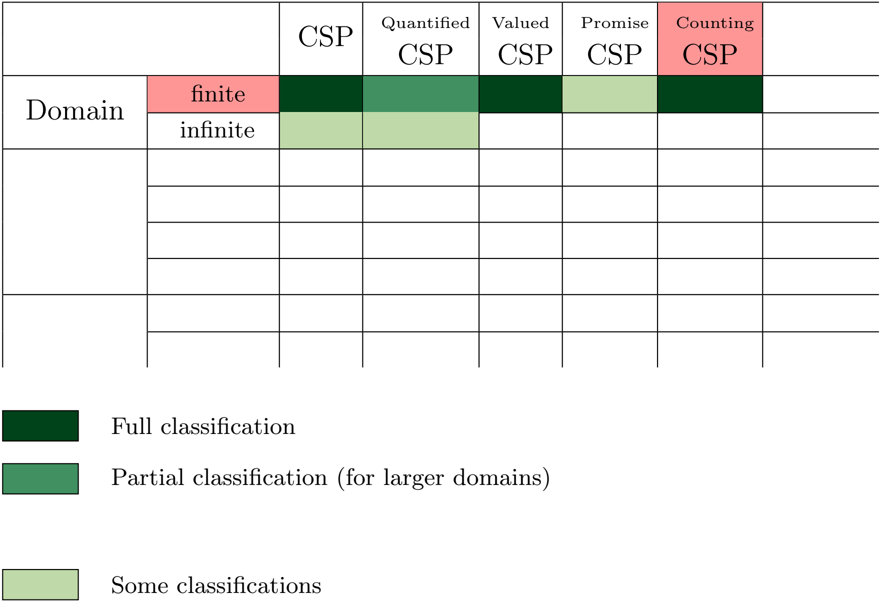
Given: a formula $R_1(x_{i_{1,1}}, \ldots, x_{i_{1,n_1}}) \wedge \cdots \wedge R_s(x_{i_{s,1}}, \ldots, x_{i_{s,n_s}})$,
where $R_1, \ldots, R_s \in \Gamma$.
Find the number of solutions.

Theorem [Bulatov, 2008]

A classification of the complexity of Counting-CSP(Γ) for every Γ.



CSP
Quantified CSP
Valued CSP
Promise CSP
Counting CSP
Domain
finite
infinite
Full classification
Partial classification (for larger domains)
Some classifications

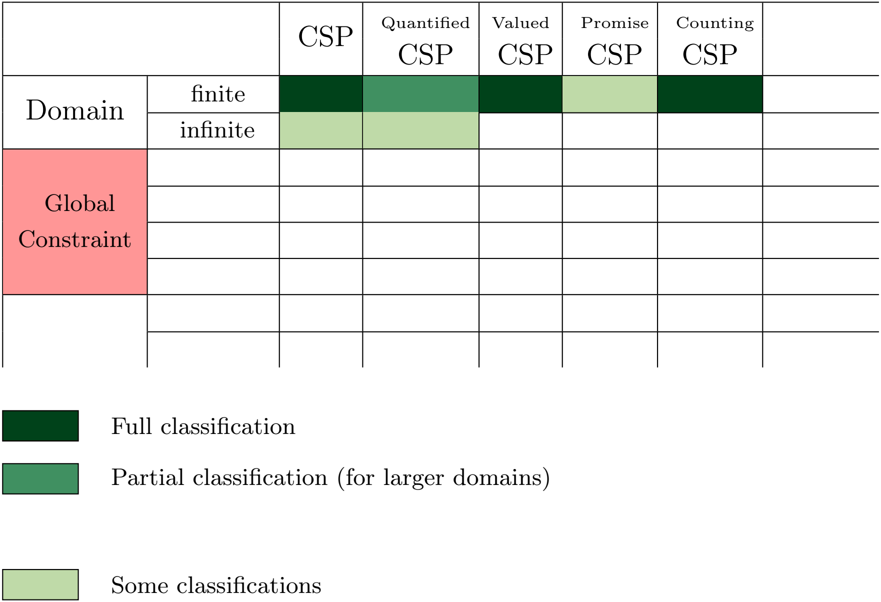

CSP
Quantified CSP
Valued CSP
Promise CSP
Counting CSP
Domain
finite
infinite
Full classification
Partial classification (for larger domains)
Some classifications

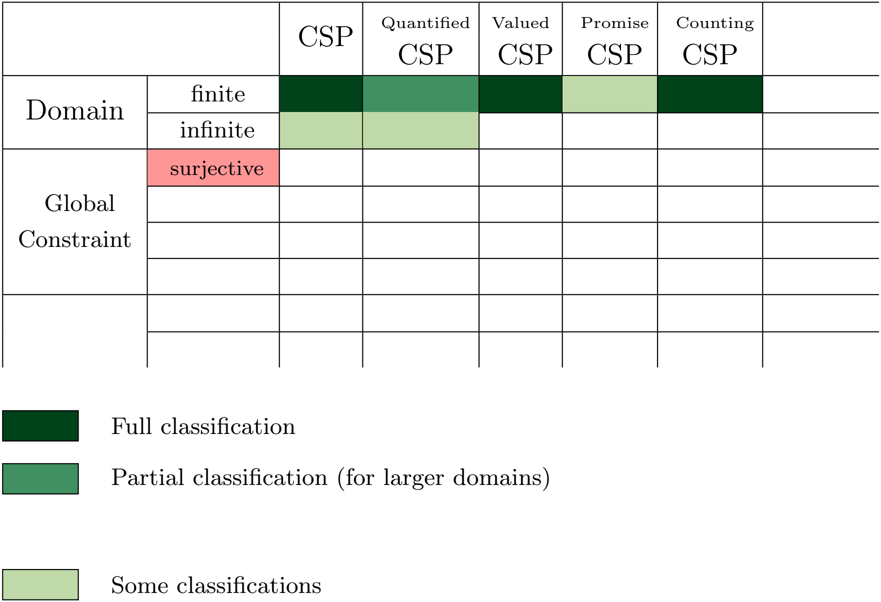
		CSP	Quantified CSP	Valued CSP	Promise CSP	Counting CSP	
Domain	finite						
	infinite						

Full classification

Partial classification (for larger domains)

Some classifications


CSP
Quantified CSP
Valued CSP
Promise CSP
Counting CSP
Domain
finite
infinite
Global Constraint
Full classification
Partial classification (for larger domains)
Some classifications



CSP
Quantified CSP
Valued CSP
Promise CSP
Counting CSP
Domain
finite
infinite
Global Constraint
surjective
Full classification
Partial classification (for larger domains)
Some classifications

Surjective Constraint Satisfaction Problem

Surjective Constraint Satisfaction Problem

Γ is a set of relations on A.

