

# Strong subalgebras: version 4

Dmitriy Zhuk

zhuk.dmitriy@gmail.com

Charles University  
Prague

The 105th Workshop on General Algebra  
Prague, Czech Republic  
May 31-June 2, 2024



Funded by the European Union ([ERC, POCOCOP, 101071674](#)).  
Views and opinions expressed are however those of the author(s) only  
and do not necessarily reflect those of the European Union or the  
European Research Council Executive Agency. Neither the European  
Union nor the granting authority can be held responsible for them.

# Introduction

## Introduction

**A** is a finite idempotent Taylor algebra.

## Introduction

$\mathbf{A}$  is a finite idempotent Taylor algebra.

Loop Lemma[Barto, Kozik, Niven, 2008]

$R \leq_{sd} \mathbf{A}^2$  is linked

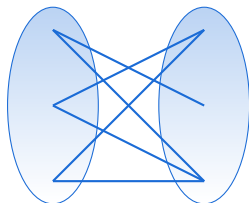
## Introduction

$\mathbf{A}$  is a finite idempotent Taylor algebra.

**Loop Lemma**[Barto, Kozik, Niven, 2008]

$R \leq_{sd} \mathbf{A}^2$  is linked

$\leq_{sd}$ :  $\forall i \text{ } pr_i(R) = A$ , linked: bipartite graph is connected



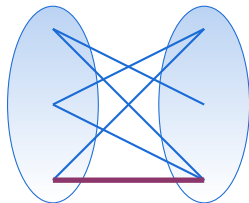
## Introduction

$\mathbf{A}$  is a finite idempotent Taylor algebra.

**Loop Lemma**[Barto, Kozik, Niven, 2008]

$R \leq_{sd} \mathbf{A}^2$  is linked  $\Rightarrow \exists a \in A: (a, a) \in R$ .

$\leq_{sd}$ :  $\forall i \text{ } pr_i(R) = A$ , linked: bipartite graph is connected



## Introduction

$\mathbf{A}$  is a finite idempotent Taylor algebra.

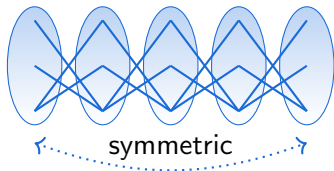
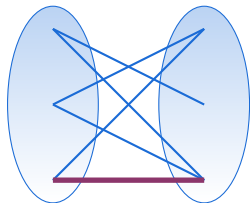
**Loop Lemma**[Barto, Kozik, Niven, 2008]

$R \leq_{sd} \mathbf{A}^2$  is linked  $\Rightarrow \exists a \in A: (a, a) \in R$ .

$\leq_{sd}: \forall i \text{ pr}_i(R) = A$ , linked: bipartite graph is connected

**Theorem** [Maróti, McKenzie, 2008]

$\emptyset \neq R \leq \mathbf{A}^p$ ,  $R$  is symmetric,  
 $p > |A|$  is a prime



## Introduction

$\mathbf{A}$  is a finite idempotent Taylor algebra.

**Loop Lemma**[Barto, Kozik, Niven, 2008]

$R \leq_{sd} \mathbf{A}^2$  is linked  $\Rightarrow \exists a \in A: (a, a) \in R$ .

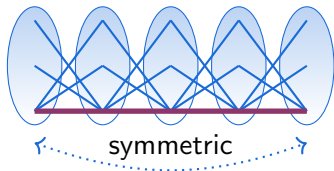
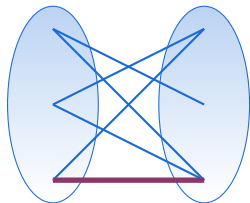
$\leq_{sd}$ :  $\forall i$   $pr_i(R) = A$ , linked: bipartite graph is connected

**Theorem** [Maróti, McKenzie, 2008]

$\emptyset \neq R \leq \mathbf{A}^p$ ,  $R$  is symmetric,

$p > |A|$  is a prime

$\Rightarrow \exists a \in A: (a, a, \dots, a) \in R$ .





## Introduction

$\mathbf{A}$  is a finite idempotent Taylor algebra.

**Loop Lemma [Barto, Kozik, Niven, 2008]**

$R \leq_{sd} \mathbf{A}^2$  is linked  $\Rightarrow \exists a \in A: (a, a) \in R$ .

$\leq_{sd}: \forall i \text{ pr}_i(R) = A$ , linked: bipartite graph is connected

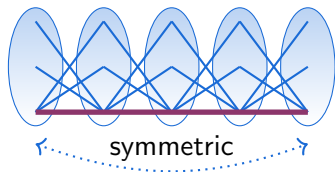
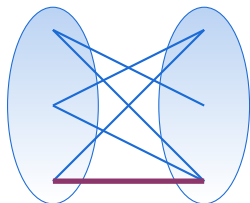
**Theorem [Maróti, McKenzie, 2008]**

$\emptyset \neq R \leq \mathbf{A}^p$ ,  $R$  is symmetric,

$p > |A|$  is a prime

$\Rightarrow \exists a \in A: (a, a, \dots, a) \in R$ .

Proof: Induction on  $|A|$ .



## Introduction

$\mathbf{A}$  is a finite idempotent Taylor algebra.

**Loop Lemma**[Barto, Kozik, Niven, 2008]

$R \leq_{sd} \mathbf{A}^2$  is linked  $\Rightarrow \exists a \in A: (a, a) \in R$ .

$\leq_{sd}$ :  $\forall i \text{ pr}_i(R) = A$ , linked: bipartite graph is connected

**Theorem** [Maróti, McKenzie, 2008]

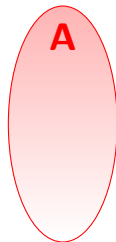
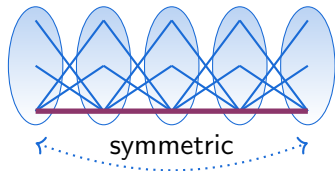
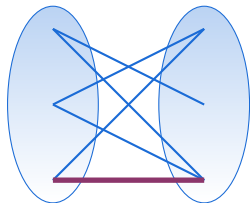
$\emptyset \neq R \leq \mathbf{A}^p$ ,  $R$  is symmetric,

$p > |A|$  is a prime

$\Rightarrow \exists a \in A: (a, a, \dots, a) \in R$ .

Proof: Induction on  $|A|$ .

**A**



## Introduction

$\mathbf{A}$  is a finite idempotent Taylor algebra.

**Loop Lemma [Barto, Kozik, Niven, 2008]**

$R \leq_{sd} \mathbf{A}^2$  is linked  $\Rightarrow \exists a \in A: (a, a) \in R$ .

$\leq_{sd}$ :  $\forall i \text{ pr}_i(R) = A$ , linked: bipartite graph is connected

**Theorem [Maróti, McKenzie, 2008]**

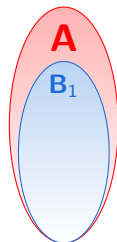
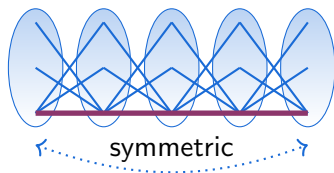
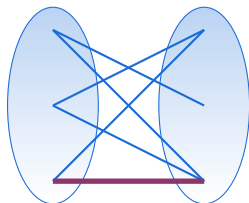
$\emptyset \neq R \leq \mathbf{A}^p$ ,  $R$  is symmetric,

$p > |A|$  is a prime

$\Rightarrow \exists a \in A: (a, a, \dots, a) \in R$ .

Proof: Induction on  $|A|$ .

$\mathbf{A} \geq \mathbf{B}_1$



## Introduction

$\mathbf{A}$  is a finite idempotent Taylor algebra.

**Loop Lemma**[Barto, Kozik, Niven, 2008]

$R \leq_{sd} \mathbf{A}^2$  is linked  $\Rightarrow \exists a \in A: (a, a) \in R$ .

$\leq_{sd}$ :  $\forall i$   $pr_i(R) = A$ , linked: bipartite graph is connected

**Theorem** [Maróti, McKenzie, 2008]

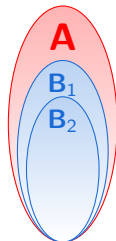
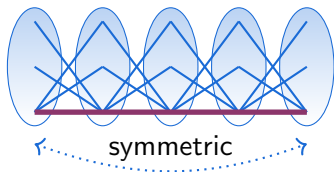
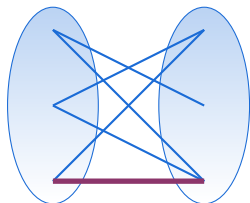
$\emptyset \neq R \leq \mathbf{A}^p$ ,  $R$  is symmetric,

$p > |A|$  is a prime

$\Rightarrow \exists a \in A: (a, a, \dots, a) \in R$ .

Proof: Induction on  $|A|$ .

$\mathbf{A} \geq \mathbf{B}_1 \geq \mathbf{B}_2$



## Introduction

$\mathbf{A}$  is a finite idempotent Taylor algebra.

**Loop Lemma [Barto, Kozik, Niven, 2008]**

$R \leq_{sd} \mathbf{A}^2$  is linked  $\Rightarrow \exists a \in A: (a, a) \in R$ .

$\leq_{sd}$ :  $\forall i$   $pr_i(R) = A$ , linked: bipartite graph is connected

**Theorem [Maróti, McKenzie, 2008]**

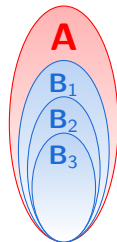
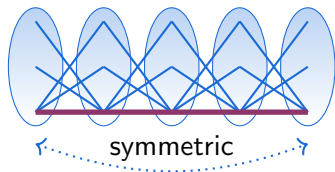
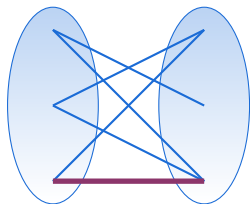
$\emptyset \neq R \leq \mathbf{A}^p$ ,  $R$  is symmetric,

$p > |A|$  is a prime

$\Rightarrow \exists a \in A: (a, a, \dots, a) \in R$ .

Proof: Induction on  $|A|$ .

$\mathbf{A} \geq \mathbf{B}_1 \geq \mathbf{B}_2 \geq \mathbf{B}_3$



## Introduction

$\mathbf{A}$  is a finite idempotent Taylor algebra.

**Loop Lemma**[Barto, Kozik, Niven, 2008]

$R \leq_{sd} \mathbf{A}^2$  is linked  $\Rightarrow \exists a \in A: (a, a) \in R$ .

$\leq_{sd}$ :  $\forall i$   $pr_i(R) = A$ , linked: bipartite graph is connected

**Theorem** [Maróti, McKenzie, 2008]

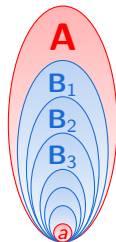
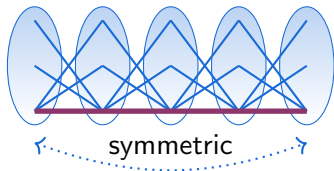
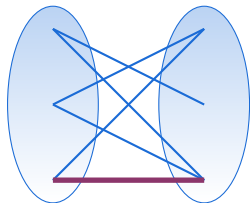
$\emptyset \neq R \leq \mathbf{A}^p$ ,  $R$  is symmetric,

$p > |A|$  is a prime

$\Rightarrow \exists a \in A: (a, a, \dots, a) \in R$ .

Proof: Induction on  $|A|$ .

$\mathbf{A} \geq \mathbf{B}_1 \geq \mathbf{B}_2 \geq \mathbf{B}_3 \geq \dots \geq \{a\}$



## Introduction

$\mathbf{A}$  is a finite idempotent Taylor algebra.

**Loop Lemma**[Barto, Kozik, Niven, 2008]

$R \leq_{sd} \mathbf{A}^2$  is linked  $\Rightarrow \exists a \in A: (a, a) \in R$ .

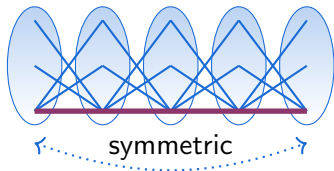
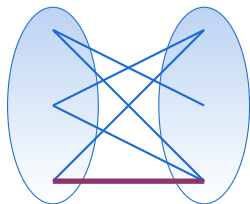
$\leq_{sd}$ :  $\forall i$   $pr_i(R) = A$ , linked: bipartite graph is connected

**Theorem** [Maróti, McKenzie, 2008]

$\emptyset \neq R \leq \mathbf{A}^p$ ,  $R$  is symmetric,

$p > |A|$  is a prime

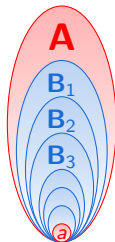
$\Rightarrow \exists a \in A: (a, a, \dots, a) \in R$ .



Proof: Induction on  $|A|$ .

$\mathbf{A} \geq \mathbf{B}_1 \geq \mathbf{B}_2 \geq \mathbf{B}_3 \geq \dots \geq \{a\}$

We need (strong) subalgebras with additional properties:



## Introduction

$\mathbf{A}$  is a finite idempotent Taylor algebra.

**Loop Lemma**[Barto, Kozik, Niven, 2008]

$R \leq_{sd} \mathbf{A}^2$  is linked  $\Rightarrow \exists a \in A: (a, a) \in R$ .

$\leq_{sd}$ :  $\forall i$   $pr_i(R) = A$ , linked: bipartite graph is connected

**Theorem** [Maróti, McKenzie, 2008]

$\emptyset \neq R \leq \mathbf{A}^p$ ,  $R$  is symmetric,

$p > |A|$  is a prime

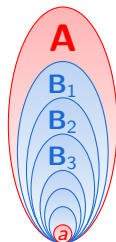
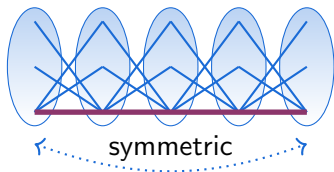
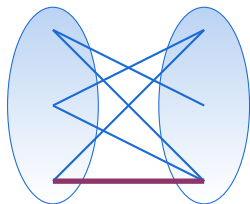
$\Rightarrow \exists a \in A: (a, a, \dots, a) \in R$ .

Proof: Induction on  $|A|$ .

$\mathbf{A} \geq \mathbf{B}_1 \geq \mathbf{B}_2 \geq \mathbf{B}_3 \geq \dots \geq \{a\}$

We need (strong) subalgebras with additional properties:

- It preserves linkedness.





## Introduction

$\mathbf{A}$  is a finite idempotent Taylor algebra.

**Loop Lemma**[Barto, Kozik, Niven, 2008]

$R \leq_{sd} \mathbf{A}^2$  is linked  $\Rightarrow \exists a \in A: (a, a) \in R$ .

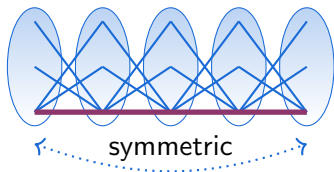
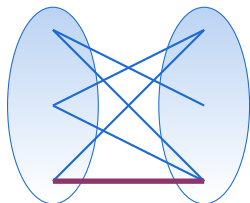
$\leq_{sd}$ :  $\forall i$   $pr_i(R) = A$ , linked: bipartite graph is connected

**Theorem** [Maróti, McKenzie, 2008]

$\emptyset \neq R \leq \mathbf{A}^p$ ,  $R$  is symmetric,

$p > |A|$  is a prime

$\Rightarrow \exists a \in A: (a, a, \dots, a) \in R$ .

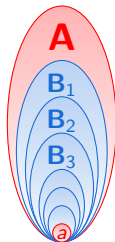


Proof: Induction on  $|A|$ .

$\mathbf{A} \geq \mathbf{B}_1 \geq \mathbf{B}_2 \geq \mathbf{B}_3 \geq \dots \geq \{a\}$

We need (strong) subalgebras with additional properties:

- ▶ It preserves linkedness.
- ▶  $R \cap \mathbf{B}_i^p \neq \emptyset$ .

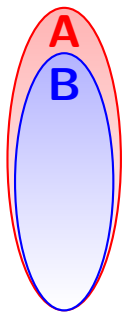


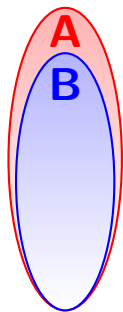


## Absorbing Subalgebras (version 1)

Barto, Kozik

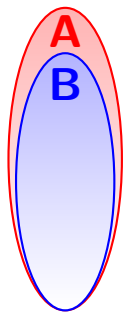
**B** absorbs **A** (write  $\mathbf{B} \trianglelefteq \mathbf{A}$ )





**B absorbs A** (write  $\mathbf{B} \trianglelefteq \mathbf{A}$ ) if  $\mathbf{B} \leq \mathbf{A}$  and  $\exists t \in \text{Clo}(\mathbf{A})$  s.t.

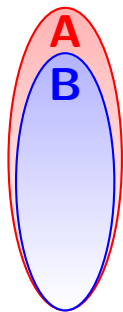
$$\forall i: t(\underbrace{B, \dots, B}_i, A, B, \dots, B) \subseteq B.$$



**B absorbs A** (write  $\mathbf{B} \trianglelefteq \mathbf{A}$ ) if  $\mathbf{B} \leq \mathbf{A}$  and  $\exists t \in \text{Clo}(\mathbf{A})$  s.t.

$$\forall i: t(\underbrace{B, \dots, B}_i, A, B, \dots, B) \subseteq B.$$

Write  $\mathbf{B} \trianglelefteq_k \mathbf{A}$  if  $t$  is of arity  $k$ .

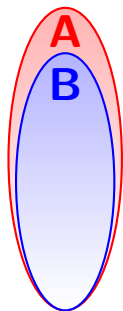


**B absorbs A** (write  $\mathbf{B} \trianglelefteq \mathbf{A}$ ) if  $\mathbf{B} \leq \mathbf{A}$  and  $\exists t \in \text{Clo}(\mathbf{A})$  s.t.

$$\forall i: t(\underbrace{B, \dots, B}_i, A, B, \dots, B) \subseteq B.$$

Write  $\mathbf{B} \trianglelefteq_k \mathbf{A}$  if  $t$  is of arity  $k$ .

### Examples



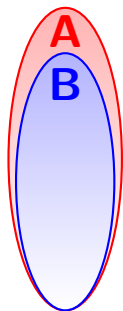
**B absorbs A** (write  $\mathbf{B} \trianglelefteq \mathbf{A}$ ) if  $\mathbf{B} \leq \mathbf{A}$  and  $\exists t \in \text{Clo}(\mathbf{A})$  s.t.

$$\forall i: t(\underbrace{B, \dots, B}_i, A, B, \dots, B) \subseteq B.$$

Write  $\mathbf{B} \trianglelefteq_k \mathbf{A}$  if  $t$  is of arity  $k$ .

### Examples

1.  $\{1\} \trianglelefteq_2 (\{0, 1\}; \vee)$ .



**B absorbs A** (write  $\mathbf{B} \trianglelefteq \mathbf{A}$ ) if  $\mathbf{B} \leq \mathbf{A}$  and  $\exists t \in \text{Clo}(\mathbf{A})$  s.t.

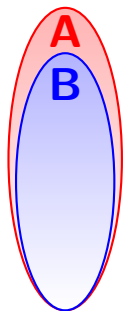
$$\forall i: t(\underbrace{B, \dots, B}_i, A, B, \dots, B) \subseteq B.$$

Write  $\mathbf{B} \trianglelefteq_k \mathbf{A}$  if  $t$  is of arity  $k$ .

### Examples

1.  $\{1\} \trianglelefteq_2 (\{0, 1\}; \vee)$ .
2.  $\{2, 3\} \trianglelefteq_2 (\{0, 1, 2, 3\}; \max)$ .





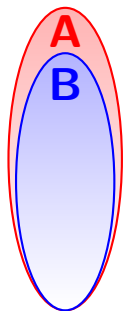
**B absorbs A** (write  $\mathbf{B} \trianglelefteq \mathbf{A}$ ) if  $\mathbf{B} \leq \mathbf{A}$  and  $\exists t \in \text{Clo}(\mathbf{A})$  s.t.

$$\forall i: t(\underbrace{B, \dots, B}_i, A, B, \dots, B) \subseteq B.$$

Write  $\mathbf{B} \trianglelefteq_k \mathbf{A}$  if  $t$  is of arity  $k$ .