SCSP(Γ)

Given: a conjunction of relations, i.e. a formula

$$R_1(x_{i_{1,1}}, \ldots, x_{i_{1,n_1}}) \wedge \cdots \wedge R_s(x_{i_{s,1}}, \ldots, x_{i_{s,n_s}}),$$

where $R_1, \ldots, R_s \in \Gamma$.
Decide: whether the formula has a surjective solution, that is, a solution such that $\{x_1, \ldots, x_n\} = A$.

Surjective Constraint Satisfaction Problem

Γ is a set of relations on A.

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Example

$A = \{0, 1, 2\}, \Gamma = \{x \leq y\}$.

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Example

$A = \{0, 1, 2\}, \Gamma = \{x \leq y\}$. Surjective CSP instances:
$x_1 \leq x_2 \wedge x_2 \leq x_3 \wedge x_3 \leq x_4$,

Surjective Constraint Satisfaction Problem

Γ is a set of relations on A.

SCSP(Γ)

Given: a conjunction of relations, i.e. a formula

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where $R_1, \ldots, R_s \in \Gamma$.

Decide: whether the formula has a surjective solution, that is, a solution such that $\{x_1, \ldots, x_n\} = A$.

Example

$A = \{0, 1, 2\}, \Gamma = \{x \leq y\}$. Surjective CSP instances:

$x_1 \leq x_2 \wedge x_2 \leq x_3 \wedge x_3 \leq x_4$, $x_1 = 0, x_2 = 1, x_3 = x_4 = 2$.

Surjective Constraint Satisfaction Problem

Γ is a set of relations on A.

SCSP(Γ)

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$x_1 \leq x_2 \wedge x_2 \leq x_3 \wedge x_3 \leq x_4$, $x_1 = 0, x_2 = 1, x_3 = x_4 = 2$.
$x_1 \leq x_2 \wedge x_2 \leq x_3 \wedge x_3 \leq x_1$,

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$A = \{0, 1, 2\}, \Gamma = \{x \leq y\}$. Surjective CSP instances:
$x_1 \leq x_2 \wedge x_2 \leq x_3 \wedge x_3 \leq x_4$, $x_1 = 0, x_2 = 1, x_3 = x_4 = 2$.
$x_1 \leq x_2 \wedge x_2 \leq x_3 \wedge x_3 \leq x_1$, No surjective solutions

Surjective Constraint Satisfaction Problem

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$x_1 \leq x_2 \wedge x_2 \leq x_3 \wedge x_3 \leq x_4$, $x_1 = 0, x_2 = 1, x_3 = x_4 = 2$.
$x_1 \leq x_2 \wedge x_2 \leq x_3 \wedge x_3 \leq x_1$, No surjective solutions

Question

What is the complexity of the SCSP(Γ)?

Surjective Graph Homomorphism Problem

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Let H be a finite graph.

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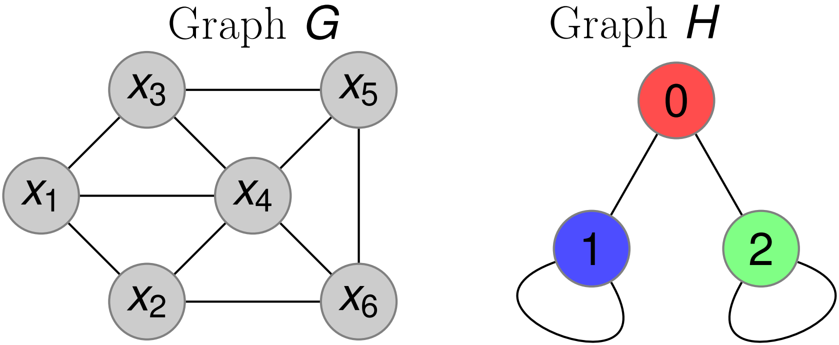

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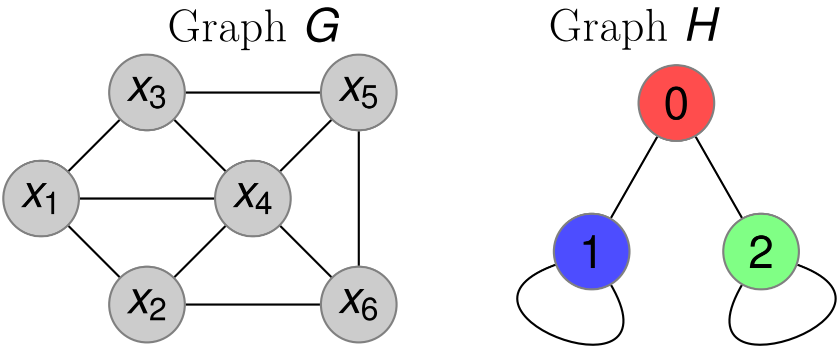

SurjHom(H) is equivalent to SCSP($\{x + y \neq 0 \mod 3\}$).

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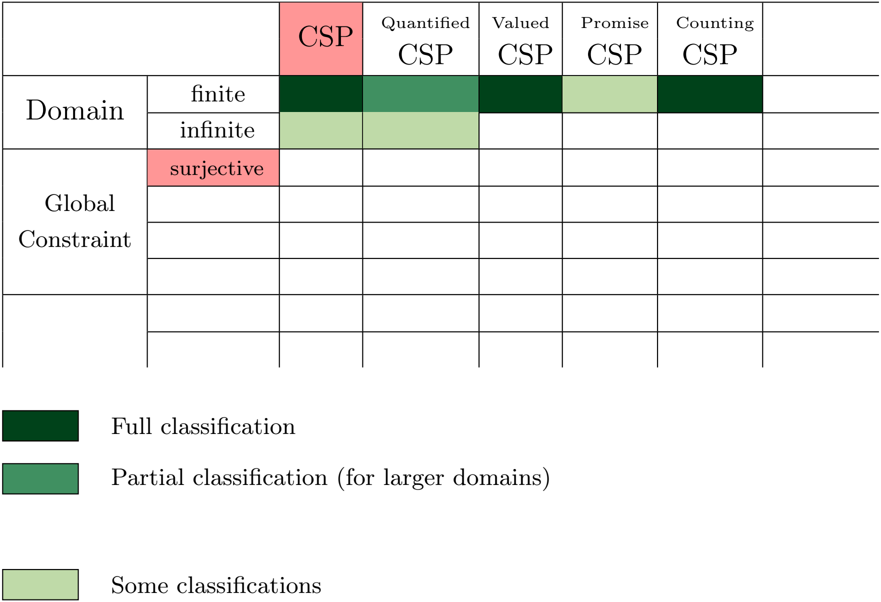
CSP
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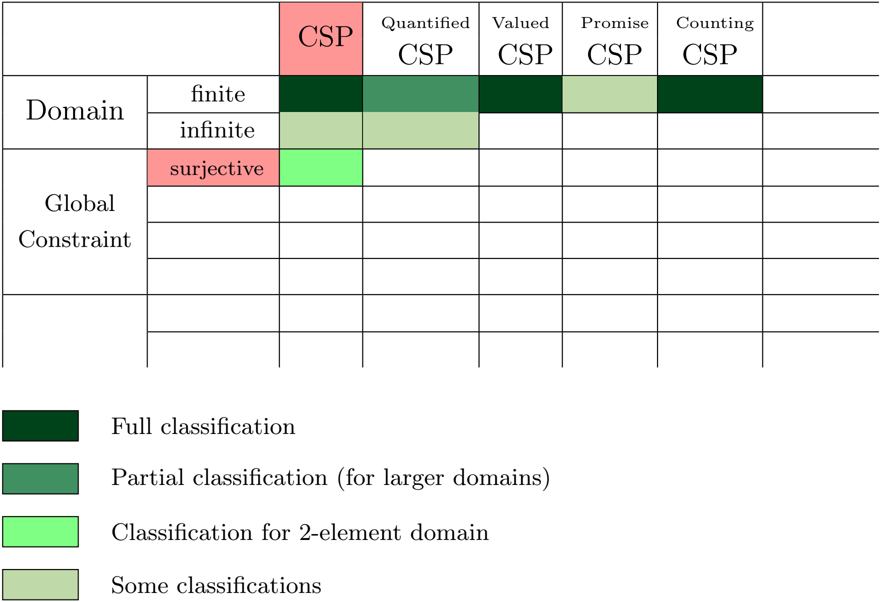
Some classifications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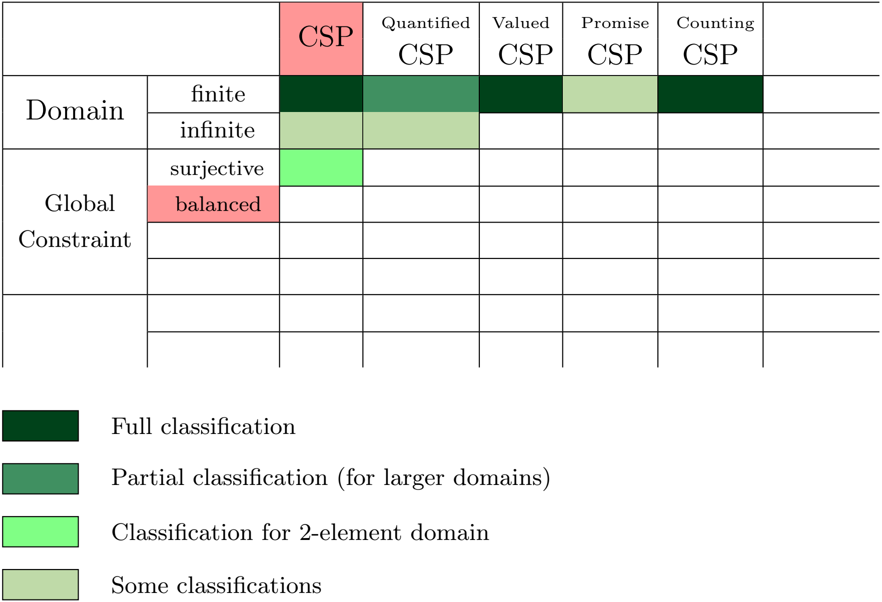

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Balanced CSP

Γ is a set of relations on a finite set A.

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Balanced-CSP(Γ)

Given: a formula $R_1(x_{i_{1,1}}, \ldots, x_{i_{1,n_1}}) \wedge \cdots \wedge R_s(x_{i_{s,1}}, \ldots, x_{i_{s,n_s}})$, where $R_1, \ldots, R_s \in \Gamma$.
Decide: whether it has a balanced solution, i.e., a solution with equal number of every element.

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Balanced-CSP($=$) on $\{0, 1\}$

Given an instance $x_{i_1} = x_{j_1} \wedge \cdots \wedge x_{i_s} = x_{j_s}$. Decide whether it has a solution with equal number of 0 and 1.

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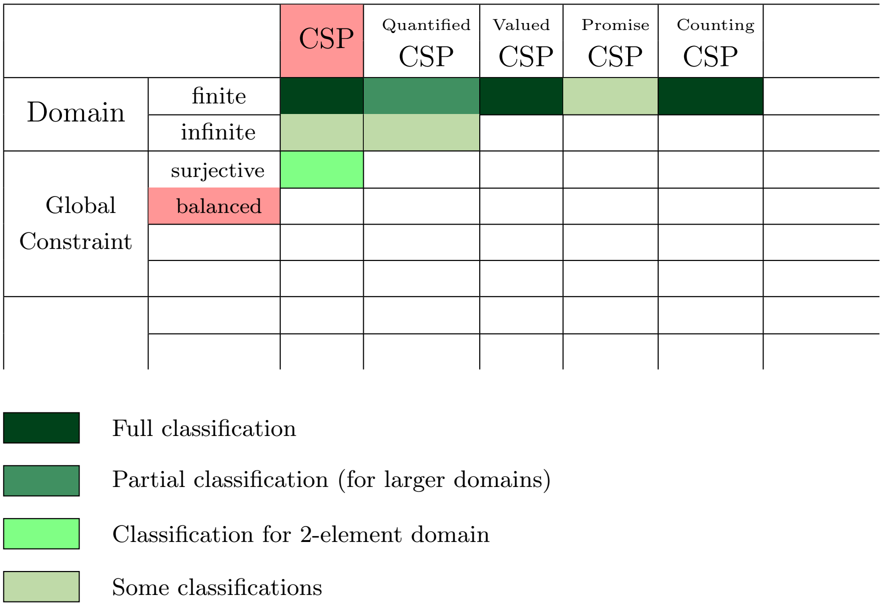
Theorem [Creignou, H. Schnoor, I. Schnoor, 2008]