### Examples

1.  $\{1\} \trianglelefteq_2 (\{0, 1\}; \vee)$ .
2.  $\{2, 3\} \trianglelefteq_2 (\{0, 1, 2, 3\}; \max)$ .
3.  $\{a\} \trianglelefteq_3 (A; \text{majority})$ .



**B absorbs A** (write  $\mathbf{B} \trianglelefteq \mathbf{A}$ ) if  $\mathbf{B} \leq \mathbf{A}$  and  $\exists t \in \text{Clo}(\mathbf{A})$  s.t.

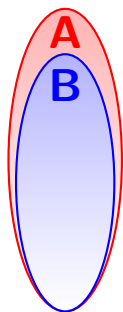
$$\forall i: t(\underbrace{B, \dots, B}_i, A, B, \dots, B) \subseteq B.$$

Write  $\mathbf{B} \trianglelefteq_k \mathbf{A}$  if  $t$  is of arity  $k$ .

### Examples

1.  $\{1\} \trianglelefteq_2 (\{0, 1\}; \vee)$ .
2.  $\{2, 3\} \trianglelefteq_2 (\{0, 1, 2, 3\}; \max)$ .
3.  $\{a\} \trianglelefteq_3 (A; \text{majority})$ .

### Properties



**B absorbs A** (write  $\mathbf{B} \trianglelefteq \mathbf{A}$ ) if  $\mathbf{B} \leq \mathbf{A}$  and  $\exists t \in \text{Clo}(\mathbf{A})$  s.t.

$$\forall i: t(\underbrace{B, \dots, B, A, B, \dots, B}_i) \subseteq B.$$

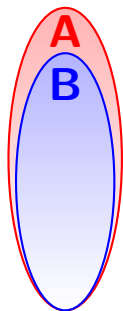
Write  $\mathbf{B} \trianglelefteq_k \mathbf{A}$  if  $t$  is of arity  $k$ .

### Examples

1.  $\{1\} \trianglelefteq_2 (\{0, 1\}; \vee)$ .
2.  $\{2, 3\} \trianglelefteq_2 (\{0, 1, 2, 3\}; \max)$ .
3.  $\{a\} \trianglelefteq_3 (A; \text{majority})$ .

### Properties

1.  $B, C \trianglelefteq \mathbf{A} \Rightarrow B \cap C \trianglelefteq \mathbf{A}$



**B absorbs A** (write  $\mathbf{B} \trianglelefteq \mathbf{A}$ ) if  $\mathbf{B} \leq \mathbf{A}$  and  $\exists t \in \text{Clo}(\mathbf{A})$  s.t.

$$\forall i: t(\underbrace{B, \dots, B, A, B, \dots, B}_i) \subseteq B.$$

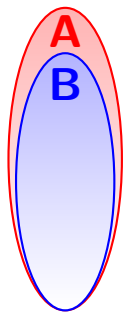
Write  $\mathbf{B} \trianglelefteq_k \mathbf{A}$  if  $t$  is of arity  $k$ .

### Examples

1.  $\{1\} \trianglelefteq_2 (\{0, 1\}; \vee)$ .
2.  $\{2, 3\} \trianglelefteq_2 (\{0, 1, 2, 3\}; \max)$ .
3.  $\{a\} \trianglelefteq_3 (A; \text{majority})$ .

### Properties

1.  $B, C \trianglelefteq \mathbf{A} \Rightarrow B \cap C \trianglelefteq \mathbf{A}$
2.  $\mathbf{C} \trianglelefteq \mathbf{B} \trianglelefteq \mathbf{A} \Rightarrow \mathbf{C} \trianglelefteq \mathbf{A}$



**B absorbs A** (write  $\mathbf{B} \trianglelefteq \mathbf{A}$ ) if  $\mathbf{B} \leq \mathbf{A}$  and  $\exists t \in \text{Clo}(\mathbf{A})$  s.t.

$$\forall i: t(\underbrace{B, \dots, B}_i, A, B, \dots, B) \subseteq B.$$

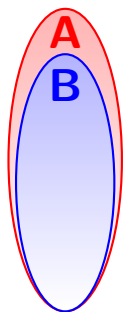
Write  $\mathbf{B} \trianglelefteq_k \mathbf{A}$  if  $t$  is of arity  $k$ .

### Examples

1.  $\{1\} \trianglelefteq_2 (\{0, 1\}; \vee)$ .
2.  $\{2, 3\} \trianglelefteq_2 (\{0, 1, 2, 3\}; \max)$ .
3.  $\{a\} \trianglelefteq_3 (A; \text{majority})$ .

### Properties

1.  $B, C \trianglelefteq \mathbf{A} \Rightarrow B \cap C \trianglelefteq \mathbf{A}$
2.  $\mathbf{C} \trianglelefteq \mathbf{B} \trianglelefteq \mathbf{A} \Rightarrow \mathbf{C} \trianglelefteq \mathbf{A}$
3.  $R \not\leq_{sd} \mathbf{A} \times \mathbf{B}$  is linked  $\Rightarrow \mathbf{A}$  or  $\mathbf{B}$  has an absorbing subalgebra



**B absorbs A** (write  $\mathbf{B} \trianglelefteq \mathbf{A}$ ) if  $\mathbf{B} \leq \mathbf{A}$  and  $\exists t \in \text{Clo}(\mathbf{A})$  s.t.

$$\forall i: t(\underbrace{B, \dots, B}_i, A, B, \dots, B) \subseteq B.$$

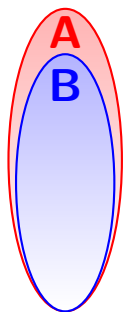
Write  $\mathbf{B} \trianglelefteq_k \mathbf{A}$  if  $t$  is of arity  $k$ .

### Examples

1.  $\{1\} \trianglelefteq_2 (\{0, 1\}; \vee)$ .
2.  $\{2, 3\} \trianglelefteq_2 (\{0, 1, 2, 3\}; \max)$ .
3.  $\{a\} \trianglelefteq_3 (A; \text{majority})$ .

### Properties

1.  $B, C \trianglelefteq \mathbf{A} \Rightarrow B \cap C \trianglelefteq \mathbf{A}$
2.  $\mathbf{C} \trianglelefteq \mathbf{B} \trianglelefteq \mathbf{A} \Rightarrow \mathbf{C} \trianglelefteq \mathbf{A}$
3.  $R \not\leq_{sd} \mathbf{A} \times \mathbf{B}$  is linked  $\Rightarrow \mathbf{A}$  or  $\mathbf{B}$  has an absorbing subalgebra
4.  $\mathbf{B} \trianglelefteq \mathbf{A}$ ,  $R \leq_{sd} \mathbf{A} \times \mathbf{A}$  is linked,  $(R \cap B^2) \leq_{sd} \mathbf{B} \times \mathbf{B}$ ,  
 $\Rightarrow (R \cap B^2)$  linked.



**B absorbs A** (write  $\mathbf{B} \trianglelefteq \mathbf{A}$ ) if  $\mathbf{B} \leq \mathbf{A}$  and  $\exists t \in \text{Clo}(\mathbf{A})$  s.t.

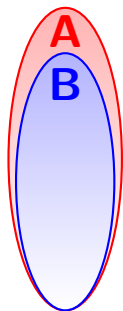
$$\forall i: t(\underbrace{B, \dots, B}_i, A, B, \dots, B) \subseteq B.$$

Write  $\mathbf{B} \trianglelefteq_k \mathbf{A}$  if  $t$  is of arity  $k$ .

### Examples

1.  $\{1\} \trianglelefteq_2 (\{0, 1\}; \vee)$ .
2.  $\{2, 3\} \trianglelefteq_2 (\{0, 1, 2, 3\}; \max)$ .
3.  $\{a\} \trianglelefteq_3 (A; \text{majority})$ .

But...



**B absorbs A** (write  $\mathbf{B} \trianglelefteq \mathbf{A}$ ) if  $\mathbf{B} \leq \mathbf{A}$  and  $\exists t \in \text{Clo}(\mathbf{A})$  s.t.

$$\forall i: t(\underbrace{B, \dots, B}_i, A, B, \dots, B) \subseteq B.$$

Write  $\mathbf{B} \trianglelefteq_k \mathbf{A}$  if  $t$  is of arity  $k$ .

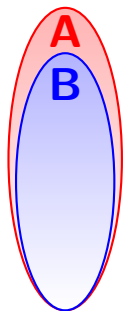
### Examples

1.  $\{1\} \trianglelefteq_2 (\{0, 1\}; \vee)$ .
2.  $\{2, 3\} \trianglelefteq_2 (\{0, 1, 2, 3\}; \max)$ .
3.  $\{a\} \trianglelefteq_3 (A; \text{majority})$ .

### But...

1. some algebras do not have absorbing subalgebras





**B absorbs A** (write  $\mathbf{B} \trianglelefteq \mathbf{A}$ ) if  $\mathbf{B} \leq \mathbf{A}$  and  $\exists t \in \text{Clo}(\mathbf{A})$  s.t.

$$\forall i: t(\underbrace{B, \dots, B}_i, A, B, \dots, B) \subseteq B.$$

Write  $\mathbf{B} \trianglelefteq_k \mathbf{A}$  if  $t$  is of arity  $k$ .

### Examples

1.  $\{1\} \trianglelefteq_2 (\{0, 1\}; \vee)$ .
2.  $\{2, 3\} \trianglelefteq_2 (\{0, 1, 2, 3\}; \max)$ .
3.  $\{a\} \trianglelefteq_3 (A; \text{majority})$ .

### But...

1. some algebras do not have absorbing subalgebras
2. sometimes we need stronger properties



1.  $\mathbf{B}$  is a **binary absorbing subalgebra** of  $\mathbf{A}$ , i.e.  $\mathbf{B} \trianglelefteq_2 \mathbf{A}$

1.  $\mathbf{B}$  is a **binary absorbing subalgebra** of  $\mathbf{A}$ , i.e.  $\mathbf{B} \trianglelefteq_2 \mathbf{A}$
2.  $\mathbf{B}$  is a **center** of  $\mathbf{A}$ , i.e. an absorbing subalgebra s.t.  
 $\forall a \in A \setminus B: (a, a) \notin \text{Sg}_{\mathbf{A}^2}((\{a\} \times B) \cup (B \times \{a\}))$

1. **B** is a **binary absorbing subalgebra** of **A**, i.e.  $\mathbf{B} \trianglelefteq_2 \mathbf{A}$
2. **B** is a **center** of **A**, i.e. an absorbing subalgebra s.t.  
 $\forall a \in A \setminus B: (a, a) \notin \text{Sg}_{\mathbf{A}^2}((\{a\} \times B) \cup (B \times \{a\}))$
3. **B** is a **linear subalgebra** of **A**, i.e a block of a congruence  $\sigma$  s.t.  
 $\mathbf{A}/\sigma$  is an affine square-free algebra, i.e.
  - ▶ there exists  $(A/\sigma; \oplus) \cong (\mathbb{Z}_{p_1} \times \cdots \times \mathbb{Z}_{p_s}; +)$
  - ▶  $(x_1 \oplus x_2 = x_3 \oplus x_4) \in \text{Inv}(\mathbf{A}/\sigma)$
  - ▶  $x_1 \oplus \cdots \oplus x_k \in \text{Clo}(\mathbf{A}/\sigma)$

1. **B** is a **binary absorbing subalgebra** of **A**, i.e.  $\mathbf{B} \trianglelefteq_2 \mathbf{A}$
2. **B** is a **center** of **A**, i.e. an absorbing subalgebra s.t.  
 $\forall a \in A \setminus B: (a, a) \notin \text{Sg}_{\mathbf{A}^2}((\{a\} \times B) \cup (B \times \{a\}))$
3. **B** is a **linear subalgebra** of **A**, i.e a block of a congruence  $\sigma$  s.t.  $\mathbf{A}/\sigma$  is an affine square-free algebra, i.e.
  - ▶ there exists  $(\mathbf{A}/\sigma; \oplus) \cong (\mathbb{Z}_{p_1} \times \cdots \times \mathbb{Z}_{p_s}; +)$
  - ▶  $(x_1 \oplus x_2 = x_3 \oplus x_4) \in \text{Inv}(\mathbf{A}/\sigma)$
  - ▶  $x_1 \oplus \cdots \oplus x_k \in \text{Clo}(\mathbf{A}/\sigma)$
4. **B** is a **PC subalgebra** of **A**, i.e. a block of a congruence  $\sigma$  s.t.  $\mathbf{A}/\sigma \cong \mathbf{A}_1 \times \cdots \times \mathbf{A}_s$ , where each  $\mathbf{A}_i$  is a Polynomially Complete (PC) algebra without binary absorption or center.

1. **B** is a **binary absorbing subalgebra** of **A**, i.e.  $\mathbf{B} \trianglelefteq_2 \mathbf{A}$
2. **B** is a **center** of **A**, i.e. an absorbing subalgebra s.t.  
 $\forall a \in A \setminus B: (a, a) \notin \text{Sg}_{\mathbf{A}^2}((\{a\} \times B) \cup (B \times \{a\}))$
3. **B** is a **linear subalgebra** of **A**, i.e a block of a congruence  $\sigma$  s.t.  $\mathbf{A}/\sigma$  is an affine square-free algebra, i.e.
  - ▶ there exists  $(\mathbf{A}/\sigma; \oplus) \cong (\mathbb{Z}_{p_1} \times \cdots \times \mathbb{Z}_{p_s}; +)$
  - ▶  $(x_1 \oplus x_2 = x_3 \oplus x_4) \in \text{Inv}(\mathbf{A}/\sigma)$
  - ▶  $x_1 \oplus \cdots \oplus x_k \in \text{Clo}(\mathbf{A}/\sigma)$
4. **B** is a **PC subalgebra** of **A**, i.e. a block of a congruence  $\sigma$  s.t.  $\mathbf{A}/\sigma \cong \mathbf{A}_1 \times \cdots \times \mathbf{A}_s$ , where each  $\mathbf{A}_i$  is a Polynomially Complete (PC) algebra without binary absorption or center.

## Advantages

1. **B** is a **binary absorbing subalgebra** of **A**, i.e.  $\mathbf{B} \trianglelefteq_2 \mathbf{A}$
2. **B** is a **center** of **A**, i.e. an absorbing subalgebra s.t.  
 $\forall a \in A \setminus B: (a, a) \notin \text{Sg}_{\mathbf{A}^2}((\{a\} \times B) \cup (B \times \{a\}))$
3. **B** is a **linear subalgebra** of **A**, i.e. a block of a congruence  $\sigma$  s.t.  $\mathbf{A}/\sigma$  is an affine square-free algebra, i.e.
  - ▶ there exists  $(\mathbf{A}/\sigma; \oplus) \cong (\mathbb{Z}_{p_1} \times \cdots \times \mathbb{Z}_{p_s}; +)$
  - ▶  $(x_1 \oplus x_2 = x_3 \oplus x_4) \in \text{Inv}(\mathbf{A}/\sigma)$
  - ▶  $x_1 \oplus \cdots \oplus x_k \in \text{Clo}(\mathbf{A}/\sigma)$
4. **B** is a **PC subalgebra** of **A**, i.e. a block of a congruence  $\sigma$  s.t.  $\mathbf{A}/\sigma \cong \mathbf{A}_1 \times \cdots \times \mathbf{A}_s$ , where each  $\mathbf{A}_i$  is a Polynomially Complete (PC) algebra without binary absorption or center.

### Advantages

1. Every algebra **A** has a strong subalgebra (for  $|A| > 1$ ).



1. **B** is a **binary absorbing subalgebra** of **A**, i.e.  $\mathbf{B} \trianglelefteq_2 \mathbf{A}$
2. **B** is a **center** of **A**, i.e. an absorbing subalgebra s.t.  
 $\forall a \in A \setminus B: (a, a) \notin \text{Sg}_{\mathbf{A}^2}((\{a\} \times B) \cup (B \times \{a\}))$
3. **B** is a **linear subalgebra** of **A**, i.e a block of a congruence  $\sigma$  s.t.  $\mathbf{A}/\sigma$  is an affine square-free algebra, i.e.
  - ▶ there exists  $(\mathbf{A}/\sigma; \oplus) \cong (\mathbb{Z}_{p_1} \times \cdots \times \mathbb{Z}_{p_s}; +)$
  - ▶  $(x_1 \oplus x_2 = x_3 \oplus x_4) \in \text{Inv}(\mathbf{A}/\sigma)$
  - ▶  $x_1 \oplus \cdots \oplus x_k \in \text{Clo}(\mathbf{A}/\sigma)$
4. **B** is a **PC subalgebra** of **A**, i.e. a block of a congruence  $\sigma$  s.t.  $\mathbf{A}/\sigma \cong \mathbf{A}_1 \times \cdots \times \mathbf{A}_s$ , where each  $\mathbf{A}_i$  is a Polynomially Complete (PC) algebra without binary absorption or center.

### Advantages

1. Every algebra **A** has a strong subalgebra (for  $|A| > 1$ ).
2. The main ingredient of the CSP Dichotomy Proof.

1. **B** is a **binary absorbing subalgebra** of **A**, i.e.  $\mathbf{B} \trianglelefteq_2 \mathbf{A}$
2. **B** is a **center** of **A**, i.e. an absorbing subalgebra s.t.  
 $\forall a \in A \setminus B: (a, a) \notin \text{Sg}_{\mathbf{A}^2}((\{a\} \times B) \cup (B \times \{a\}))$
3. **B** is a **linear subalgebra** of **A**, i.e a block of a congruence  $\sigma$  s.t.  $\mathbf{A}/\sigma$  is an affine square-free algebra, i.e.
  - ▶ there exists  $(\mathbf{A}/\sigma; \oplus) \cong (\mathbb{Z}_{p_1} \times \cdots \times \mathbb{Z}_{p_s}; +)$
  - ▶  $(x_1 \oplus x_2 = x_3 \oplus x_4) \in \text{Inv}(\mathbf{A}/\sigma)$
  - ▶  $x_1 \oplus \cdots \oplus x_k \in \text{Clo}(\mathbf{A}/\sigma)$
4. **B** is a **PC subalgebra** of **A**, i.e. a block of a congruence  $\sigma$  s.t.  $\mathbf{A}/\sigma \cong \mathbf{A}_1 \times \cdots \times \mathbf{A}_s$ , where each  $\mathbf{A}_i$  is a Polynomially Complete (PC) algebra without binary absorption or center.

### Advantages

1. Every algebra **A** has a strong subalgebra (for  $|A| > 1$ ).
2. The main ingredient of the CSP Dichotomy Proof.
3. An easy proof of the existence of WNU and characterization of CSPs with bounded width.

1. **B** is a **binary absorbing subalgebra** of **A**, i.e.  $\mathbf{B} \trianglelefteq_2 \mathbf{A}$
2. **B** is a **center** of **A**, i.e. an absorbing subalgebra s.t.  
 $\forall a \in A \setminus B: (a, a) \notin \text{Sg}_{\mathbf{A}^2}((\{a\} \times B) \cup (B \times \{a\}))$
3. **B** is a **linear subalgebra** of **A**, i.e a block of a congruence  $\sigma$  s.t.  $\mathbf{A}/\sigma$  is an affine square-free algebra, i.e.
  - ▶ there exists  $(\mathbf{A}/\sigma; \oplus) \cong (\mathbb{Z}_{p_1} \times \cdots \times \mathbb{Z}_{p_s}; +)$
  - ▶  $(x_1 \oplus x_2 = x_3 \oplus x_4) \in \text{Inv}(\mathbf{A}/\sigma)$
  - ▶  $x_1 \oplus \cdots \oplus x_k \in \text{Clo}(\mathbf{A}/\sigma)$
4. **B** is a **PC subalgebra** of **A**, i.e. a block of a congruence  $\sigma$  s.t.  $\mathbf{A}/\sigma \cong \mathbf{A}_1 \times \cdots \times \mathbf{A}_s$ , where each  $\mathbf{A}_i$  is a Polynomially Complete (PC) algebra without binary absorption or center.

But...

1. **B** is a **binary absorbing subalgebra** of **A**, i.e.  $\mathbf{B} \trianglelefteq_2 \mathbf{A}$
2. **B** is a **center** of **A**, i.e. an absorbing subalgebra s.t.  
 $\forall a \in A \setminus B: (a, a) \notin \text{Sg}_{\mathbf{A}^2}((\{a\} \times B) \cup (B \times \{a\}))$
3. **B** is a **linear subalgebra** of **A**, i.e a block of a congruence  $\sigma$  s.t.  $\mathbf{A}/\sigma$  is an affine square-free algebra, i.e.
  - ▶ there exists  $(\mathbf{A}/\sigma; \oplus) \cong (\mathbb{Z}_{p_1} \times \cdots \times \mathbb{Z}_{p_s}; +)$
  - ▶  $(x_1 \oplus x_2 = x_3 \oplus x_4) \in \text{Inv}(\mathbf{A}/\sigma)$
  - ▶  $x_1 \oplus \cdots \oplus x_k \in \text{Clo}(\mathbf{A}/\sigma)$
4. **B** is a **PC subalgebra** of **A**, i.e. a block of a congruence  $\sigma$  s.t.  $\mathbf{A}/\sigma \cong \mathbf{A}_1 \times \cdots \times \mathbf{A}_s$ , where each  $\mathbf{A}_i$  is a Polynomially Complete (PC) algebra without binary absorption or center.

### But...

1. No transitivity: we must remember previous reductions.

1. **B** is a **binary absorbing subalgebra** of **A**, i.e.  $\mathbf{B} \trianglelefteq_2 \mathbf{A}$
2. **B** is a **center** of **A**, i.e. an absorbing subalgebra s.t.  
 $\forall a \in A \setminus B: (a, a) \notin \text{Sg}_{\mathbf{A}^2}((\{a\} \times B) \cup (B \times \{a\}))$
3. **B** is a **linear subalgebra** of **A**, i.e a block of a congruence  $\sigma$  s.t.  $\mathbf{A}/\sigma$  is an affine square-free algebra, i.e.
  - ▶ there exists  $(A/\sigma; \oplus) \cong (\mathbb{Z}_{p_1} \times \cdots \times \mathbb{Z}_{p_s}; +)$
  - ▶  $(x_1 \oplus x_2 = x_3 \oplus x_4) \in \text{Inv}(\mathbf{A}/\sigma)$
  - ▶  $x_1 \oplus \cdots \oplus x_k \in \text{Clo}(\mathbf{A}/\sigma)$
4. **B** is a **PC subalgebra** of **A**, i.e. a block of a congruence  $\sigma$  s.t.  $\mathbf{A}/\sigma \cong \mathbf{A}_1 \times \cdots \times \mathbf{A}_s$ , where each  $\mathbf{A}_i$  is a Polynomially Complete (PC) algebra without binary absorption or center.

### But...

1. No transitivity: we must remember previous reductions.
2. Linear subalgebras are not strong enough.



## Stable Subalgebras (version 3)

Zarathustra Brady

(Subalgebra)  $\mathbf{B} \prec \mathbf{A} \Rightarrow \mathbf{B} \leq \mathbf{A}$

## Stable Subalgebras (version 3)

Zarathustra Brady

(Subalgebra)  $\mathbf{B} \prec \mathbf{A} \Rightarrow \mathbf{B} \leq \mathbf{A}$

(Transitivity)  $\mathbf{C} \prec \mathbf{B} \prec \mathbf{A} \Rightarrow \mathbf{C} \prec \mathbf{A}$



## Stable Subalgebras (version 3)

Zarathustra Brady

(Subalgebra)  $\mathbf{B} \prec \mathbf{A} \Rightarrow \mathbf{B} \leq \mathbf{A}$

(Transitivity)  $\mathbf{C} \prec \mathbf{B} \prec \mathbf{A} \Rightarrow \mathbf{C} \prec \mathbf{A}$

(Subalgebra)  $\mathbf{B} \prec \mathbf{A} \Rightarrow \mathbf{B} \leq \mathbf{A}$

(Transitivity)  $\mathbf{C} \prec \mathbf{B} \prec \mathbf{A} \Rightarrow \mathbf{C} \prec \mathbf{A}$

(Intersection)  $\mathbf{C}, \mathbf{B} \prec \mathbf{A} \wedge \mathbf{B} \cap \mathbf{C} \neq 0 \Rightarrow \mathbf{B} \cap \mathbf{C} \prec \mathbf{A}$

(Subalgebra)  $\mathbf{B} \prec \mathbf{A} \Rightarrow \mathbf{B} \leq \mathbf{A}$

(Transitivity)  $\mathbf{C} \prec \mathbf{B} \prec \mathbf{A} \Rightarrow \mathbf{C} \prec \mathbf{A}$

(Intersection)  $\mathbf{C}, \mathbf{B} \prec \mathbf{A} \wedge \mathbf{B} \cap \mathbf{C} \neq 0 \Rightarrow \mathbf{B} \cap \mathbf{C} \prec \mathbf{A}$

(Subalgebra)  $\mathbf{B} \prec \mathbf{A} \Rightarrow \mathbf{B} \leq \mathbf{A}$

(Transitivity)  $\mathbf{C} \prec \mathbf{B} \prec \mathbf{A} \Rightarrow \mathbf{C} \prec \mathbf{A}$

(Intersection)  $\mathbf{C}, \mathbf{B} \prec \mathbf{A} \wedge \mathbf{B} \cap \mathbf{C} \neq 0 \Rightarrow \mathbf{B} \cap \mathbf{C} \prec \mathbf{A}$

(Propagation) if  $f: \mathbf{A} \rightarrow \mathbf{B}$  is a surjective homomorphism,

(Pushforward)  $\mathbf{C} \prec \mathbf{A} \Rightarrow f(\mathbf{C}) \prec \mathbf{B}$

(Pullback)  $\mathbf{D} \prec \mathbf{B} \Rightarrow f^{-1}(\mathbf{D}) \prec \mathbf{A}$

(Subalgebra)  $\mathbf{B} \prec \mathbf{A} \Rightarrow \mathbf{B} \leq \mathbf{A}$

(Transitivity)  $\mathbf{C} \prec \mathbf{B} \prec \mathbf{A} \Rightarrow \mathbf{C} \prec \mathbf{A}$

(Intersection)  $\mathbf{C}, \mathbf{B} \prec \mathbf{A} \wedge \mathbf{B} \cap \mathbf{C} \neq 0 \Rightarrow \mathbf{B} \cap \mathbf{C} \prec \mathbf{A}$

(Propagation) if  $f: \mathbf{A} \rightarrow \mathbf{B}$  is a surjective homomorphism,

(Pushforward)  $\mathbf{C} \prec \mathbf{A} \Rightarrow f(\mathbf{C}) \prec \mathbf{B}$

(Pullback)  $\mathbf{D} \prec \mathbf{B} \Rightarrow f^{-1}(\mathbf{D}) \prec \mathbf{A}$

(Helly) if  $\mathbf{B}_1, \mathbf{B}_2, \mathbf{B}_3 \prec \mathbf{A}, \forall i, j: \mathbf{B}_i \cap \mathbf{B}_j \neq \emptyset \Rightarrow \mathbf{B}_1 \cap \mathbf{B}_2 \cap \mathbf{B}_3 \neq \emptyset$ .

(Subalgebra)  $\mathbf{B} \prec \mathbf{A} \Rightarrow \mathbf{B} \leq \mathbf{A}$

(Transitivity)  $\mathbf{C} \prec \mathbf{B} \prec \mathbf{A} \Rightarrow \mathbf{C} \prec \mathbf{A}$

(Intersection)  $\mathbf{C}, \mathbf{B} \prec \mathbf{A} \wedge \mathbf{B} \cap \mathbf{C} \neq 0 \Rightarrow \mathbf{B} \cap \mathbf{C} \prec \mathbf{A}$

(Propagation) if  $f: \mathbf{A} \rightarrow \mathbf{B}$  is a surjective homomorphism,

(Pushforward)  $\mathbf{C} \prec \mathbf{A} \Rightarrow f(\mathbf{C}) \prec \mathbf{B}$

(Pullback)  $\mathbf{D} \prec \mathbf{B} \Rightarrow f^{-1}(\mathbf{D}) \prec \mathbf{A}$

(Helly) if  $\mathbf{B}_1, \mathbf{B}_2, \mathbf{B}_3 \prec \mathbf{A}, \forall i, j: \mathbf{B}_i \cap \mathbf{B}_j \neq \emptyset \Rightarrow \mathbf{B}_1 \cap \mathbf{B}_2 \cap \mathbf{B}_3 \neq \emptyset$ .

(Ubiquity) if  $|\mathbf{A}| > 1$  then either

(Subalgebra)  $\mathbf{B} \prec \mathbf{A} \Rightarrow \mathbf{B} \leq \mathbf{A}$

(Transitivity)  $\mathbf{C} \prec \mathbf{B} \prec \mathbf{A} \Rightarrow \mathbf{C} \prec \mathbf{A}$

(Intersection)  $\mathbf{C}, \mathbf{B} \prec \mathbf{A} \wedge \mathbf{B} \cap \mathbf{C} \neq 0 \Rightarrow \mathbf{B} \cap \mathbf{C} \prec \mathbf{A}$

(Propagation) if  $f: \mathbf{A} \rightarrow \mathbf{B}$  is a surjective homomorphism,

(Pushforward)  $\mathbf{C} \prec \mathbf{A} \Rightarrow f(\mathbf{C}) \prec \mathbf{B}$

(Pullback)  $\mathbf{D} \prec \mathbf{B} \Rightarrow f^{-1}(\mathbf{D}) \prec \mathbf{A}$

(Helly) if  $\mathbf{B}_1, \mathbf{B}_2, \mathbf{B}_3 \prec \mathbf{A}, \forall i, j: \mathbf{B}_i \cap \mathbf{B}_j \neq \emptyset \Rightarrow \mathbf{B}_1 \cap \mathbf{B}_2 \cap \mathbf{B}_3 \neq \emptyset$ .

(Ubiquity) if  $|\mathbf{A}| > 1$  then either

▶  $\exists \mathbf{B} \prec \mathbf{A}$  s.t.  $\emptyset \neq \mathbf{B} \neq \mathbf{A}$ , or

(Subalgebra)  $\mathbf{B} \prec \mathbf{A} \Rightarrow \mathbf{B} \leq \mathbf{A}$

(Transitivity)  $\mathbf{C} \prec \mathbf{B} \prec \mathbf{A} \Rightarrow \mathbf{C} \prec \mathbf{A}$

(Intersection)  $\mathbf{C}, \mathbf{B} \prec \mathbf{A} \wedge \mathbf{B} \cap \mathbf{C} \neq 0 \Rightarrow \mathbf{B} \cap \mathbf{C} \prec \mathbf{A}$

(Propagation) if  $f: \mathbf{A} \rightarrow \mathbf{B}$  is a surjective homomorphism,

(Pushforward)  $\mathbf{C} \prec \mathbf{A} \Rightarrow f(\mathbf{C}) \prec \mathbf{B}$

(Pullback)  $\mathbf{D} \prec \mathbf{B} \Rightarrow f^{-1}(\mathbf{D}) \prec \mathbf{A}$

(Helly) if  $\mathbf{B}_1, \mathbf{B}_2, \mathbf{B}_3 \prec \mathbf{A}, \forall i, j: \mathbf{B}_i \cap \mathbf{B}_j \neq \emptyset \Rightarrow \mathbf{B}_1 \cap \mathbf{B}_2 \cap \mathbf{B}_3 \neq \emptyset$ .

(Ubiquity) if  $|\mathbf{A}| > 1$  then either

- ▶  $\exists \mathbf{B} \prec \mathbf{A}$  s.t.  $\emptyset \neq \mathbf{B} \neq \mathbf{A}$ , or
- ▶  $\exists \theta \in \text{Con}(\mathbf{A})$  s.t.  $\mathbf{A}/\theta$  is affine.



(Subalgebra)  $\mathbf{B} \prec \mathbf{A} \Rightarrow \mathbf{B} \leq \mathbf{A}$

(Transitivity)  $\mathbf{C} \prec \mathbf{B} \prec \mathbf{A} \Rightarrow \mathbf{C} \prec \mathbf{A}$

(Intersection)  $\mathbf{C}, \mathbf{B} \prec \mathbf{A} \wedge \mathbf{B} \cap \mathbf{C} \neq 0 \Rightarrow \mathbf{B} \cap \mathbf{C} \prec \mathbf{A}$

(Propagation) if  $f: \mathbf{A} \rightarrow \mathbf{B}$  is a surjective homomorphism,

(Pushforward)  $\mathbf{C} \prec \mathbf{A} \Rightarrow f(\mathbf{C}) \prec \mathbf{B}$

(Pullback)  $\mathbf{D} \prec \mathbf{B} \Rightarrow f^{-1}(\mathbf{D}) \prec \mathbf{A}$

(Helly) if  $\mathbf{B}_1, \mathbf{B}_2, \mathbf{B}_3 \prec \mathbf{A}, \forall i, j: \mathbf{B}_i \cap \mathbf{B}_j \neq \emptyset \Rightarrow \mathbf{B}_1 \cap \mathbf{B}_2 \cap \mathbf{B}_3 \neq \emptyset$ .

(Ubiquity) if  $|\mathbf{A}| > 1$  then either

- ▶  $\exists \mathbf{B} \prec \mathbf{A}$  s.t.  $\emptyset \neq \mathbf{B} \neq \mathbf{A}$ , or
- ▶  $\exists \theta \in \text{Con}(\mathbf{A})$  s.t.  $\mathbf{A}/\theta$  is affine.

## Advantages

(Subalgebra)  $\mathbf{B} \prec \mathbf{A} \Rightarrow \mathbf{B} \leq \mathbf{A}$

(Transitivity)  $\mathbf{C} \prec \mathbf{B} \prec \mathbf{A} \Rightarrow \mathbf{C} \prec \mathbf{A}$

(Intersection)  $\mathbf{C}, \mathbf{B} \prec \mathbf{A} \wedge \mathbf{B} \cap \mathbf{C} \neq 0 \Rightarrow \mathbf{B} \cap \mathbf{C} \prec \mathbf{A}$

(Propagation) if  $f: \mathbf{A} \rightarrow \mathbf{B}$  is a surjective homomorphism,

(Pushforward)  $\mathbf{C} \prec \mathbf{A} \Rightarrow f(\mathbf{C}) \prec \mathbf{B}$

(Pullback)  $\mathbf{D} \prec \mathbf{B} \Rightarrow f^{-1}(\mathbf{D}) \prec \mathbf{A}$

(Helly) if  $\mathbf{B}_1, \mathbf{B}_2, \mathbf{B}_3 \prec \mathbf{A}, \forall i, j: \mathbf{B}_i \cap \mathbf{B}_j \neq \emptyset \Rightarrow \mathbf{B}_1 \cap \mathbf{B}_2 \cap \mathbf{B}_3 \neq \emptyset$ .

(Ubiquity) if  $|\mathbf{A}| > 1$  then either

- ▶  $\exists \mathbf{B} \prec \mathbf{A}$  s.t.  $\emptyset \neq \mathbf{B} \neq \mathbf{A}$ , or
- ▶  $\exists \theta \in \text{Con}(\mathbf{A})$  s.t.  $\mathbf{A}/\theta$  is affine.

## Advantages

1. Even weaker consistency condition for the bounded width CSP

(Subalgebra)  $\mathbf{B} \prec \mathbf{A} \Rightarrow \mathbf{B} \leq \mathbf{A}$

(Transitivity)  $\mathbf{C} \prec \mathbf{B} \prec \mathbf{A} \Rightarrow \mathbf{C} \prec \mathbf{A}$

(Intersection)  $\mathbf{C}, \mathbf{B} \prec \mathbf{A} \wedge \mathbf{B} \cap \mathbf{C} \neq 0 \Rightarrow \mathbf{B} \cap \mathbf{C} \prec \mathbf{A}$

(Propagation) if  $f: \mathbf{A} \rightarrow \mathbf{B}$  is a surjective homomorphism,

(Pushforward)  $\mathbf{C} \prec \mathbf{A} \Rightarrow f(\mathbf{C}) \prec \mathbf{B}$

(Pullback)  $\mathbf{D} \prec \mathbf{B} \Rightarrow f^{-1}(\mathbf{D}) \prec \mathbf{A}$

(Helly) if  $\mathbf{B}_1, \mathbf{B}_2, \mathbf{B}_3 \prec \mathbf{A}, \forall i, j: \mathbf{B}_i \cap \mathbf{B}_j \neq \emptyset \Rightarrow \mathbf{B}_1 \cap \mathbf{B}_2 \cap \mathbf{B}_3 \neq \emptyset$ .

(Ubiquity) if  $|\mathbf{A}| > 1$  then either

- ▶  $\exists \mathbf{B} \prec \mathbf{A}$  s.t.  $\emptyset \neq \mathbf{B} \neq \mathbf{A}$ , or
- ▶  $\exists \theta \in \text{Con}(\mathbf{A})$  s.t.  $\mathbf{A}/\theta$  is affine.

## Advantages

1. Even weaker consistency condition for the bounded width CSP
2. Characterization of identities satisfied by any bounded width algebra.

(Subalgebra)  $\mathbf{B} \prec \mathbf{A} \Rightarrow \mathbf{B} \leq \mathbf{A}$

(Transitivity)  $\mathbf{C} \prec \mathbf{B} \prec \mathbf{A} \Rightarrow \mathbf{C} \prec \mathbf{A}$

(Intersection)  $\mathbf{C}, \mathbf{B} \prec \mathbf{A} \wedge \mathbf{B} \cap \mathbf{C} \neq 0 \Rightarrow \mathbf{B} \cap \mathbf{C} \prec \mathbf{A}$

(Propagation) if  $f: \mathbf{A} \rightarrow \mathbf{B}$  is a surjective homomorphism,

(Pushforward)  $\mathbf{C} \prec \mathbf{A} \Rightarrow f(\mathbf{C}) \prec \mathbf{B}$

(Pullback)  $\mathbf{D} \prec \mathbf{B} \Rightarrow f^{-1}(\mathbf{D}) \prec \mathbf{A}$

(Helly) if  $\mathbf{B}_1, \mathbf{B}_2, \mathbf{B}_3 \prec \mathbf{A}, \forall i, j: \mathbf{B}_i \cap \mathbf{B}_j \neq \emptyset \Rightarrow \mathbf{B}_1 \cap \mathbf{B}_2 \cap \mathbf{B}_3 \neq \emptyset$ .

(Ubiquity) if  $|\mathbf{A}| > 1$  then either

- ▶  $\exists \mathbf{B} \prec \mathbf{A}$  s.t.  $\emptyset \neq \mathbf{B} \neq \mathbf{A}$ , or
- ▶  $\exists \theta \in \text{Con}(\mathbf{A})$  s.t.  $\mathbf{A}/\theta$  is affine.

But...

(Subalgebra)  $\mathbf{B} \prec \mathbf{A} \Rightarrow \mathbf{B} \leq \mathbf{A}$

(Transitivity)  $\mathbf{C} \prec \mathbf{B} \prec \mathbf{A} \Rightarrow \mathbf{C} \prec \mathbf{A}$

(Intersection)  $\mathbf{C}, \mathbf{B} \prec \mathbf{A} \wedge \mathbf{B} \cap \mathbf{C} \neq 0 \Rightarrow \mathbf{B} \cap \mathbf{C} \prec \mathbf{A}$

(Propagation) if  $f: \mathbf{A} \rightarrow \mathbf{B}$  is a surjective homomorphism,

(Pushforward)  $\mathbf{C} \prec \mathbf{A} \Rightarrow f(\mathbf{C}) \prec \mathbf{B}$

(Pullback)  $\mathbf{D} \prec \mathbf{B} \Rightarrow f^{-1}(\mathbf{D}) \prec \mathbf{A}$

(Helly) if  $\mathbf{B}_1, \mathbf{B}_2, \mathbf{B}_3 \prec \mathbf{A}, \forall i, j: \mathbf{B}_i \cap \mathbf{B}_j \neq \emptyset \Rightarrow \mathbf{B}_1 \cap \mathbf{B}_2 \cap \mathbf{B}_3 \neq \emptyset$ .

(Ubiquity) if  $|\mathbf{A}| > 1$  then either

- ▶  $\exists \mathbf{B} \prec \mathbf{A}$  s.t.  $\emptyset \neq \mathbf{B} \neq \mathbf{A}$ , or
- ▶  $\exists \theta \in \text{Con}(\mathbf{A})$  s.t.  $\mathbf{A}/\theta$  is affine.