A classification of the complexity of Balanced-CSP(Γ) and Cardinality-CSP(Γ) for each Γ on $\{0, 1\}$.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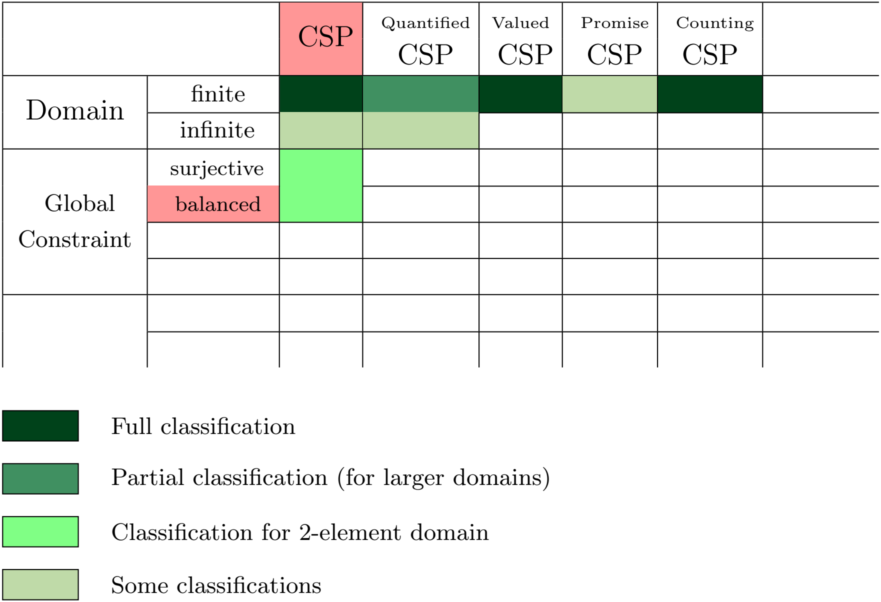
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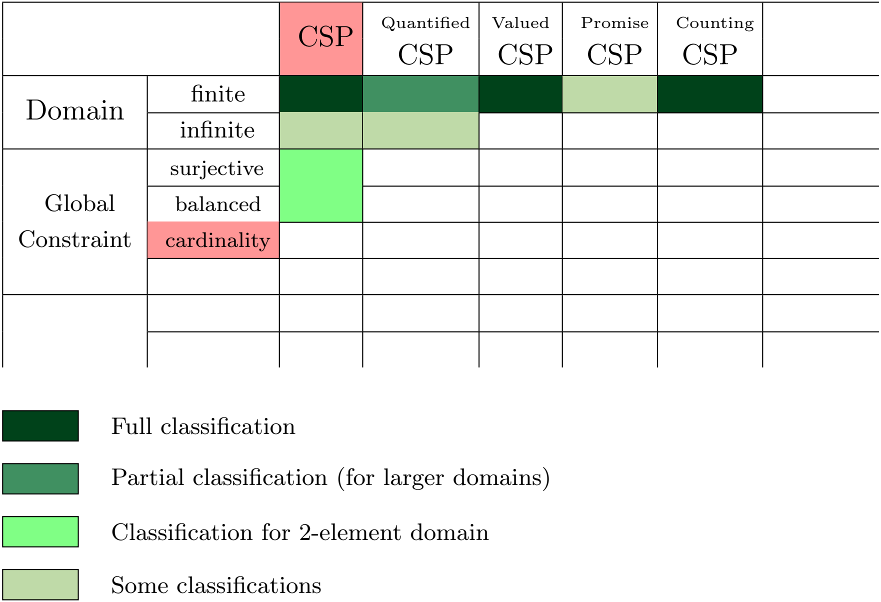
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Global cardinality constraint

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Cardinality-CSP(Γ)

Given: a mapping $\pi\colon A \to \mathbb{N}$ and a formula $R_1(x_{i_{1,1}}, \ldots, x_{i_{1,n_1}}) \wedge \cdots \wedge R_s(x_{i_{s,1}}, \ldots, x_{i_{s,n_s}})$, where $R_1, \ldots, R_s \in \Gamma$.
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Cardinality-CSP(Linear Equations in $\mathbb{Z}_2$)

Given a system of linear equations in $\mathbb{Z}_2$ and $k \in \mathbb{N}$.
Decide whether there exists a solution with exactly k 1s.

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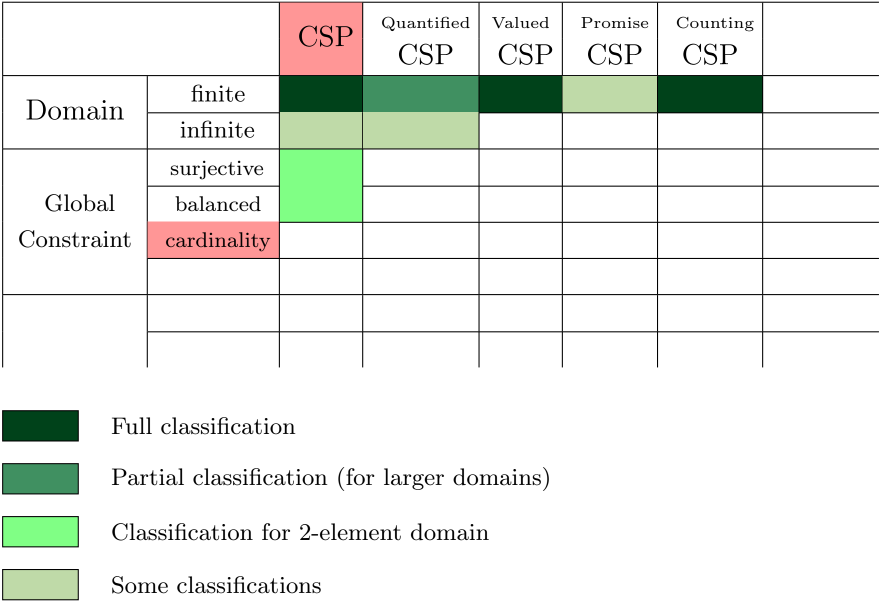
Theorem [Bulatov, Marx, 2009]