### But...

1. No explicit definition

(Subalgebra)  $\mathbf{B} \prec \mathbf{A} \Rightarrow \mathbf{B} \leq \mathbf{A}$

(Transitivity)  $\mathbf{C} \prec \mathbf{B} \prec \mathbf{A} \Rightarrow \mathbf{C} \prec \mathbf{A}$

(Intersection)  $\mathbf{C}, \mathbf{B} \prec \mathbf{A} \wedge \mathbf{B} \cap \mathbf{C} \neq 0 \Rightarrow \mathbf{B} \cap \mathbf{C} \prec \mathbf{A}$

(Propagation) if  $f: \mathbf{A} \rightarrow \mathbf{B}$  is a surjective homomorphism,

(Pushforward)  $\mathbf{C} \prec \mathbf{A} \Rightarrow f(\mathbf{C}) \prec \mathbf{B}$

(Pullback)  $\mathbf{D} \prec \mathbf{B} \Rightarrow f^{-1}(\mathbf{D}) \prec \mathbf{A}$

(Helly) if  $\mathbf{B}_1, \mathbf{B}_2, \mathbf{B}_3 \prec \mathbf{A}, \forall i, j: \mathbf{B}_i \cap \mathbf{B}_j \neq \emptyset \Rightarrow \mathbf{B}_1 \cap \mathbf{B}_2 \cap \mathbf{B}_3 \neq \emptyset$ .

(Ubiquity) if  $|\mathbf{A}| > 1$  then either

- ▶  $\exists \mathbf{B} \prec \mathbf{A}$  s.t.  $\emptyset \neq \mathbf{B} \neq \mathbf{A}$ , or
- ▶  $\exists \theta \in \text{Con}(\mathbf{A})$  s.t.  $\mathbf{A}/\theta$  is affine.

### But...

1. No explicit definition
2. Affine case is too weak and cannot be used. This works mostly for the bounded width case.



New definitions are super nice and simple!



New definitions are super nice and simple!

But...

New definitions are super nice and simple!

But... to simplify even more we define them only for Taylor  
Minimal Algebras:

New definitions are super nice and simple!

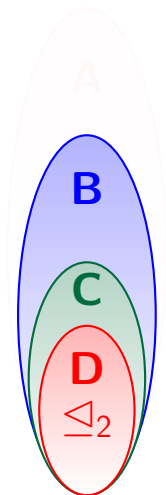
But... to simplify even more we define them only for Taylor Minimal Algebras:

**A** is **Taylor minimal** if  $\text{Clo}(\mathbf{A})$  is an inclusion minimal clone containing a Taylor operation.



I. Contains a Binary Absorbing Subalgebra.

$$\mathbf{C} \triangleleft_2 \mathbf{B} \Leftrightarrow \mathbf{C} \leq \mathbf{B} \wedge \exists \mathbf{D} \triangleleft_2 \mathbf{B} \text{ s.t. } \mathbf{D} \subseteq \mathbf{C}.$$

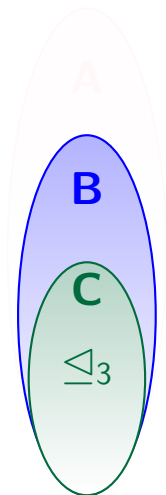


I. Contains a Binary Absorbing Subalgebra.

$$\mathbf{C} \triangleleft_2 \mathbf{B} \Leftrightarrow \mathbf{C} \leq \mathbf{B} \wedge \exists \mathbf{D} \triangleleft_2 \mathbf{B} \text{ s.t. } \mathbf{D} \subseteq \mathbf{C}.$$

II. Ternary absorbing subalgebra.

$$\mathbf{C} \triangleleft_3 \mathbf{B}.$$



I. Contains a Binary Absorbing Subalgebra.

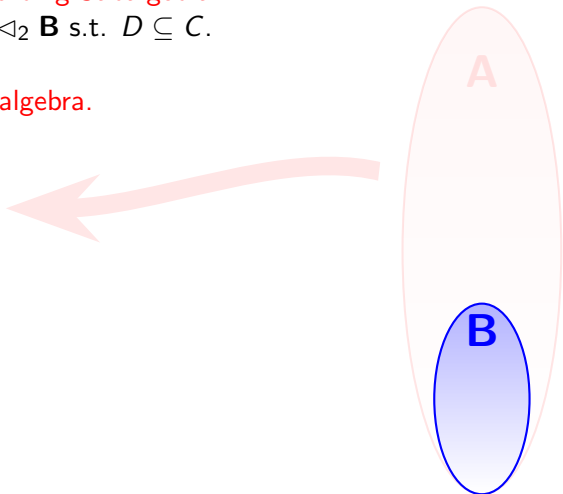
$$\mathbf{C} \prec_2 \mathbf{B} \Leftrightarrow \mathbf{C} \leq \mathbf{B} \wedge \exists \mathbf{D} \triangleleft_2 \mathbf{B} \text{ s.t. } \mathbf{D} \subseteq \mathbf{C}.$$

II. Ternary absorbing subalgebra.

$$\mathbf{C} \triangleleft_3 \mathbf{B}.$$

III. Dividing subalgebra.

$$\mathbf{C} \triangleleft^A \mathbf{B}$$



I. Contains a Binary Absorbing Subalgebra.

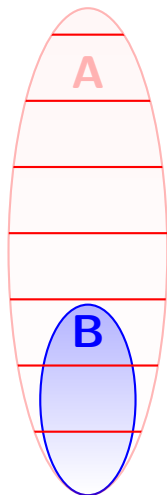
$$\mathbf{C} \prec_2 \mathbf{B} \Leftrightarrow \mathbf{C} \leq \mathbf{B} \wedge \exists \mathbf{D} \triangleleft_2 \mathbf{B} \text{ s.t. } \mathbf{D} \subseteq \mathbf{C}.$$

II. Ternary absorbing subalgebra.

$$\mathbf{C} \triangleleft_3 \mathbf{B}.$$

III. Dividing subalgebra.

$$\mathbf{C} \triangleleft^A \mathbf{B}$$





I. Contains a Binary Absorbing Subalgebra.

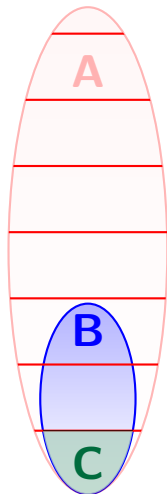
$$\mathbf{C} \prec_2 \mathbf{B} \Leftrightarrow \mathbf{C} \leq \mathbf{B} \wedge \exists \mathbf{D} \triangleleft_2 \mathbf{B} \text{ s.t. } \mathbf{D} \subseteq \mathbf{C}.$$

II. Ternary absorbing subalgebra.

$$\mathbf{C} \triangleleft_3 \mathbf{B}.$$

III. Dividing subalgebra.

$$\mathbf{C} \triangleleft^A \mathbf{B}$$



I. Contains a Binary Absorbing Subalgebra.

$$\mathbf{C} \prec_2 \mathbf{B} \Leftrightarrow \mathbf{C} \leq \mathbf{B} \wedge \exists \mathbf{D} \triangleleft_2 \mathbf{B} \text{ s.t. } \mathbf{D} \subseteq \mathbf{C}.$$

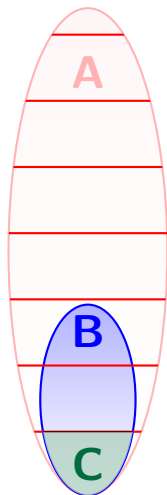
II. Ternary absorbing subalgebra.

$$\mathbf{C} \triangleleft_3 \mathbf{B}.$$

III. Dividing subalgebra.

$$\mathbf{C} \triangleleft^A \mathbf{B} \Leftrightarrow \mathbf{C} = \mathbf{B} \cap \mathbf{E},$$

where  $\mathbf{E}$  is a block of a congruence  $\sigma$  on  $\mathbf{A}$  s.t.



I. Contains a Binary Absorbing Subalgebra.

$$\mathbf{C} \prec_2 \mathbf{B} \Leftrightarrow \mathbf{C} \leq \mathbf{B} \wedge \exists \mathbf{D} \triangleleft_2 \mathbf{B} \text{ s.t. } \mathbf{D} \subseteq \mathbf{C}.$$

II. Ternary absorbing subalgebra.

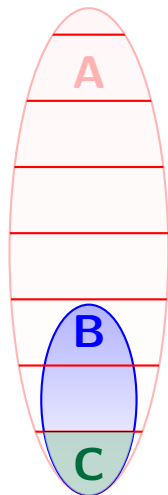
$$\mathbf{C} \triangleleft_3 \mathbf{B}.$$

III. Dividing subalgebra.

$$\mathbf{C} \triangleleft^A \mathbf{B} \Leftrightarrow \mathbf{C} = \mathbf{B} \cap \mathbf{E},$$

where  $\mathbf{E}$  is a block of a congruence  $\sigma$  on  $\mathbf{A}$  s.t.

1.  $\sigma$  is  $\wedge$ -irreducible



I. Contains a Binary Absorbing Subalgebra.

$$\mathbf{C} \triangleleft_2 \mathbf{B} \Leftrightarrow \mathbf{C} \leq \mathbf{B} \wedge \exists \mathbf{D} \triangleleft_2 \mathbf{B} \text{ s.t. } \mathbf{D} \subseteq \mathbf{C}.$$

II. Ternary absorbing subalgebra.

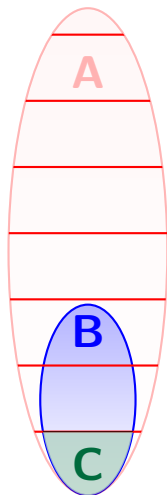
$$\mathbf{C} \triangleleft_3 \mathbf{B}.$$

III. Dividing subalgebra.

$$\mathbf{C} \triangleleft^A \mathbf{B} \Leftrightarrow \mathbf{C} = \mathbf{B} \cap \mathbf{E},$$

where  $\mathbf{E}$  is a block of a congruence  $\sigma$  on  $\mathbf{A}$  s.t.

1.  $\sigma$  is  $\wedge$ -irreducible
2.  $\sigma^* \supseteq \mathbf{B}^2$ , where  $\sigma^*$  is the cover of  $\sigma$



I. Contains a Binary Absorbing Subalgebra.

$$\mathbf{C} \triangleleft_2 \mathbf{B} \Leftrightarrow \mathbf{C} \leq \mathbf{B} \wedge \exists \mathbf{D} \triangleleft_2 \mathbf{B} \text{ s.t. } \mathbf{D} \subseteq \mathbf{C}.$$

II. Ternary absorbing subalgebra.

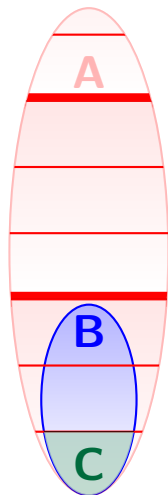
$$\mathbf{C} \triangleleft_3 \mathbf{B}.$$

III. Dividing subalgebra.

$$\mathbf{C} \triangleleft^A \mathbf{B} \Leftrightarrow \mathbf{C} = \mathbf{B} \cap \mathbf{E},$$

where  $\mathbf{E}$  is a block of a congruence  $\sigma$  on  $\mathbf{A}$  s.t.

1.  $\sigma$  is  $\wedge$ -irreducible
2.  $\sigma^* \supseteq \mathbf{B}^2$ , where  $\sigma^*$  is the cover of  $\sigma$



I. Contains a Binary Absorbing Subalgebra.

$$\mathbf{C} \triangleleft_2 \mathbf{B} \Leftrightarrow \mathbf{C} \leq \mathbf{B} \wedge \exists \mathbf{D} \triangleleft_2 \mathbf{B} \text{ s.t. } \mathbf{D} \subseteq \mathbf{C}.$$

II. Ternary absorbing subalgebra.

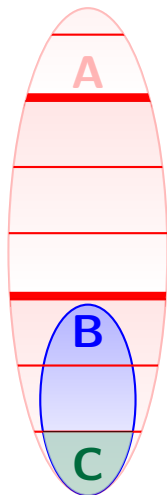
$$\mathbf{C} \triangleleft_3 \mathbf{B}.$$

III. Dividing subalgebra.

$$\mathbf{C} \triangleleft^A \mathbf{B} \Leftrightarrow \mathbf{C} = \mathbf{B} \cap \mathbf{E},$$

where  $\mathbf{E}$  is a block of a congruence  $\sigma$  on  $\mathbf{A}$  s.t.

1.  $\sigma$  is  $\wedge$ -irreducible
2.  $\sigma^* \supseteq \mathbf{B}^2$ , where  $\sigma^*$  is the cover of  $\sigma$
3.  $\mathbf{B}/\sigma$  is ternary absorption free



I. Contains a Binary Absorbing Subalgebra.

$$\mathbf{C} \triangleleft_2 \mathbf{B} \Leftrightarrow \mathbf{C} \leq \mathbf{B} \wedge \exists \mathbf{D} \triangleleft_2 \mathbf{B} \text{ s.t. } D \subseteq C.$$

II. Ternary absorbing subalgebra.

$$\mathbf{C} \triangleleft_3 \mathbf{B}.$$

III. Dividing subalgebra.

$$\mathbf{C} \triangleleft^A \mathbf{B} \Leftrightarrow C = B \cap E,$$

where  $E$  is a block of a congruence  $\sigma$  on  $\mathbf{A}$  s.t.

1.  $\sigma$  is  $\wedge$ -irreducible
2.  $\sigma^* \supseteq B^2$ , where  $\sigma^*$  is the cover of  $\sigma$
3.  $\mathbf{B}/\sigma$  is ternary absorption free

Transitive closure.

$$\mathbf{B}_k \lll^A \mathbf{B}_0$$

I. Contains a Binary Absorbing Subalgebra.

$$\mathbf{C} \prec_2 \mathbf{B} \Leftrightarrow \mathbf{C} \leq \mathbf{B} \wedge \exists \mathbf{D} \triangleleft_2 \mathbf{B} \text{ s.t. } \mathbf{D} \subseteq \mathbf{C}.$$

II. Ternary absorbing subalgebra.

$$\mathbf{C} \triangleleft_3 \mathbf{B}.$$

III. Dividing subalgebra.

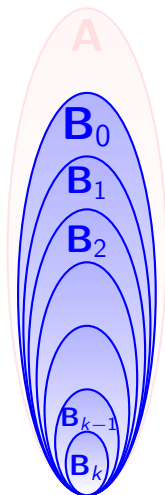
$$\mathbf{C} \triangleleft^A \mathbf{B} \Leftrightarrow \mathbf{C} = \mathbf{B} \cap \mathbf{E},$$

where  $\mathbf{E}$  is a block of a congruence  $\sigma$  on  $\mathbf{A}$  s.t.

1.  $\sigma$  is  $\wedge$ -irreducible
2.  $\sigma^* \supseteq \mathbf{B}^2$ , where  $\sigma^*$  is the cover of  $\sigma$
3.  $\mathbf{B}/\sigma$  is ternary absorption free

Transitive closure.

$$\mathbf{B}_k \lll^A \mathbf{B}_0 \Leftrightarrow \mathbf{B}_k <_k \mathbf{B}_{k-1} <_{k-1} \cdots <_2 \mathbf{B}_1 <_1 \mathbf{B}_0, \text{ where } <_i \in \{\prec_2, \triangleleft_3, \triangleleft^A\}.$$





# Properties of $\lll^A$

## Properties of $\lll^A$

(Transitivity)  $D \lll^A C \lll^A B \Rightarrow D \lll^A B$

## Properties of $\lll^A$

(Transitivity)  $D \lll^A C \lll^A B \Rightarrow D \lll^A B$

(Intersection)  $C, D \lll^A B \wedge C \cap D \neq 0 \Rightarrow C \cap D \lll^A B$

## Properties of $\lll^A$

(Transitivity)  $D \lll^A C \lll^A B \Rightarrow D \lll^A B$

(Intersection)  $C, D \lll^A B \wedge C \cap D \neq 0 \Rightarrow C \cap D \lll^A B$

## Properties of $\lll^A$

(Transitivity)  $\mathbf{D} \lll^A \mathbf{C} \lll^A \mathbf{B} \Rightarrow \mathbf{D} \lll^A \mathbf{B}$

(Intersection)  $\mathbf{C}, \mathbf{D} \lll^A \mathbf{B} \wedge \mathbf{C} \cap \mathbf{D} \neq 0 \Rightarrow \mathbf{C} \cap \mathbf{D} \lll^A \mathbf{B}$

(Propagation) if  $f: \mathbf{A} \rightarrow \mathbf{A}'$  is a surjective homomorphism,

(Pushforward)  $\mathbf{C} \lll^A \mathbf{B} \Rightarrow f(\mathbf{C}) \lll^{A'} f(\mathbf{B})$

(Pullback)  $\mathbf{C}' \lll^{A'} \mathbf{B}' \Rightarrow f^{-1}(\mathbf{C}') \lll^A f^{-1}(\mathbf{B}')$

## Properties of $\lll^A$

(Transitivity)  $\mathbf{D} \lll^A \mathbf{C} \lll^A \mathbf{B} \Rightarrow \mathbf{D} \lll^A \mathbf{B}$

(Intersection)  $\mathbf{C}, \mathbf{D} \lll^A \mathbf{B} \wedge \mathbf{C} \cap \mathbf{D} \neq 0 \Rightarrow \mathbf{C} \cap \mathbf{D} \lll^A \mathbf{B}$

(Propagation) if  $f: \mathbf{A} \rightarrow \mathbf{A}'$  is a surjective homomorphism,

(Pushforward)  $\mathbf{C} \lll^A \mathbf{B} \Rightarrow f(\mathbf{C}) \lll^{A'} f(\mathbf{B})$

(Pullback)  $\mathbf{C}' \lll^{A'} \mathbf{B}' \Rightarrow f^{-1}(\mathbf{C}') \lll^A f^{-1}(\mathbf{B}')$

(Ubiquity) if  $\mathbf{B} \lll^A \mathbf{A}$  and  $|\mathbf{B}| > 1$  then  $\exists \mathbf{C} \lll^A \mathbf{B}$ .

## Properties of $\lll^A$

(Transitivity)  $\mathbf{D} \lll^A \mathbf{C} \lll^A \mathbf{B} \Rightarrow \mathbf{D} \lll^A \mathbf{B}$

(Intersection)  $\mathbf{C}, \mathbf{D} \lll^A \mathbf{B} \wedge \mathbf{C} \cap \mathbf{D} \neq 0 \Rightarrow \mathbf{C} \cap \mathbf{D} \lll^A \mathbf{B}$

(Propagation) if  $f: \mathbf{A} \rightarrow \mathbf{A}'$  is a surjective homomorphism,

(Pushforward)  $\mathbf{C} \lll^A \mathbf{B} \Rightarrow f(\mathbf{C}) \lll^{A'} f(\mathbf{B})$

(Pullback)  $\mathbf{C}' \lll^{A'} \mathbf{B}' \Rightarrow f^{-1}(\mathbf{C}') \lll^A f^{-1}(\mathbf{B}')$

(Ubiquity) if  $\mathbf{B} \lll^A \mathbf{A}$  and  $|\mathbf{B}| > 1$  then  $\exists \mathbf{C} \lll^A \mathbf{B}$ .