A classification of the complexity of Cardinality-CSP(Γ) for each Γ.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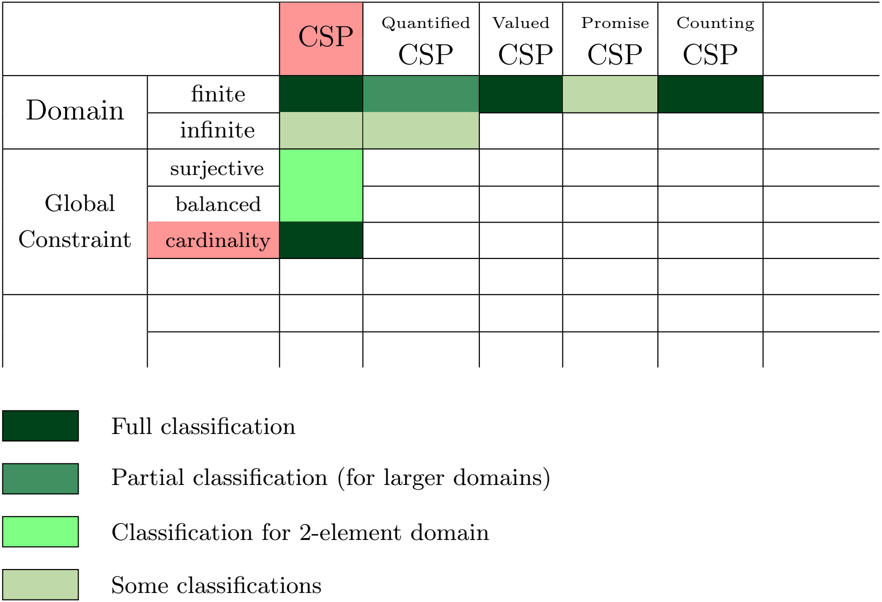
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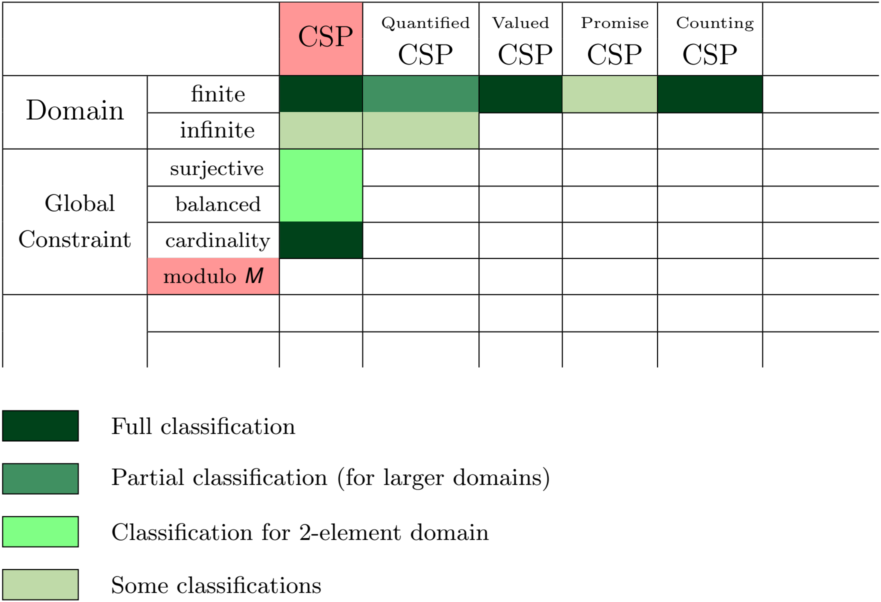
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Global modular constraint

Mod_M-CSP(Γ)

Given: a formula $R_1(x_{i_{1,1}}, \dots, x_{i_{1,n_1}}) \wedge \dots \wedge R_s(x_{i_{s,1}}, \dots, x_{i_{s,n_s}})$, where $R_1, \dots, R_s \in \Gamma$.
Decide: whether it has a solution satisfying $x_1 + \dots + x_n = 0 \mod M$.

Global modular constraint

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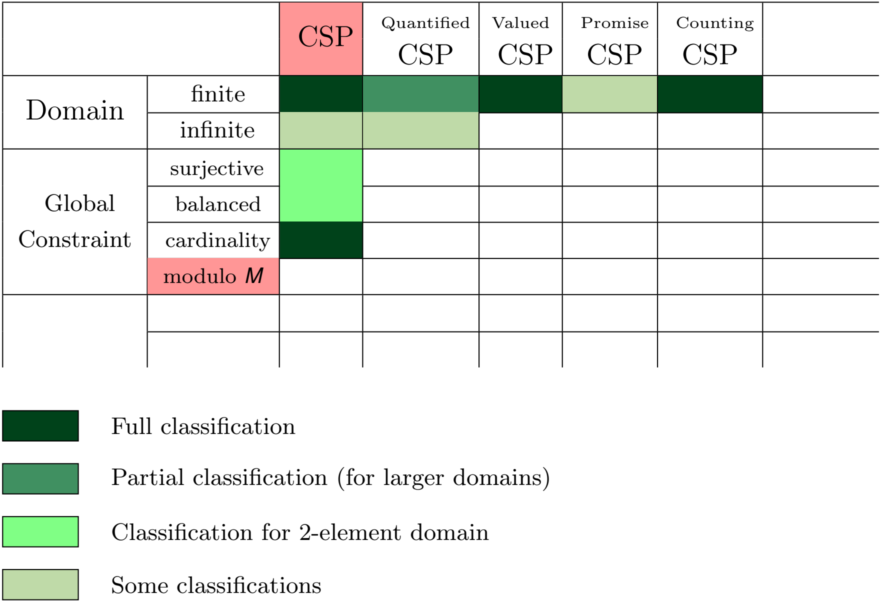
- If Γ consists of linear equations on $\{0,1\}$ and $M = 15$ then Mod_M-CSP(Γ) is not tractable
- If Γ consists of linear equations on $\{0,1\}$ and $M = 24$ then the complexity of Mod_M-CSP(Γ) is not known.