(Helly)

## Helly property

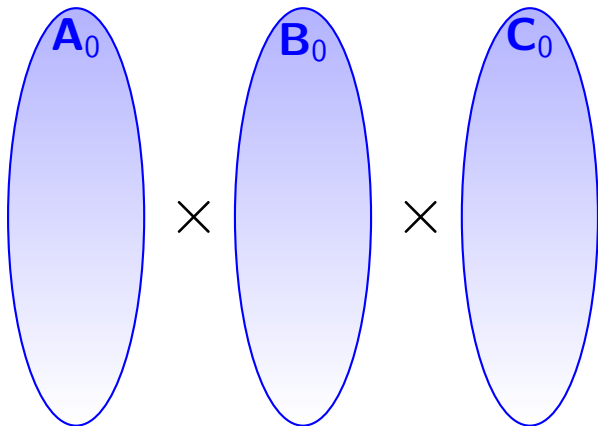


## Helly property

$$\mathbf{R} \leq_{sd} \mathbf{A}_0 \times \mathbf{B}_0 \times \mathbf{C}_0,$$

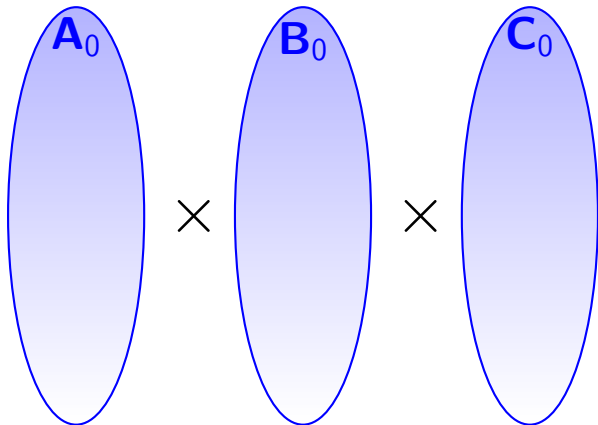
## Helly property

$$\mathbf{R} \leq_{sd} \mathbf{A}_0 \times \mathbf{B}_0 \times \mathbf{C}_0,$$



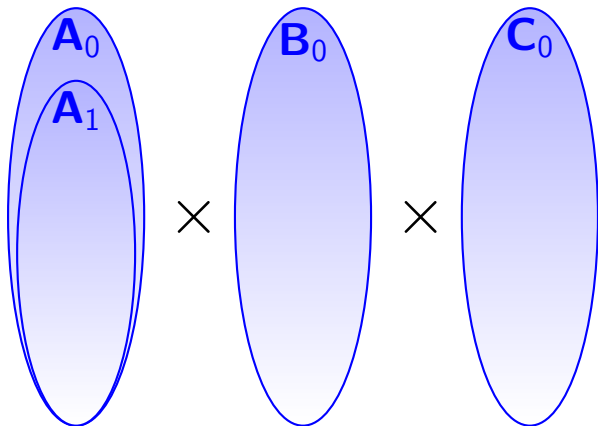
## Helly property

$$\mathbf{R} \leq_{sd} \mathbf{A}_0 \times \mathbf{B}_0 \times \mathbf{C}_0, \quad \mathbf{A}_k \lll^{A_0} \mathbf{A}_0,$$



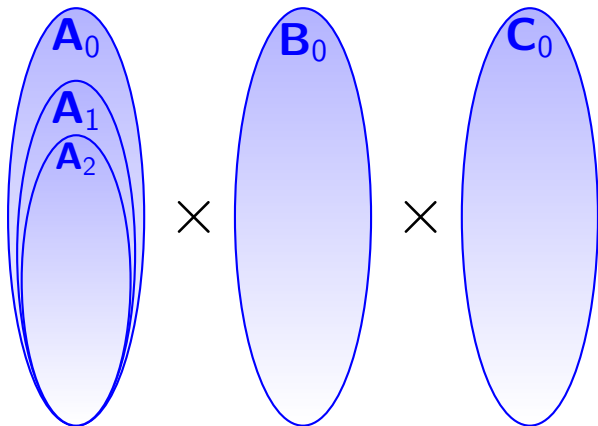
## Helly property

$$\mathbf{R} \leq_{sd} \mathbf{A}_0 \times \mathbf{B}_0 \times \mathbf{C}_0, \quad \mathbf{A}_k \lll^{A_0} \mathbf{A}_0,$$



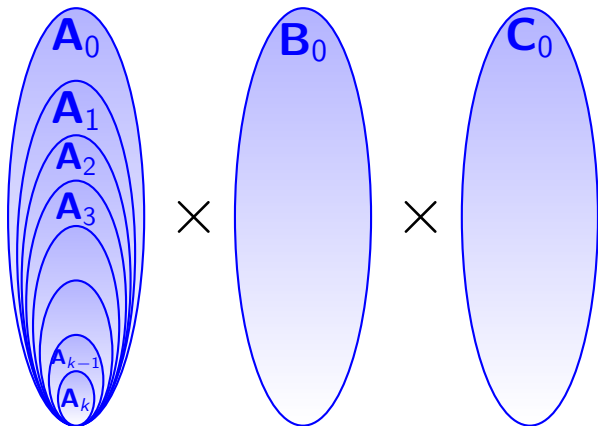
## Helly property

$$\mathbf{R} \leq_{sd} \mathbf{A}_0 \times \mathbf{B}_0 \times \mathbf{C}_0, \quad \mathbf{A}_k \lll^{A_0} \mathbf{A}_0,$$



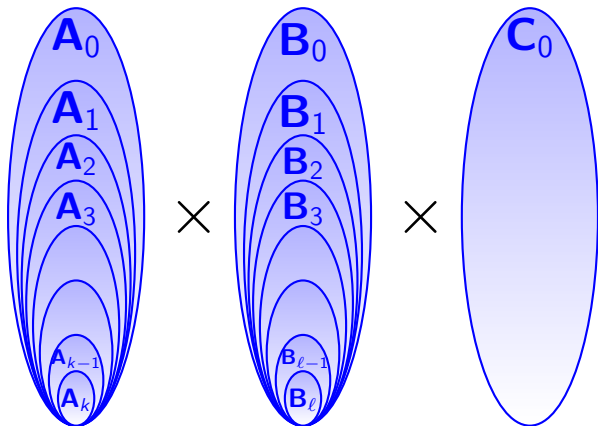
## Helly property

$$\mathbf{R} \leq_{sd} \mathbf{A}_0 \times \mathbf{B}_0 \times \mathbf{C}_0, \quad \mathbf{A}_k \lll^{A_0} \mathbf{A}_0,$$



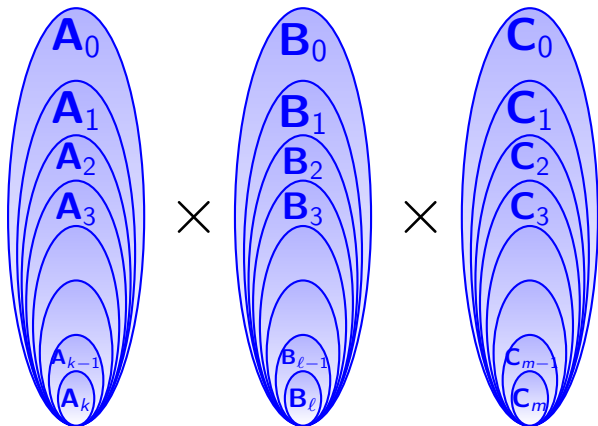
## Helly property

$$R \leq_{sd} \mathbf{A}_0 \times \mathbf{B}_0 \times \mathbf{C}_0, \quad \mathbf{A}_k \lll^{A_0} \mathbf{A}_0, \quad \mathbf{B}_\ell \lll^{B_0} \mathbf{B}_0,$$



## Helly property

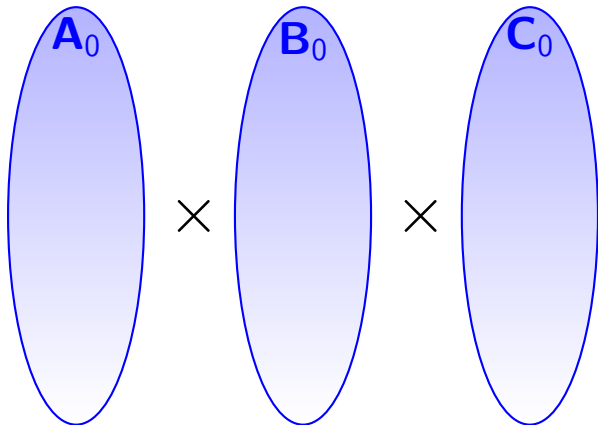
$$\mathbf{R} \leq_{sd} \mathbf{A}_0 \times \mathbf{B}_0 \times \mathbf{C}_0, \quad \mathbf{A}_k \lll^{A_0} \mathbf{A}_0, \quad \mathbf{B}_\ell \lll^{B_0} \mathbf{B}_0, \quad \mathbf{C}_m \lll^{C_0} \mathbf{C}_0.$$





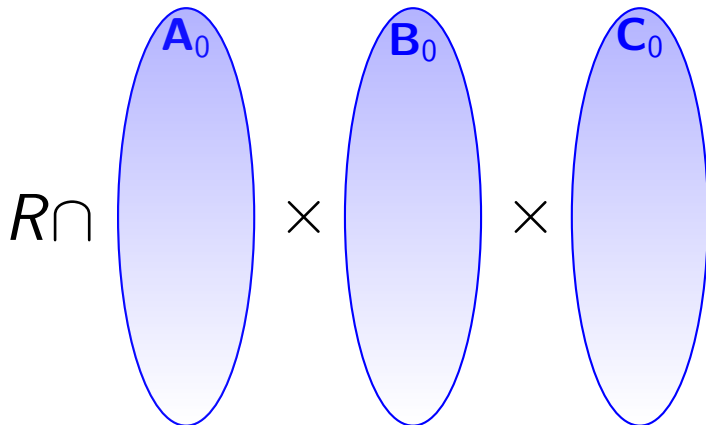
## Helly property

$$\mathbf{R} \leq_{sd} \mathbf{A}_0 \times \mathbf{B}_0 \times \mathbf{C}_0, \quad \mathbf{A}_k \lll^{A_0} \mathbf{A}_0, \quad \mathbf{B}_\ell \lll^{B_0} \mathbf{B}_0, \quad \mathbf{C}_m \lll^{C_0} \mathbf{C}_0.$$



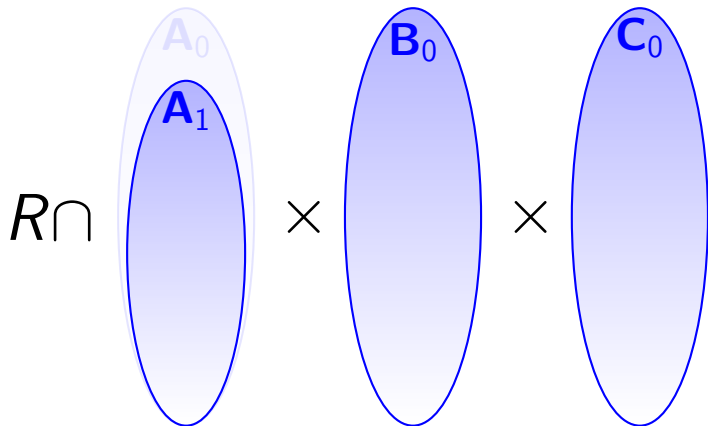
## Helly property

$$R \leq_{sd} A_0 \times B_0 \times C_0, \quad A_k \lll^{A_0} A_0, \quad B_\ell \lll^{B_0} B_0, \quad C_m \lll^{C_0} C_0.$$



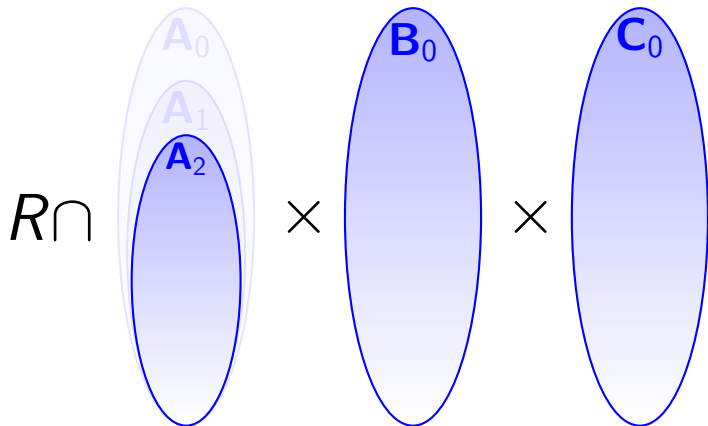
## Helly property

$$R \leq_{sd} \mathbf{A}_0 \times \mathbf{B}_0 \times \mathbf{C}_0, \quad \mathbf{A}_k \lll^{A_0} \mathbf{A}_0, \quad \mathbf{B}_\ell \lll^{B_0} \mathbf{B}_0, \quad \mathbf{C}_m \lll^{C_0} \mathbf{C}_0.$$



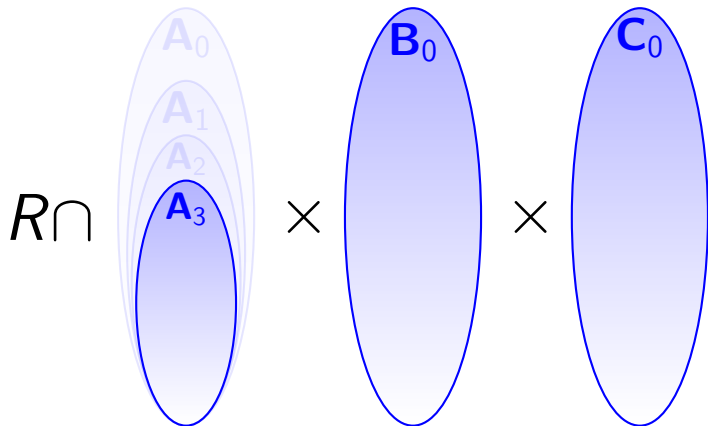
## Helly property

$$R \leq_{sd} \mathbf{A}_0 \times \mathbf{B}_0 \times \mathbf{C}_0, \quad \mathbf{A}_k \lll^{A_0} \mathbf{A}_0, \quad \mathbf{B}_\ell \lll^{B_0} \mathbf{B}_0, \quad \mathbf{C}_m \lll^{C_0} \mathbf{C}_0.$$



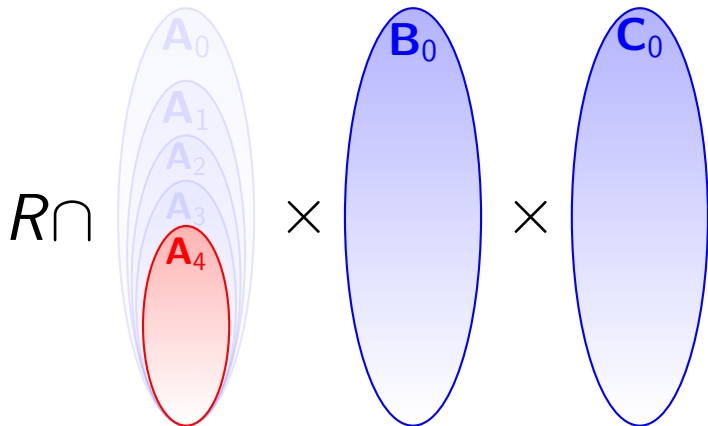
## Helly property

$$R \leq_{sd} A_0 \times B_0 \times C_0, \quad A_k \lll^{A_0} A_0, \quad B_\ell \lll^{B_0} B_0, \quad C_m \lll^{C_0} C_0.$$



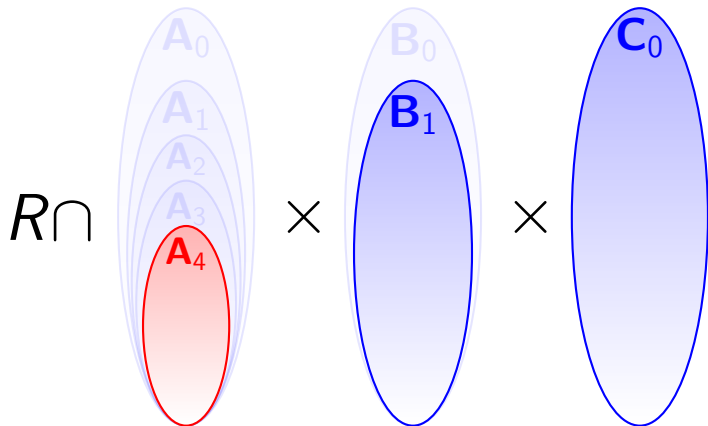
## Helly property

$$R \leq_{sd} \mathbf{A}_0 \times \mathbf{B}_0 \times \mathbf{C}_0, \quad \mathbf{A}_k \lll^{A_0} \mathbf{A}_0, \quad \mathbf{B}_\ell \lll^{B_0} \mathbf{B}_0, \quad \mathbf{C}_m \lll^{C_0} \mathbf{C}_0.$$



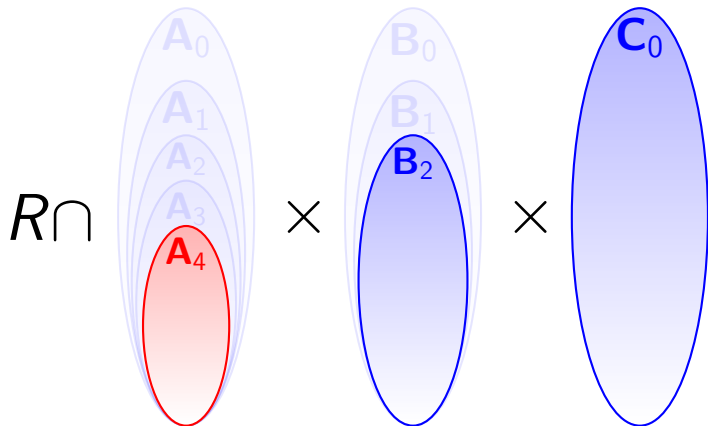
## Helly property

$$R \leq_{sd} \mathbf{A}_0 \times \mathbf{B}_0 \times \mathbf{C}_0, \quad \mathbf{A}_k \lll^{A_0} \mathbf{A}_0, \quad \mathbf{B}_\ell \lll^{B_0} \mathbf{B}_0, \quad \mathbf{C}_m \lll^{C_0} \mathbf{C}_0.$$



## Helly property

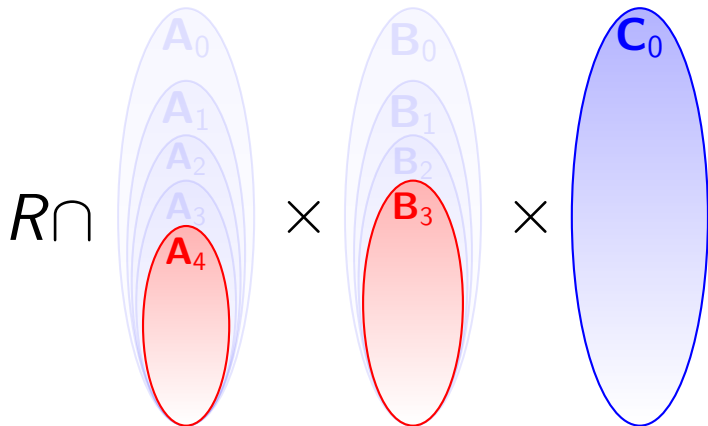
$$R \leq_{sd} \mathbf{A}_0 \times \mathbf{B}_0 \times \mathbf{C}_0, \quad \mathbf{A}_k \lll^{A_0} \mathbf{A}_0, \quad \mathbf{B}_\ell \lll^{B_0} \mathbf{B}_0, \quad \mathbf{C}_m \lll^{C_0} \mathbf{C}_0.$$





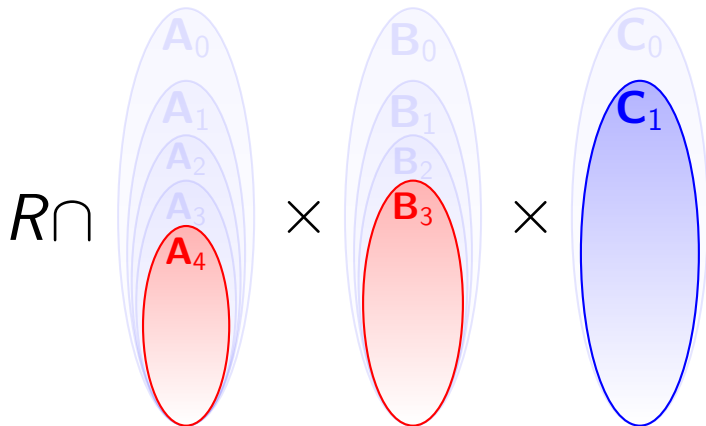
## Helly property

$$R \leq_{sd} \mathbf{A}_0 \times \mathbf{B}_0 \times \mathbf{C}_0, \quad \mathbf{A}_k \lll^{A_0} \mathbf{A}_0, \quad \mathbf{B}_\ell \lll^{B_0} \mathbf{B}_0, \quad \mathbf{C}_m \lll^{C_0} \mathbf{C}_0.$$