$$\begin{cases} x_1 + x_2 + x_3 = 0 & \mod 2 \\ x_1 + x_3 + x_5 = 0 & \mod 2 \\ x_2 + x_4 + x_5 = 1 & \mod 2 \\ x_2 + x_3 + x_5 = 0 & \mod 24 \end{cases}$$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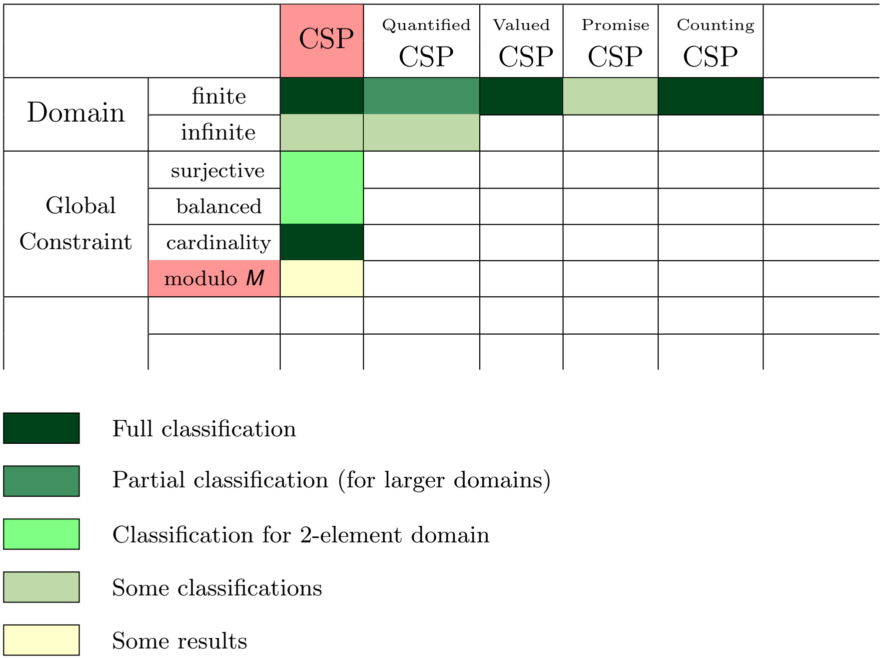

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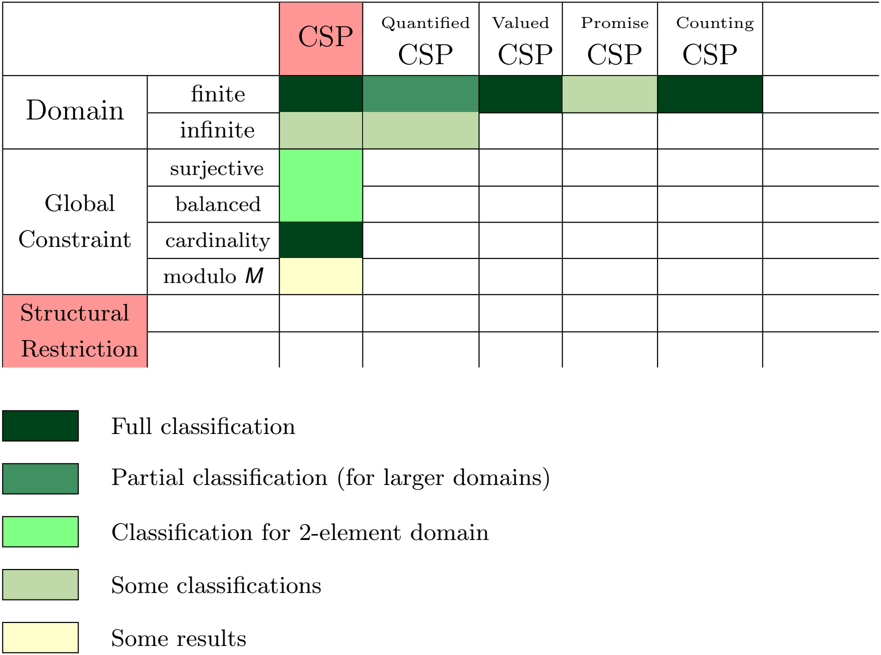
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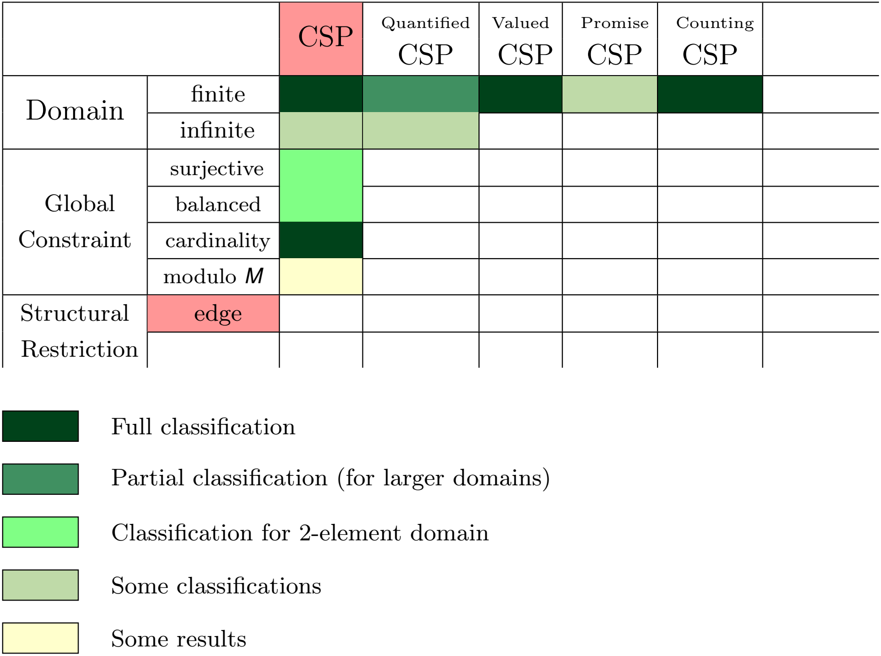

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Edge Constraint Satisfaction Problem

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Edge-CSP(Γ)

Given: a formula $R_1(x_{i_{1,1}}, \ldots, x_{i_{1,n_1}}) \wedge \cdots \wedge R_s(x_{i_{s,1}}, \ldots, x_{i_{s,n_s}})$, where $R_1, \ldots, R_s \in \Gamma$ and every variable appears exactly twice.

Decide: whether it has a solution.

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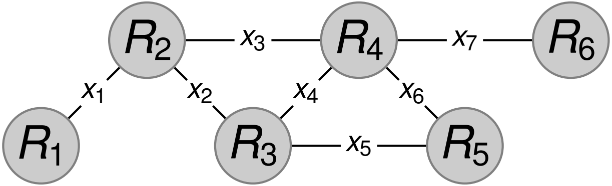

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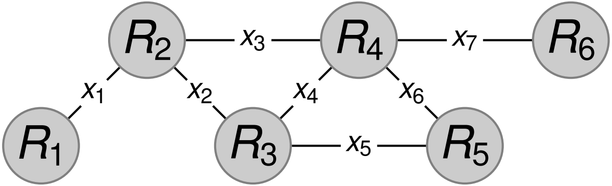
Edge-CSP(Γ)

Given: a formula $R_1(x_{i_{1,1}}, \dots, x_{i_{1,n_1}}) \wedge \dots \wedge R_s(x_{i_{s,1}}, \dots, x_{i_{s,n_s}})$, where $R_1, \dots, R_s \in \Gamma$ and every variable appears exactly twice.
Decide: whether it has a solution.



- Edge-CSP($\{1\text{IN}2, 1\text{IN}3, 1\text{IN}4, \dots\}$) is equivalent to the Perfect Matching Problem.

Edge Constraint Satisfaction Problem

Γ is a set of relations on a finite set A.

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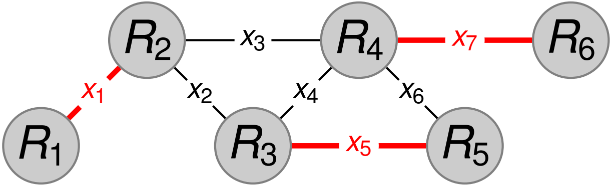

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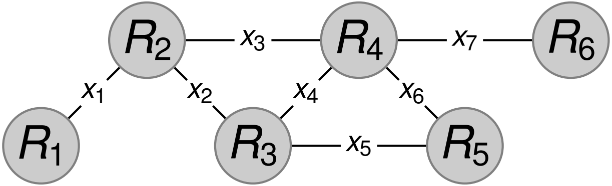
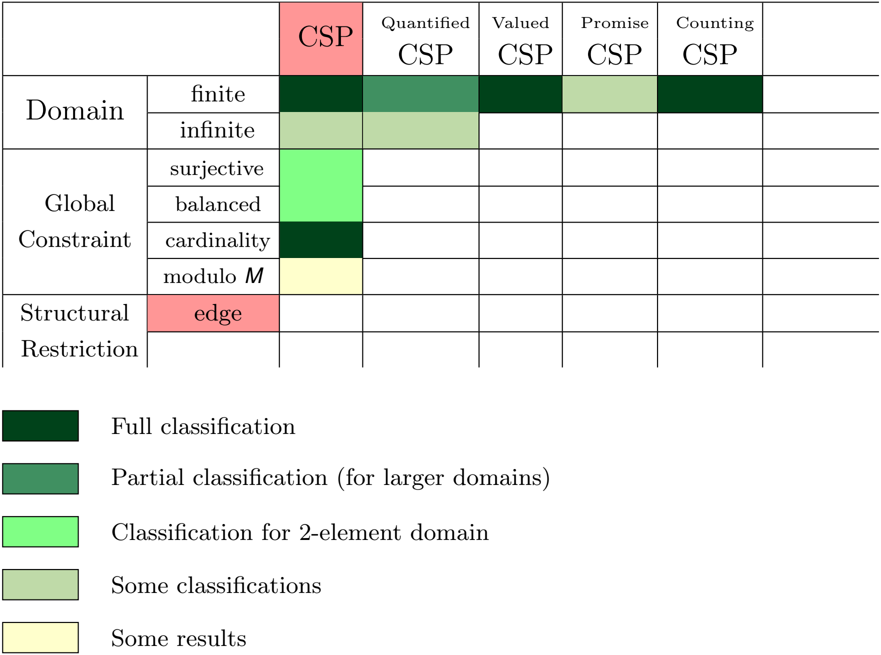


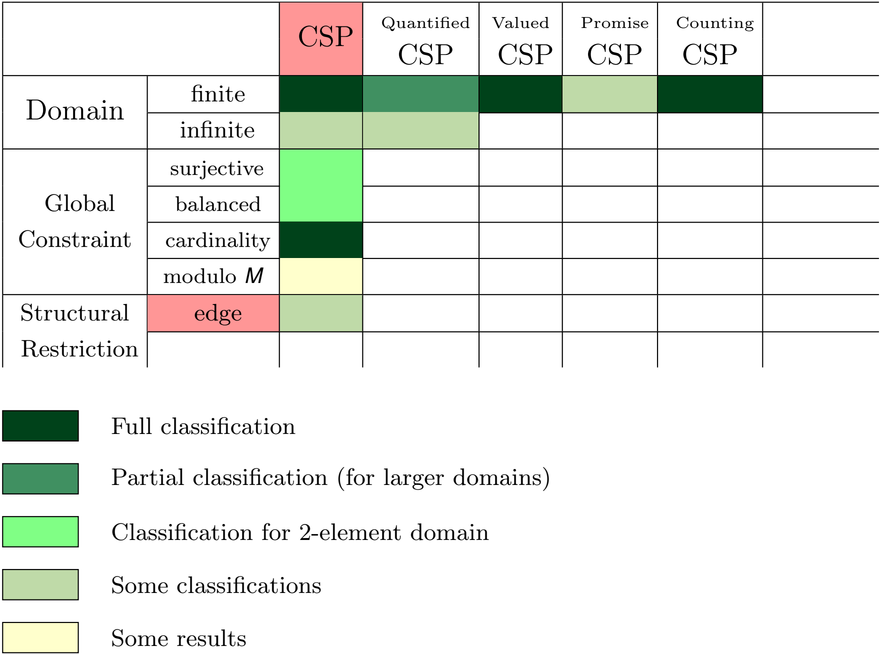
- Edge-CSP($\{1IN2, 1IN3, 1IN4, \ldots\}$) is equivalent to the Perfect Matching Problem.