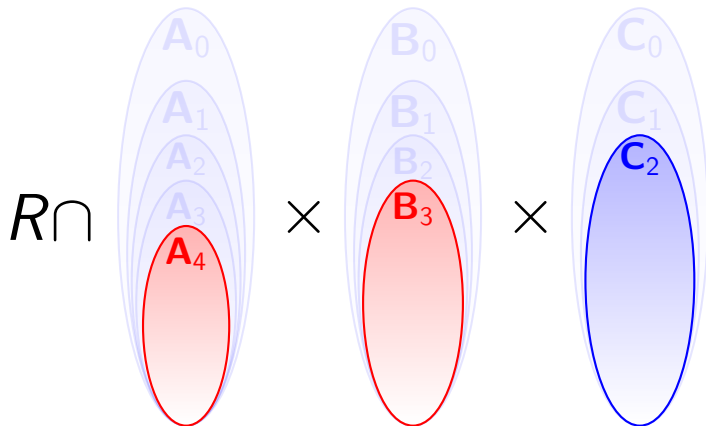
## Helly property

$$R \leq_{sd} \mathbf{A}_0 \times \mathbf{B}_0 \times \mathbf{C}_0, \quad \mathbf{A}_k \lll^{A_0} \mathbf{A}_0, \quad \mathbf{B}_\ell \lll^{B_0} \mathbf{B}_0, \quad \mathbf{C}_m \lll^{C_0} \mathbf{C}_0.$$



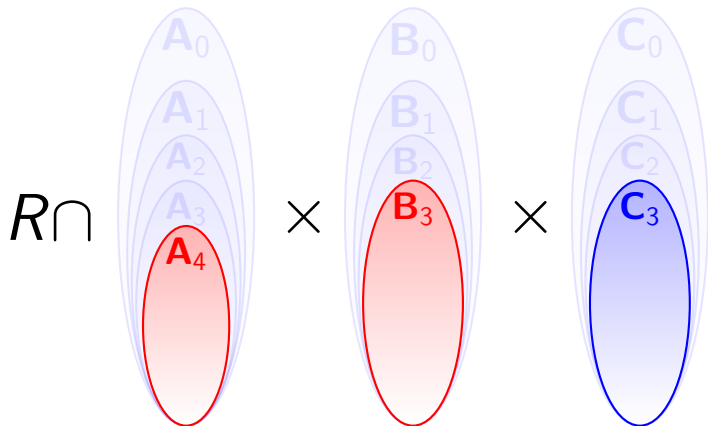
## Helly property

$$R \leq_{sd} \mathbf{A}_0 \times \mathbf{B}_0 \times \mathbf{C}_0, \quad \mathbf{A}_k \lll^{A_0} \mathbf{A}_0, \quad \mathbf{B}_\ell \lll^{B_0} \mathbf{B}_0, \quad \mathbf{C}_m \lll^{C_0} \mathbf{C}_0.$$



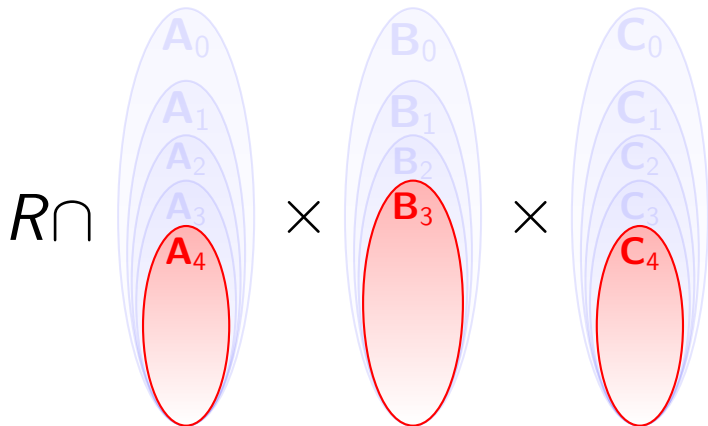
## Helly property

$$R \leq_{sd} \mathbf{A}_0 \times \mathbf{B}_0 \times \mathbf{C}_0, \quad \mathbf{A}_k \lll^{A_0} \mathbf{A}_0, \quad \mathbf{B}_\ell \lll^{B_0} \mathbf{B}_0, \quad \mathbf{C}_m \lll^{C_0} \mathbf{C}_0.$$



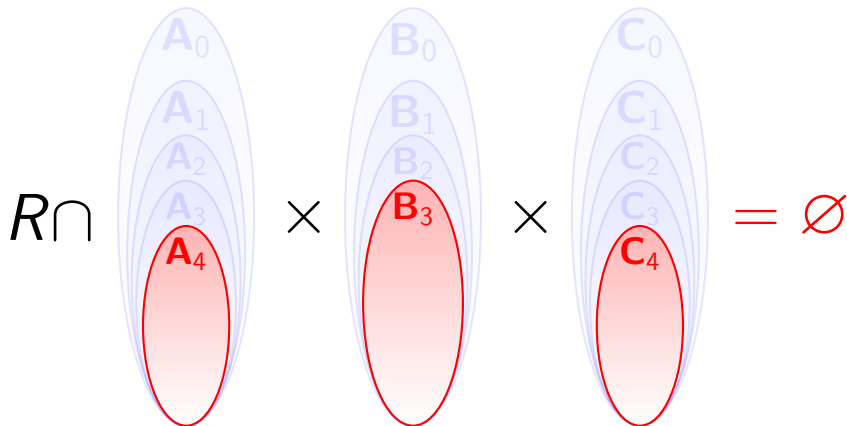
## Helly property

$$R \leq_{sd} \mathbf{A}_0 \times \mathbf{B}_0 \times \mathbf{C}_0, \quad \mathbf{A}_k \lll^{A_0} \mathbf{A}_0, \quad \mathbf{B}_\ell \lll^{B_0} \mathbf{B}_0, \quad \mathbf{C}_m \lll^{C_0} \mathbf{C}_0.$$



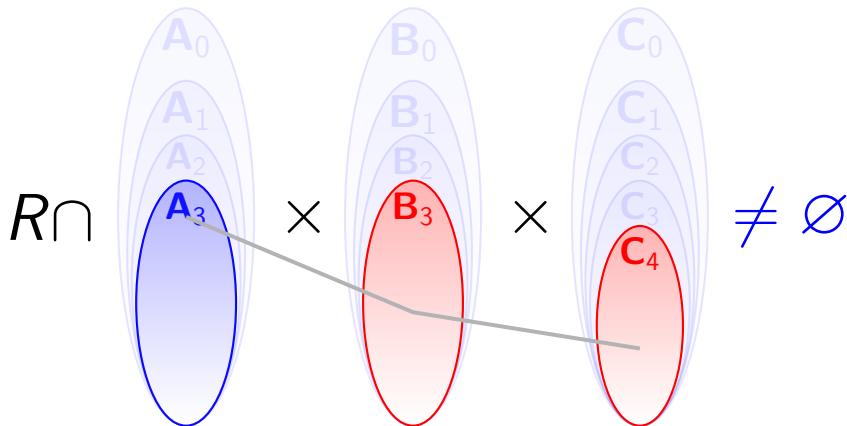
## Helly property

$$R \leq_{sd} \mathbf{A}_0 \times \mathbf{B}_0 \times \mathbf{C}_0, \quad \mathbf{A}_k \lll^{A_0} \mathbf{A}_0, \quad \mathbf{B}_\ell \lll^{B_0} \mathbf{B}_0, \quad \mathbf{C}_m \lll^{C_0} \mathbf{C}_0.$$



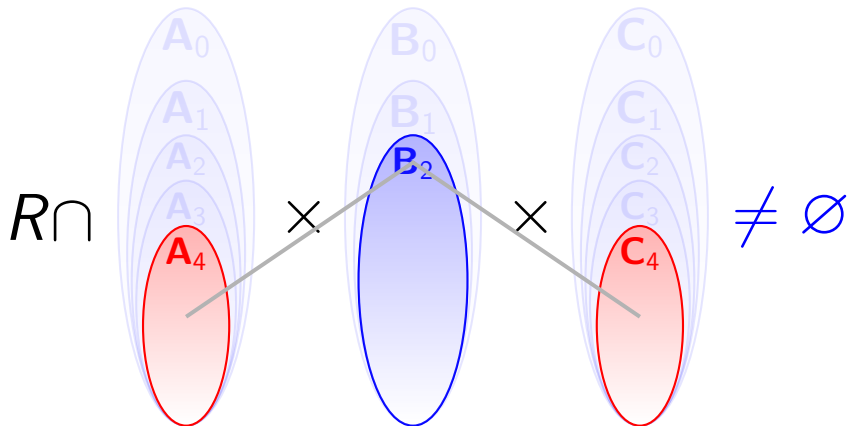
## Helly property

$$R \leq_{sd} \mathbf{A}_0 \times \mathbf{B}_0 \times \mathbf{C}_0, \quad \mathbf{A}_k \lll^{A_0} \mathbf{A}_0, \quad \mathbf{B}_\ell \lll^{B_0} \mathbf{B}_0, \quad \mathbf{C}_m \lll^{C_0} \mathbf{C}_0.$$



## Helly property

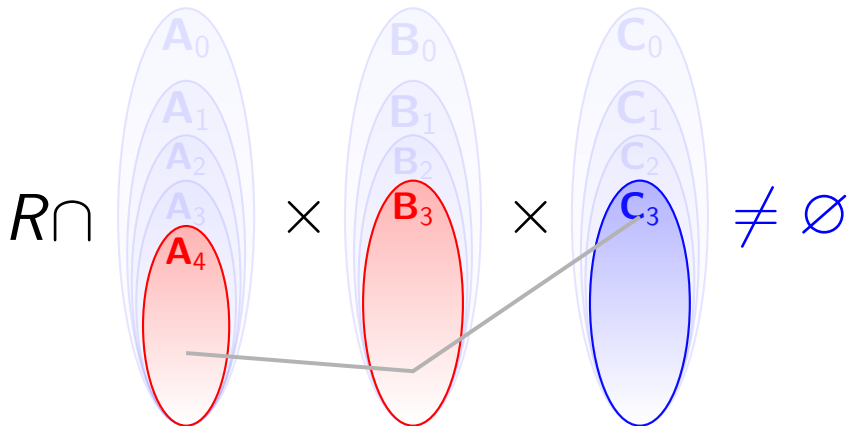
$$R \leq_{sd} \mathbf{A}_0 \times \mathbf{B}_0 \times \mathbf{C}_0, \quad \mathbf{A}_k \lll^{A_0} \mathbf{A}_0, \quad \mathbf{B}_\ell \lll^{B_0} \mathbf{B}_0, \quad \mathbf{C}_m \lll^{C_0} \mathbf{C}_0.$$





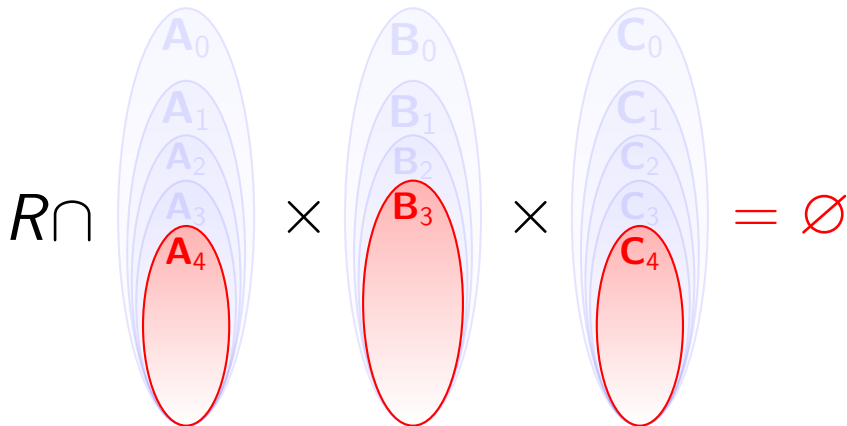
## Helly property

$$R \leq_{sd} \mathbf{A}_0 \times \mathbf{B}_0 \times \mathbf{C}_0, \quad \mathbf{A}_k \lll^{A_0} \mathbf{A}_0, \quad \mathbf{B}_\ell \lll^{B_0} \mathbf{B}_0, \quad \mathbf{C}_m \lll^{C_0} \mathbf{C}_0.$$



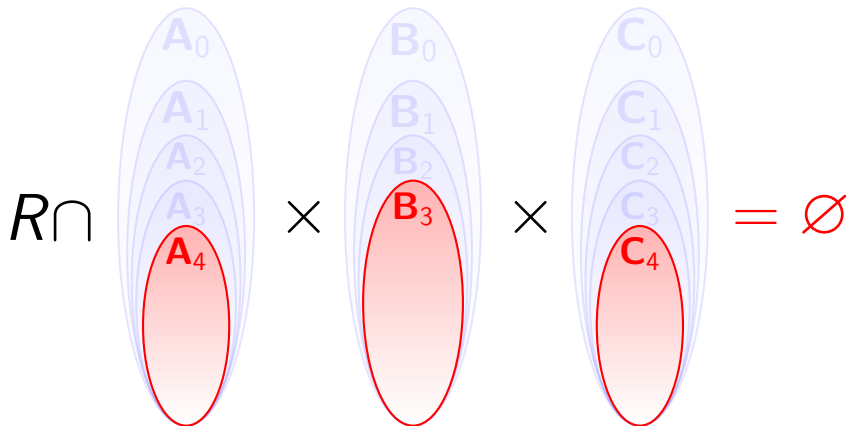
## Helly property

$$R \leq_{sd} \mathbf{A}_0 \times \mathbf{B}_0 \times \mathbf{C}_0, \quad \mathbf{A}_k \lll^{A_0} \mathbf{A}_0, \quad \mathbf{B}_\ell \lll^{B_0} \mathbf{B}_0, \quad \mathbf{C}_m \lll^{C_0} \mathbf{C}_0.$$



## Helly property

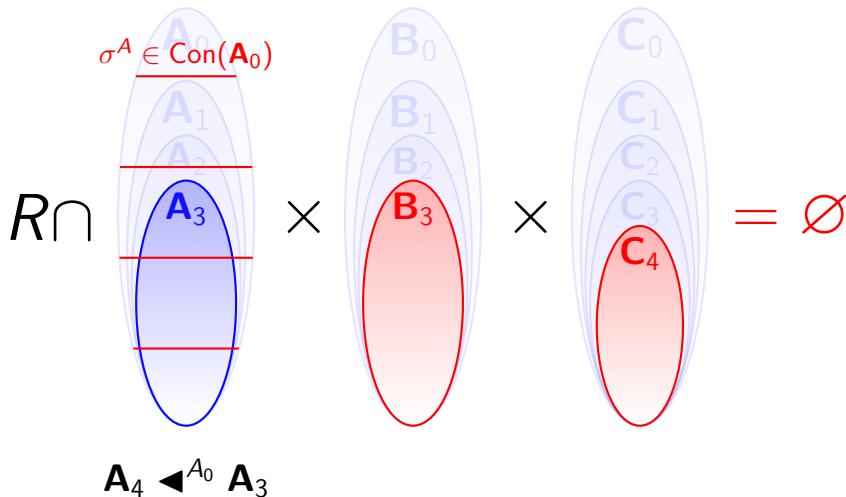
$$R \leq_{sd} \mathbf{A}_0 \times \mathbf{B}_0 \times \mathbf{C}_0, \quad \mathbf{A}_k \lll^{A_0} \mathbf{A}_0, \quad \mathbf{B}_\ell \lll^{B_0} \mathbf{B}_0, \quad \mathbf{C}_m \lll^{C_0} \mathbf{C}_0.$$



$$\mathbf{A}_4 \lll^{A_0} \mathbf{A}_3$$

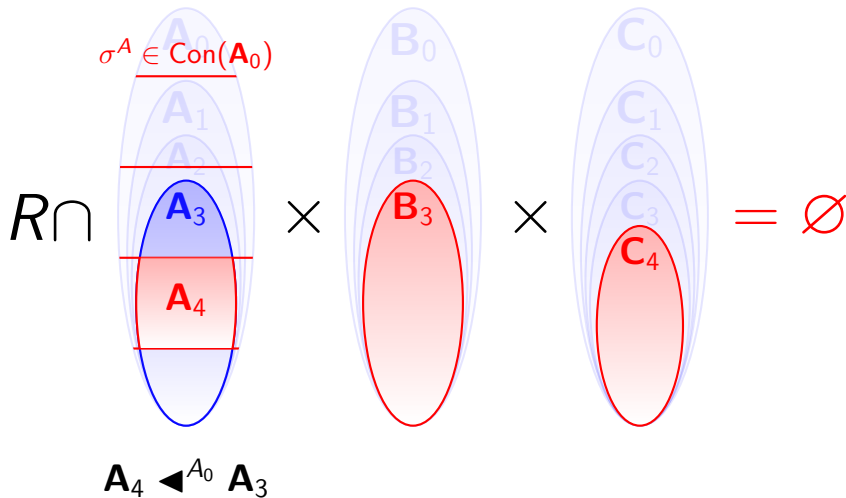
## Helly property

$$R \leq_{sd} \mathbf{A}_0 \times \mathbf{B}_0 \times \mathbf{C}_0, \quad \mathbf{A}_k \lll^{A_0} \mathbf{A}_0, \quad \mathbf{B}_\ell \lll^{B_0} \mathbf{B}_0, \quad \mathbf{C}_m \lll^{C_0} \mathbf{C}_0.$$



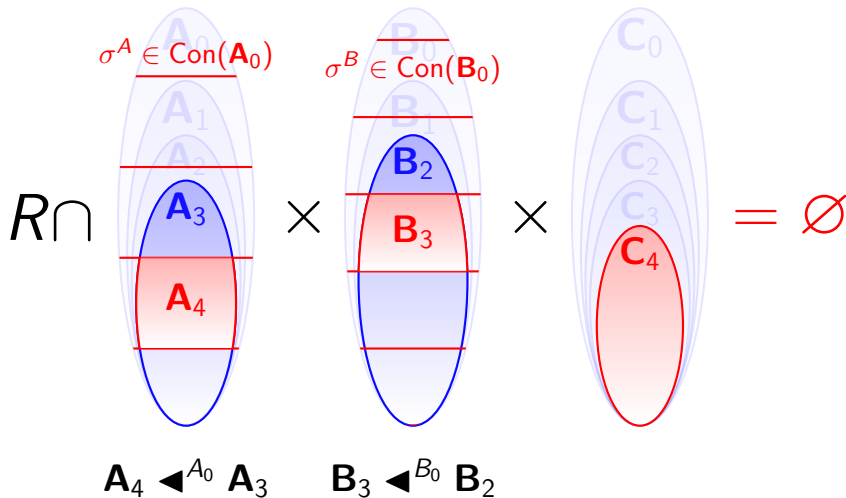
## Helly property

$$R \leq_{sd} \mathbf{A}_0 \times \mathbf{B}_0 \times \mathbf{C}_0, \quad \mathbf{A}_k \lll^{A_0} \mathbf{A}_0, \quad \mathbf{B}_\ell \lll^{B_0} \mathbf{B}_0, \quad \mathbf{C}_m \lll^{C_0} \mathbf{C}_0.$$



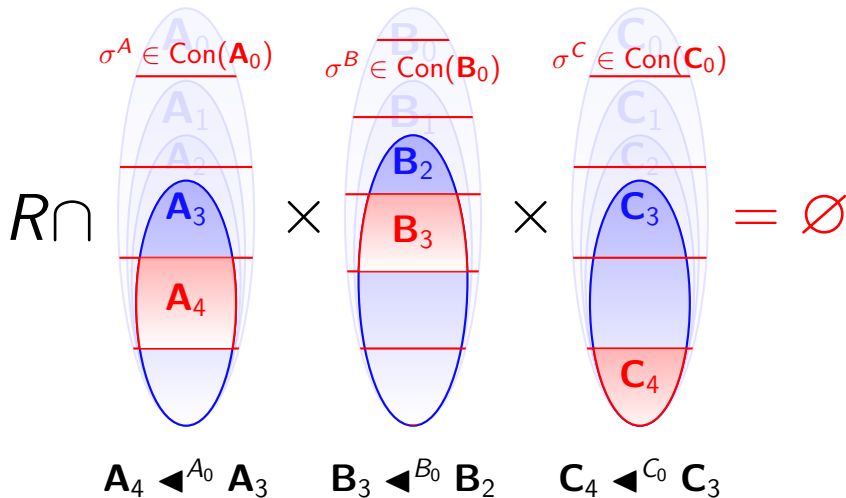
## Helly property

$$R \leq_{sd} \mathbf{A}_0 \times \mathbf{B}_0 \times \mathbf{C}_0, \quad \mathbf{A}_k \lll^{A_0} \mathbf{A}_0, \quad \mathbf{B}_\ell \lll^{B_0} \mathbf{B}_0, \quad \mathbf{C}_m \lll^{C_0} \mathbf{C}_0.$$



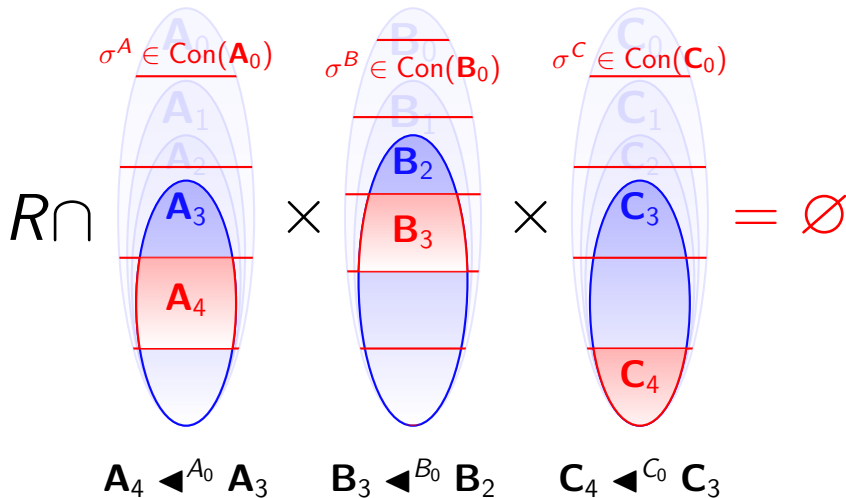
## Helly property

$$R \leq_{sd} \mathbf{A}_0 \times \mathbf{B}_0 \times \mathbf{C}_0, \quad \mathbf{A}_k \lll^{A_0} \mathbf{A}_0, \quad \mathbf{B}_\ell \lll^{B_0} \mathbf{B}_0, \quad \mathbf{C}_m \lll^{C_0} \mathbf{C}_0.$$



## Helly property

$$R \leq_{sd} \mathbf{A}_0 \times \mathbf{B}_0 \times \mathbf{C}_0, \quad \mathbf{A}_k \lll^{A_0} \mathbf{A}_0, \quad \mathbf{B}_\ell \lll^{B_0} \mathbf{B}_0, \quad \mathbf{C}_m \lll^{C_0} \mathbf{C}_0.$$



There exist bridges between congruences  $\sigma^A$ ,  $\sigma^B$ , and  $\sigma^C$



## Properties of $\lll^A$

(Transitivity)  $\mathbf{D} \lll^A \mathbf{C} \lll^A \mathbf{B} \Rightarrow \mathbf{D} \lll^A \mathbf{B}$

(Intersection)  $\mathbf{C}, \mathbf{D} \lll^A \mathbf{B} \wedge \mathbf{C} \cap \mathbf{D} \neq 0 \Rightarrow \mathbf{C} \cap \mathbf{D} \lll^A \mathbf{B}$

(Propagation) if  $f: \mathbf{A} \rightarrow \mathbf{A}'$  is a surjective homomorphism,

(Pushforward)  $\mathbf{C} \lll^A \mathbf{B} \Rightarrow f(\mathbf{C}) \lll^{A'} f(\mathbf{B})$

(Pullback)  $\mathbf{C}' \lll^{A'} \mathbf{B}' \Rightarrow f^{-1}(\mathbf{C}') \lll^A f^{-1}(\mathbf{B}')$

(Ubiquity) if  $\mathbf{B} \lll^A \mathbf{A}$  and  $|\mathbf{B}| > 1$  then  $\exists \mathbf{C} \lll^A \mathbf{B}$ .

(Helly)

## Properties of $\lll^A$

(Transitivity)  $\mathbf{D} \lll^A \mathbf{C} \lll^A \mathbf{B} \Rightarrow \mathbf{D} \lll^A \mathbf{B}$

(Intersection)  $\mathbf{C}, \mathbf{D} \lll^A \mathbf{B} \wedge \mathbf{C} \cap \mathbf{D} \neq 0 \Rightarrow \mathbf{C} \cap \mathbf{D} \lll^A \mathbf{B}$

(Propagation) if  $f: \mathbf{A} \rightarrow \mathbf{A}'$  is a surjective homomorphism,

(Pushforward)  $\mathbf{C} \lll^A \mathbf{B} \Rightarrow f(\mathbf{C}) \lll^{A'} f(\mathbf{B})$

(Pullback)  $\mathbf{C}' \lll^{A'} \mathbf{B}' \Rightarrow f^{-1}(\mathbf{C}') \lll^A f^{-1}(\mathbf{B}')$

(Ubiquity) if  $\mathbf{B} \lll^A \mathbf{A}$  and  $|\mathbf{B}| > 1$  then  $\exists \mathbf{C} \lll^A \mathbf{B}$ .

(Helly) Suppose

- ▶  $\mathbf{C}_i \lll^A \mathbf{B}_i \lll^A \mathbf{A}$ ,  $\lll^A \in \{\prec_2, \prec_3, \blacktriangleleft^A\}$ ,  $i \in [n]$ ,  $n \geq 2$ ,

## Properties of $\lll^A$

(Transitivity)  $\mathbf{D} \lll^A \mathbf{C} \lll^A \mathbf{B} \Rightarrow \mathbf{D} \lll^A \mathbf{B}$

(Intersection)  $\mathbf{C}, \mathbf{D} \lll^A \mathbf{B} \wedge \mathbf{C} \cap \mathbf{D} \neq \emptyset \Rightarrow \mathbf{C} \cap \mathbf{D} \lll^A \mathbf{B}$

(Propagation) if  $f: \mathbf{A} \rightarrow \mathbf{A}'$  is a surjective homomorphism,

(Pushforward)  $\mathbf{C} \lll^A \mathbf{B} \Rightarrow f(\mathbf{C}) \lll^{A'} f(\mathbf{B})$

(Pullback)  $\mathbf{C}' \lll^{A'} \mathbf{B}' \Rightarrow f^{-1}(\mathbf{C}') \lll^A f^{-1}(\mathbf{B}')$

(Ubiquity) if  $\mathbf{B} \lll^A \mathbf{A}$  and  $|\mathbf{B}| > 1$  then  $\exists \mathbf{C} \lll^A \mathbf{B}$ .

(Helly) Suppose

▶  $\mathbf{C}_i \lll^A \mathbf{B}_i \lll^A \mathbf{A}$ ,  $\lll_i \in \{\lll_2, \lll_3, \lll^A\}$ ,  $i \in [n]$ ,  $n \geq 2$ ,

▶  $\bigcap \mathbf{C}_i = \emptyset$ ,

## Properties of $\lll^A$

(Transitivity)  $\mathbf{D} \lll^A \mathbf{C} \lll^A \mathbf{B} \Rightarrow \mathbf{D} \lll^A \mathbf{B}$

(Intersection)  $\mathbf{C}, \mathbf{D} \lll^A \mathbf{B} \wedge \mathbf{C} \cap \mathbf{D} \neq \emptyset \Rightarrow \mathbf{C} \cap \mathbf{D} \lll^A \mathbf{B}$

(Propagation) if  $f: \mathbf{A} \rightarrow \mathbf{A}'$  is a surjective homomorphism,

(Pushforward)  $\mathbf{C} \lll^A \mathbf{B} \Rightarrow f(\mathbf{C}) \lll^{A'} f(\mathbf{B})$

(Pullback)  $\mathbf{C}' \lll^{A'} \mathbf{B}' \Rightarrow f^{-1}(\mathbf{C}') \lll^A f^{-1}(\mathbf{B}')$

(Ubiquity) if  $\mathbf{B} \lll^A \mathbf{A}$  and  $|\mathbf{B}| > 1$  then  $\exists \mathbf{C} \lll^A \mathbf{B}$ .

(Helly) Suppose

- ▶  $\mathbf{C}_i \lll^A \mathbf{B}_i \lll^A \mathbf{A}$ ,  $\lll_i \in \{\lll_2, \lll_3, \lll^A\}$ ,  $i \in [n]$ ,  $n \geq 2$ ,
- ▶  $\bigcap \mathbf{C}_i = \emptyset$ ,
- ▶  $\forall j: \mathbf{B}_j \cap \bigcap_{i \neq j} \mathbf{C}_i \neq \emptyset$ .

## Properties of $\lll^A$

(Transitivity)  $\mathbf{D} \lll^A \mathbf{C} \lll^A \mathbf{B} \Rightarrow \mathbf{D} \lll^A \mathbf{B}$

(Intersection)  $\mathbf{C}, \mathbf{D} \lll^A \mathbf{B} \wedge \mathbf{C} \cap \mathbf{D} \neq \emptyset \Rightarrow \mathbf{C} \cap \mathbf{D} \lll^A \mathbf{B}$

(Propagation) if  $f: \mathbf{A} \rightarrow \mathbf{A}'$  is a surjective homomorphism,

(Pushforward)  $\mathbf{C} \lll^A \mathbf{B} \Rightarrow f(\mathbf{C}) \lll^{A'} f(\mathbf{B})$

(Pullback)  $\mathbf{C}' \lll^{A'} \mathbf{B}' \Rightarrow f^{-1}(\mathbf{C}') \lll^A f^{-1}(\mathbf{B}')$

(Ubiquity) if  $\mathbf{B} \lll^A \mathbf{A}$  and  $|\mathbf{B}| > 1$  then  $\exists \mathbf{C} \lll^A \mathbf{B}$ .

(Helly) Suppose

▶  $\mathbf{C}_i \lll^A \mathbf{B}_i \lll^A \mathbf{A}$ ,  $\langle_i \in \{\prec_2, \triangleleft_3, \blacktriangleleft^A\}$ ,  $i \in [n]$ ,  $n \geq 2$ ,

▶  $\bigcap \mathbf{C}_i = \emptyset$ ,

▶  $\forall j: \mathbf{B}_j \cap \bigcap_{i \neq j} \mathbf{C}_i \neq \emptyset$ .

then

1. either  $\langle_i = \triangleleft_3$  for every  $i$  and  $n = 2$ ,

## Properties of $\lll^A$

(Transitivity)  $\mathbf{D} \lll^A \mathbf{C} \lll^A \mathbf{B} \Rightarrow \mathbf{D} \lll^A \mathbf{B}$

(Intersection)  $\mathbf{C}, \mathbf{D} \lll^A \mathbf{B} \wedge \mathbf{C} \cap \mathbf{D} \neq \emptyset \Rightarrow \mathbf{C} \cap \mathbf{D} \lll^A \mathbf{B}$

(Propagation) if  $f: \mathbf{A} \rightarrow \mathbf{A}'$  is a surjective homomorphism,

(Pushforward)  $\mathbf{C} \lll^A \mathbf{B} \Rightarrow f(\mathbf{C}) \lll^{A'} f(\mathbf{B})$

(Pullback)  $\mathbf{C}' \lll^{A'} \mathbf{B}' \Rightarrow f^{-1}(\mathbf{C}') \lll^A f^{-1}(\mathbf{B}')$

(Ubiquity) if  $\mathbf{B} \lll^A \mathbf{A}$  and  $|\mathbf{B}| > 1$  then  $\exists \mathbf{C} \lll^A \mathbf{B}$ .

(Helly) Suppose

- ▶  $\mathbf{C}_i \lll^A \mathbf{A}$ ,  $\lll_i \in \{\lll_2, \lll_3, \lll^A\}$ ,  $i \in [n]$ ,  $n \geq 2$ ,
- ▶  $\bigcap \mathbf{C}_i = \emptyset$ ,
- ▶  $\forall j: \mathbf{B}_j \cap \bigcap_{i \neq j} \mathbf{C}_i \neq \emptyset$ .

then

1. either  $\lll_i = \lll_3$  for every  $i$  and  $n = 2$ ,
2. or  $\lll_i = \lll^A$  for every  $i$

## Properties of $\lll^A$

(Transitivity)  $\mathbf{D} \lll^A \mathbf{C} \lll^A \mathbf{B} \Rightarrow \mathbf{D} \lll^A \mathbf{B}$

(Intersection)  $\mathbf{C}, \mathbf{D} \lll^A \mathbf{B} \wedge \mathbf{C} \cap \mathbf{D} \neq \emptyset \Rightarrow \mathbf{C} \cap \mathbf{D} \lll^A \mathbf{B}$

(Propagation) if  $f: \mathbf{A} \rightarrow \mathbf{A}'$  is a surjective homomorphism,

(Pushforward)  $\mathbf{C} \lll^A \mathbf{B} \Rightarrow f(\mathbf{C}) \lll^{A'} f(\mathbf{B})$

(Pullback)  $\mathbf{C}' \lll^{A'} \mathbf{B}' \Rightarrow f^{-1}(\mathbf{C}') \lll^A f^{-1}(\mathbf{B}')$

(Ubiquity) if  $\mathbf{B} \lll^A \mathbf{A}$  and  $|\mathbf{B}| > 1$  then  $\exists \mathbf{C} \lll^A \mathbf{B}$ .

(Helly) Suppose

▶  $\mathbf{C}_i \lll^A \mathbf{B}_i \lll^A \mathbf{A}$ ,  $\lll_i \in \{\lll_2, \lll_3, \lll^A\}$ ,  $i \in [n]$ ,  $n \geq 2$ ,

▶  $\bigcap \mathbf{C}_i = \emptyset$ ,

▶  $\forall j: \mathbf{B}_j \cap \bigcap_{i \neq j} \mathbf{C}_i \neq \emptyset$ .

then

1. either  $\lll_i = \lll_3$  for every  $i$  and  $n = 2$ ,

2. or  $\lll_i = \lll^A$  for every  $i$  + exist nice bridges

# Bridges



## Bridges

$\sigma_1 \in \text{Con}(\mathbf{A}_1), \sigma_2 \in \text{Con}(\mathbf{A}_2).$

## Bridges

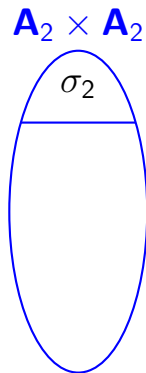
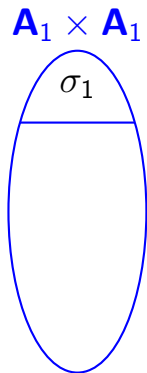
$\sigma_1 \in \text{Con}(\mathbf{A}_1)$ ,  $\sigma_2 \in \text{Con}(\mathbf{A}_2)$ .

$\delta \leq \mathbf{A}_1^2 \times \mathbf{A}_2^2$  is a **bridge** from  $\sigma_1$  to  $\sigma_2$  if

## Bridges

$\sigma_1 \in \text{Con}(\mathbf{A}_1)$ ,  $\sigma_2 \in \text{Con}(\mathbf{A}_2)$ .

$\delta \leq \mathbf{A}_1^2 \times \mathbf{A}_2^2$  is a **bridge** from  $\sigma_1$  to  $\sigma_2$  if

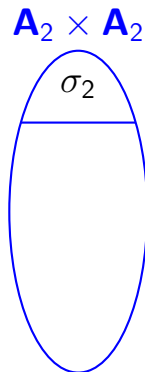
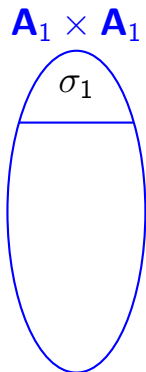


## Bridges

$\sigma_1 \in \text{Con}(\mathbf{A}_1)$ ,  $\sigma_2 \in \text{Con}(\mathbf{A}_2)$ .

$\delta \subseteq \mathbf{A}_1^2 \times \mathbf{A}_2^2$  is a **bridge** from  $\sigma_1$  to  $\sigma_2$  if

1.  $(a_1, a_2, a_3, a_4) \in \delta$



## Bridges

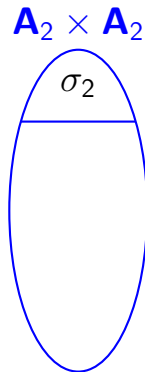
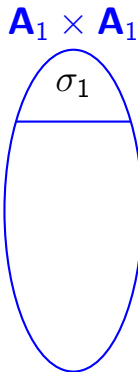
$\sigma_1 \in \text{Con}(\mathbf{A}_1)$ ,  $\sigma_2 \in \text{Con}(\mathbf{A}_2)$ .

$\delta \leq \mathbf{A}_1^2 \times \mathbf{A}_2^2$  is a **bridge** from  $\sigma_1$  to  $\sigma_2$  if

1.  $(a_1, a_2, a_3, a_4) \in \delta$

$\sigma_1 \left\{ \begin{array}{l} a_1 \\ a_2 \end{array} \right\} \left\{ \begin{array}{l} a_3 \\ a_4 \end{array} \right\} \sigma_2$

$(b_1, b_2, b_3, b_4)$

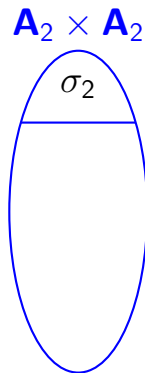
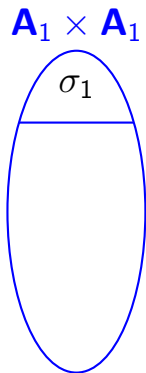


## Bridges

$\sigma_1 \in \text{Con}(\mathbf{A}_1)$ ,  $\sigma_2 \in \text{Con}(\mathbf{A}_2)$ .

$\delta \leq \mathbf{A}_1^2 \times \mathbf{A}_2^2$  is a **bridge** from  $\sigma_1$  to  $\sigma_2$  if

- $(a_1, a_2, a_3, a_4) \in \delta$   
 $\sigma_1 \{ \sigma_1 \} \{ \sigma_2 \} \sigma_2 \quad \Downarrow$   
 $(b_1, b_2, b_3, b_4) \in \delta$

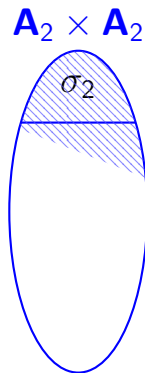
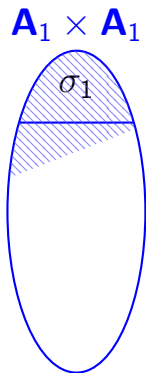


## Bridges

$\sigma_1 \in \text{Con}(\mathbf{A}_1)$ ,  $\sigma_2 \in \text{Con}(\mathbf{A}_2)$ .

$\delta \leq \mathbf{A}_1^2 \times \mathbf{A}_2^2$  is a **bridge** from  $\sigma_1$  to  $\sigma_2$  if

- $(a_1, a_2, a_3, a_4) \in \delta$   
 $\sigma_1 \upharpoonright_{\{a_1, a_2\}} \upharpoonright_{\{a_3, a_4\}} \sigma_2 \downarrow$   
 $(b_1, b_2, b_3, b_4) \in \delta$
- $\text{pr}_{1,2}(\delta) \supseteq \sigma_1$ ,  
 $\text{pr}_{3,4}(\delta) \supseteq \sigma_2$

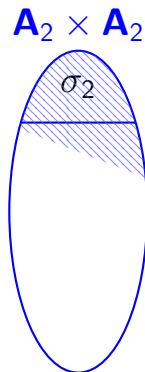
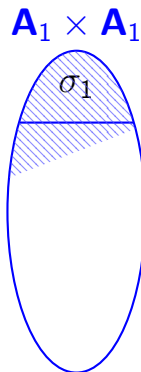


## Bridges

$\sigma_1 \in \text{Con}(\mathbf{A}_1)$ ,  $\sigma_2 \in \text{Con}(\mathbf{A}_2)$ .