Theorem [Kazda, Kolmogorov, Rolinek, 2018]

A classification of the complexity for planar Edge-CSP(Γ) for every Γ on $\{0, 1\}$.



CSP
Quantified CSP
Valued CSP
Promise CSP
Counting CSP
Domain
finite
infinite
Global Constraint
surjective
balanced
cardinality
modulo M
Structural Restriction
edge
Full classification
Partial classification (for larger domains)
Classification for 2-element domain
Some classifications
Some results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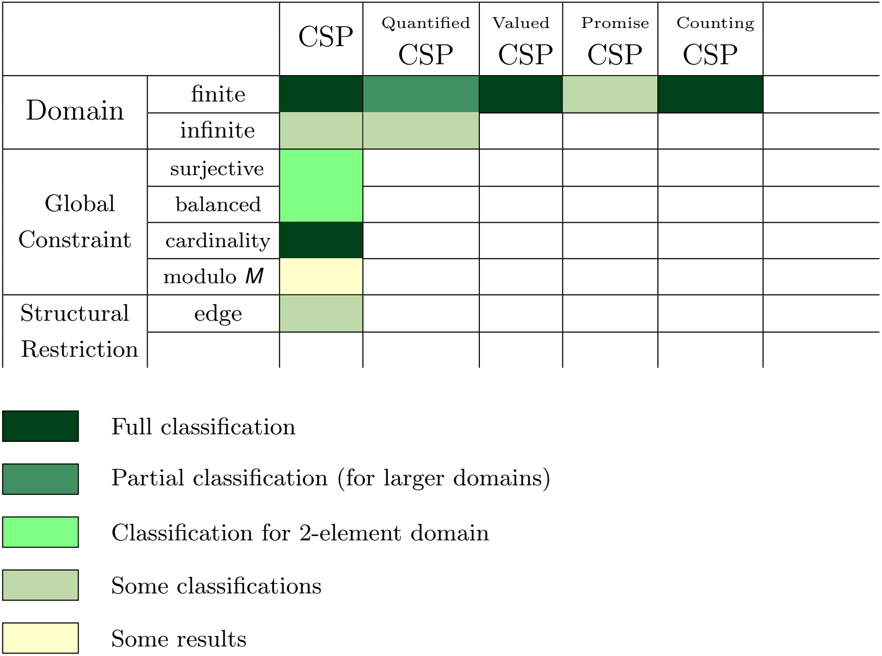
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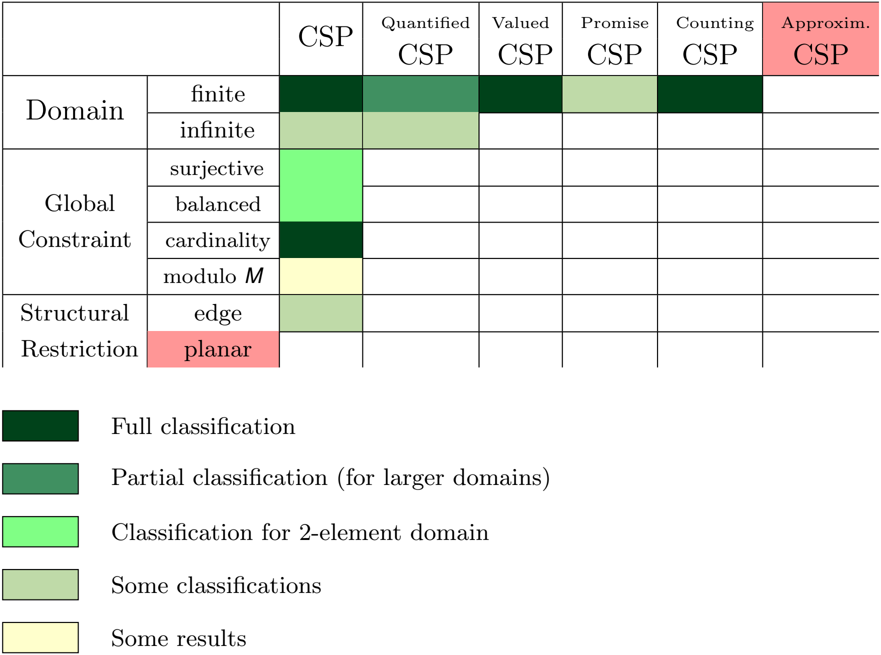

CSP
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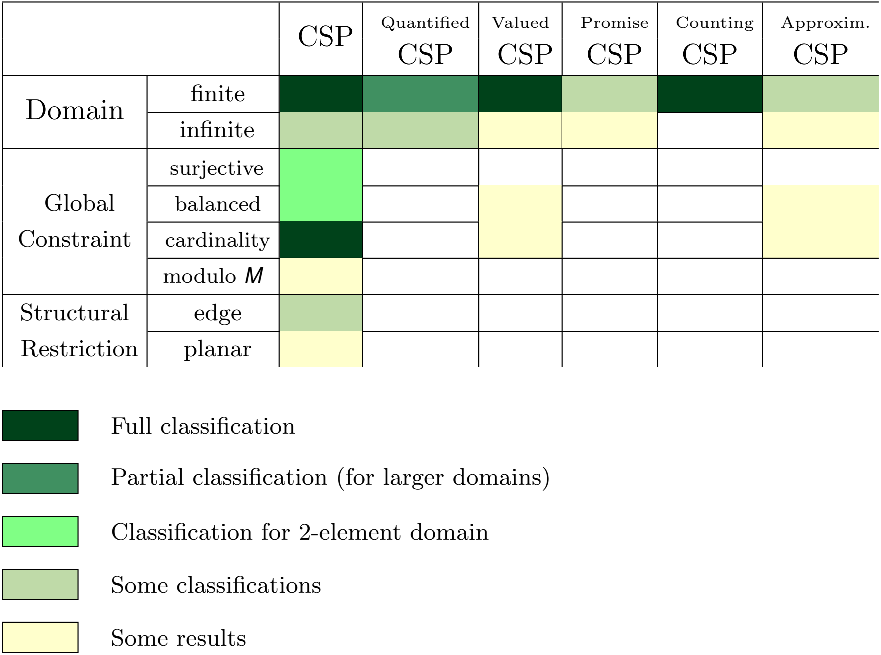
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