$\delta \leq \mathbf{A}_1^2 \times \mathbf{A}_2^2$  is a **bridge** from  $\sigma_1$  to  $\sigma_2$  if

- $(a_1, a_2, a_3, a_4) \in \delta$   
 $\sigma_1 \upharpoonright_{\{a_1, a_2\}} \upharpoonright_{\{a_3, a_4\}} \sigma_2 \downarrow$   
 $(b_1, b_2, b_3, b_4) \in \delta$
- $\text{pr}_{1,2}(\delta) \supseteq \sigma_1$ ,  
 $\text{pr}_{3,4}(\delta) \supseteq \sigma_2$
- $(a_1, a_2, a_3, a_4) \in \delta$   
 $\downarrow$





## Bridges

$\sigma_1 \in \text{Con}(\mathbf{A}_1)$ ,  $\sigma_2 \in \text{Con}(\mathbf{A}_2)$ .

$\delta \leq \mathbf{A}_1^2 \times \mathbf{A}_2^2$  is a **bridge** from  $\sigma_1$  to  $\sigma_2$  if

1.  $(a_1, a_2, a_3, a_4) \in \delta$

$$\sigma_1 \upharpoonright_{\sigma_1} \upharpoonright_{\sigma_2} \downarrow$$

$(b_1, b_2, b_3, b_4) \in \delta$

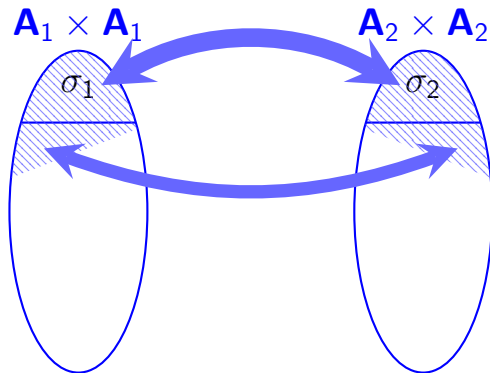
2.  $\text{pr}_{1,2}(\delta) \supseteq \sigma_1$ ,

$\text{pr}_{3,4}(\delta) \supseteq \sigma_2$

3.  $(a_1, a_2, a_3, a_4) \in \delta$

$$\Downarrow$$

$$(a_1, a_2) \in \sigma_1 \Leftrightarrow (a_3, a_4) \in \sigma_2.$$



## Bridges

$\sigma_1 \in \text{Con}(\mathbf{A}_1)$ ,  $\sigma_2 \in \text{Con}(\mathbf{A}_2)$ .

$\delta \leq \mathbf{A}_1^2 \times \mathbf{A}_2^2$  is a **bridge** from  $\sigma_1$  to  $\sigma_2$  if

1.  $(a_1, a_2, a_3, a_4) \in \delta$

$$\sigma_1 \upharpoonright_{\sigma_1} \upharpoonright_{\sigma_2} \downarrow$$

$(b_1, b_2, b_3, b_4) \in \delta$

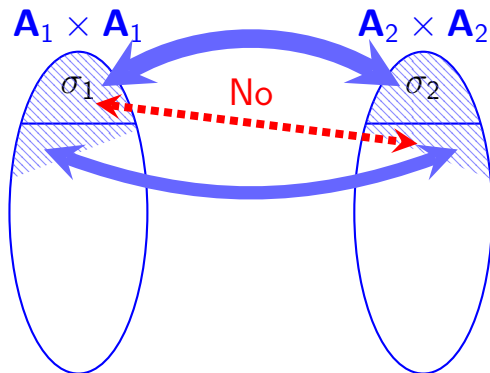
2.  $\text{pr}_{1,2}(\delta) \supseteq \sigma_1$ ,

$\text{pr}_{3,4}(\delta) \supseteq \sigma_2$

3.  $(a_1, a_2, a_3, a_4) \in \delta$

$$\Downarrow$$

$$(a_1, a_2) \in \sigma_1 \Leftrightarrow (a_3, a_4) \in \sigma_2.$$



## Bridges

$\sigma_1 \in \text{Con}(\mathbf{A}_1)$ ,  $\sigma_2 \in \text{Con}(\mathbf{A}_2)$ .

$\delta \leq \mathbf{A}_1^2 \times \mathbf{A}_2^2$  is a **bridge** from  $\sigma_1$  to  $\sigma_2$  if

1.  $(a_1, a_2, a_3, a_4) \in \delta$

$$\sigma_1 \upharpoonright_{\sigma_1} \upharpoonright_{\sigma_2} \downarrow$$

$(b_1, b_2, b_3, b_4) \in \delta$

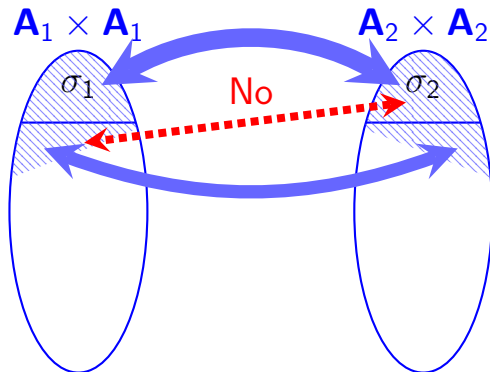
2.  $\text{pr}_{1,2}(\delta) \supsetneq \sigma_1$ ,

$\text{pr}_{3,4}(\delta) \supsetneq \sigma_2$

3.  $(a_1, a_2, a_3, a_4) \in \delta$

$$\Downarrow$$

$$(a_1, a_2) \in \sigma_1 \Leftrightarrow (a_3, a_4) \in \sigma_2.$$



## Bridges

$\sigma_1 \in \text{Con}(\mathbf{A}_1)$ ,  $\sigma_2 \in \text{Con}(\mathbf{A}_2)$ .

$\delta \leq \mathbf{A}_1^2 \times \mathbf{A}_2^2$  is a **bridge** from  $\sigma_1$  to  $\sigma_2$  if

1.  $(a_1, a_2, a_3, a_4) \in \delta$

$$\sigma_1 \upharpoonright_{\{a_1, a_2\}} \upharpoonright_{\{a_3, a_4\}} \sigma_2 \downarrow$$

$(b_1, b_2, b_3, b_4) \in \delta$

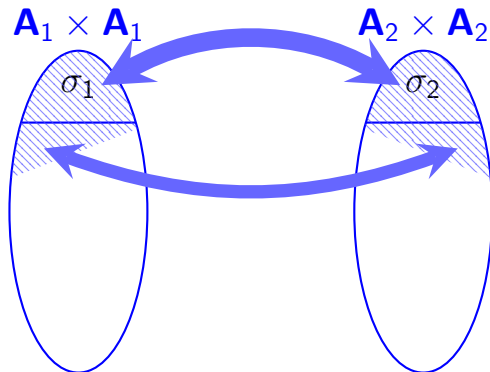
2.  $\text{pr}_{1,2}(\delta) \supseteq \sigma_1$ ,

$\text{pr}_{3,4}(\delta) \supseteq \sigma_2$

3.  $(a_1, a_2, a_3, a_4) \in \delta$

$$\Downarrow$$

$$(a_1, a_2) \in \sigma_1 \Leftrightarrow (a_3, a_4) \in \sigma_2.$$



### Example

## Bridges

$\sigma_1 \in \text{Con}(\mathbf{A}_1)$ ,  $\sigma_2 \in \text{Con}(\mathbf{A}_2)$ .

$\delta \leq \mathbf{A}_1^2 \times \mathbf{A}_2^2$  is a **bridge** from  $\sigma_1$  to  $\sigma_2$  if

1.  $(a_1, a_2, a_3, a_4) \in \delta$

$$\sigma_1 \upharpoonright_{\{a_1, a_2\}} \upharpoonright_{\{a_3, a_4\}} \sigma_2 \downarrow$$

$(b_1, b_2, b_3, b_4) \in \delta$

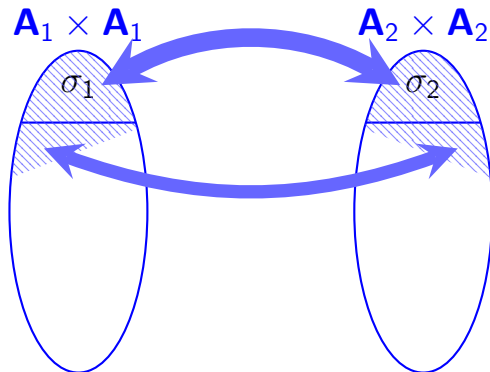
2.  $\text{pr}_{1,2}(\delta) \supseteq \sigma_1$ ,

$\text{pr}_{3,4}(\delta) \supseteq \sigma_2$

3.  $(a_1, a_2, a_3, a_4) \in \delta$

$$\Downarrow$$

$$(a_1, a_2) \in \sigma_1 \Leftrightarrow (a_3, a_4) \in \sigma_2.$$



### Example

$$A_1 = A_2 = \mathbb{Z}_4$$

## Bridges

$\sigma_1 \in \text{Con}(\mathbf{A}_1)$ ,  $\sigma_2 \in \text{Con}(\mathbf{A}_2)$ .

$\delta \leq \mathbf{A}_1^2 \times \mathbf{A}_2^2$  is a **bridge** from  $\sigma_1$  to  $\sigma_2$  if

1.  $(a_1, a_2, a_3, a_4) \in \delta$

$$\sigma_1 \upharpoonright_{\{a_1, a_2\}} \upharpoonright_{\{a_3, a_4\}} \sigma_2 \downarrow$$

$(b_1, b_2, b_3, b_4) \in \delta$

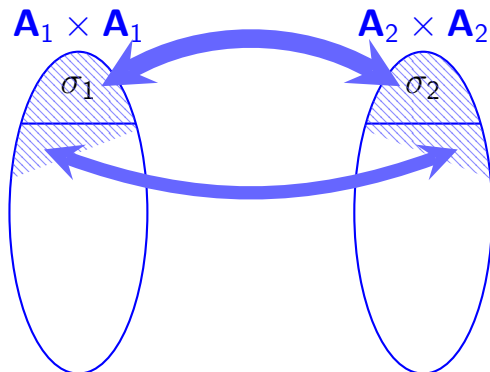
2.  $\text{pr}_{1,2}(\delta) \supseteq \sigma_1$ ,

$\text{pr}_{3,4}(\delta) \supseteq \sigma_2$

3.  $(a_1, a_2, a_3, a_4) \in \delta$

$$\Downarrow$$

$$(a_1, a_2) \in \sigma_1 \Leftrightarrow (a_3, a_4) \in \sigma_2.$$



### Example

$$A_1 = A_2 = \mathbb{Z}_4$$

$$\delta := \{(x_1, x_2, x_3, x_4) \in \mathbb{Z}_4^4 \mid x_1 - x_2 = 2x_3 - 2x_4\},$$

## Bridges

$\sigma_1 \in \text{Con}(\mathbf{A}_1)$ ,  $\sigma_2 \in \text{Con}(\mathbf{A}_2)$ .

$\delta \leq \mathbf{A}_1^2 \times \mathbf{A}_2^2$  is a **bridge** from  $\sigma_1$  to  $\sigma_2$  if

1.  $(a_1, a_2, a_3, a_4) \in \delta$

$$\sigma_1 \upharpoonright_{\{a_1, a_2\}} \upharpoonright_{\{a_3, a_4\}} \sigma_2 \downarrow$$

$(b_1, b_2, b_3, b_4) \in \delta$

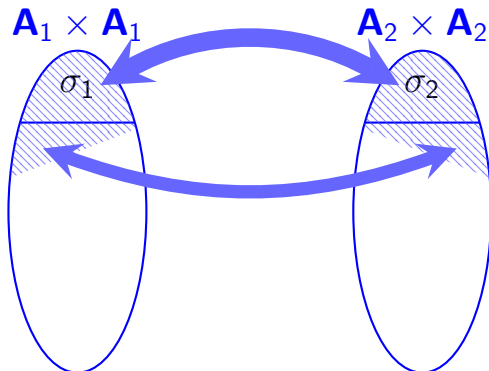
2.  $\text{pr}_{1,2}(\delta) \supseteq \sigma_1$ ,

$\text{pr}_{3,4}(\delta) \supseteq \sigma_2$

3.  $(a_1, a_2, a_3, a_4) \in \delta$

$$\Downarrow$$

$$(a_1, a_2) \in \sigma_1 \Leftrightarrow (a_3, a_4) \in \sigma_2.$$



### Example

$$A_1 = A_2 = \mathbb{Z}_4$$

$$\delta := \{(x_1, x_2, x_3, x_4) \in \mathbb{Z}_4^4 \mid x_1 - x_2 = 2x_3 - 2x_4\},$$

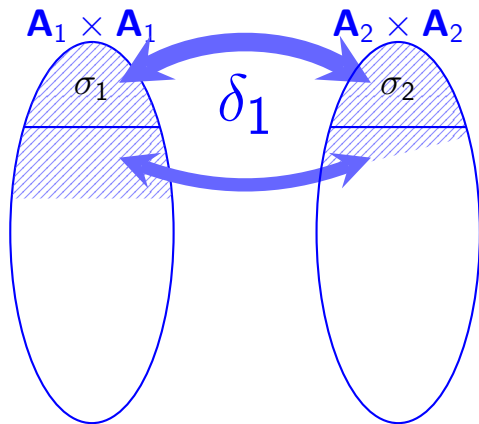
$\delta$  is a bridge between 0-congruence and (mod 2)-congruence.

## Composition of bridges



## Composition of bridges

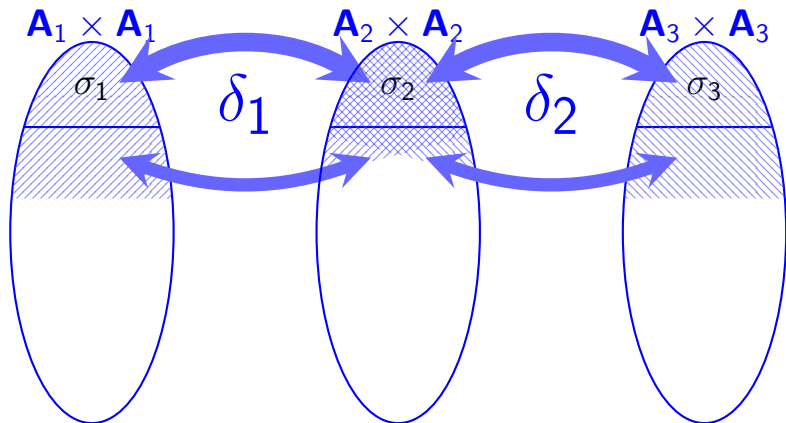
$\delta_1$  is a bridge from  $\sigma_1$  to  $\sigma_2$



## Composition of bridges

$\delta_1$  is a bridge from  $\sigma_1$  to  $\sigma_2$

$\delta_2$  is a bridge from  $\sigma_2$  to  $\sigma_3$



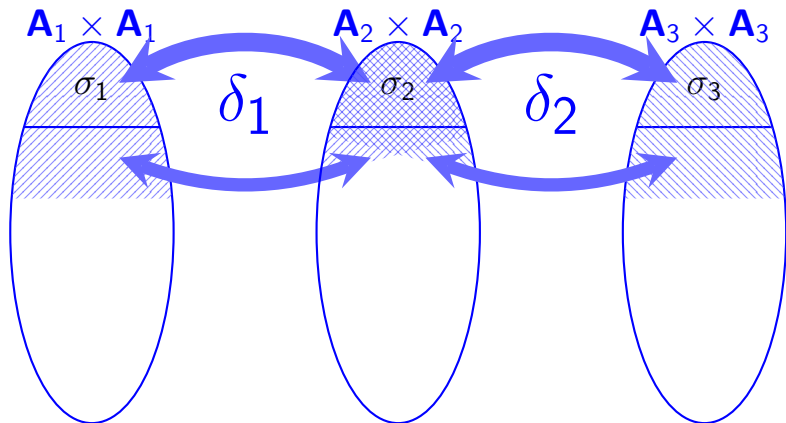
## Composition of bridges

$\delta_1$  is a bridge from  $\sigma_1$  to  $\sigma_2$

$\delta_2$  is a bridge from  $\sigma_2$  to  $\sigma_3$

$\delta := \delta_1 \circ \delta_2$  is a bridge from  $\sigma_1$  to  $\sigma_3$  (if  $\sigma_2$  is irreducible)

$$\delta(x_1, x_2, z_1, z_2) := \exists y_1 \exists y_2 \delta_1(x_1, x_2, y_1, y_2) \wedge \delta_2(y_1, y_2, z_1, z_2)$$



## Nice bridges

### Theorem

## Nice bridges

### Theorem

Suppose

- ▶  $\mathbf{A} = (A; w)$ , where  $w$  is a WNU operation.

## Nice bridges

### Theorem

Suppose

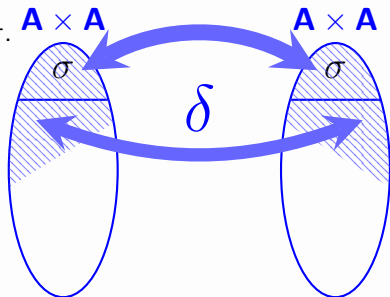
- ▶  $\mathbf{A} = (A; w)$ , where  $w$  is a WNU operation.
- ▶  $\sigma \in \text{Con}(\mathbf{A})$  is  $\wedge$ -irreducible.

## Nice bridges

### Theorem

Suppose

- ▶  $\mathbf{A} = (A; w)$ , where  $w$  is a WNU operation.
- ▶  $\sigma \in \text{Con}(\mathbf{A})$  is  $\wedge$ -irreducible.
- ▶  $\delta \leq \mathbf{A}^4$  is a bridge from  $\sigma$  to  $\sigma$ .

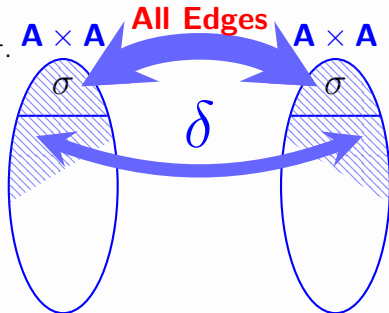


## Nice bridges

### Theorem

Suppose

- ▶  $\mathbf{A} = (A; w)$ , where  $w$  is a WNU operation.
- ▶  $\sigma \in \text{Con}(\mathbf{A})$  is  $\wedge$ -irreducible.
- ▶  $\delta \leq \mathbf{A}^4$  is a bridge from  $\sigma$  to  $\sigma$ .
- ▶  $\delta(x, x, y, y)$  defines  $\mathbf{A} \times \mathbf{A}$ .



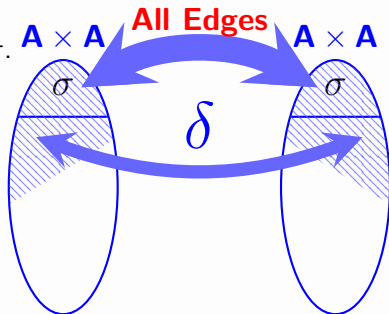


## Nice bridges

### Theorem

Suppose

- ▶  $\mathbf{A} = (A; w)$ , where  $w$  is a WNU operation.
- ▶  $\sigma \in \text{Con}(\mathbf{A})$  is  $\wedge$ -irreducible.
- ▶  $\delta \leq \mathbf{A}^4$  is a bridge from  $\sigma$  to  $\sigma$ .
- ▶  $\delta(x, x, y, y)$  defines  $\mathbf{A} \times \mathbf{A}$ .



$\sigma$  is called a **perfect linear congruence**.

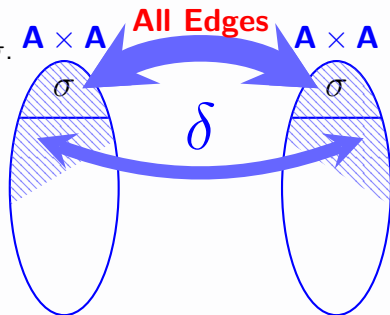
## Nice bridges

### Theorem

Suppose

- ▶  $\mathbf{A} = (A; w)$ , where  $w$  is a WNU operation.
- ▶  $\sigma \in \text{Con}(\mathbf{A})$  is  $\wedge$ -irreducible.
- ▶  $\delta \leq \mathbf{A}^4$  is a bridge from  $\sigma$  to  $\sigma$ .
- ▶  $\delta(x, x, y, y)$  defines  $\mathbf{A} \times \mathbf{A}$ .

Then  $\mathbf{A}/\sigma \in \mathbf{A}/\sigma^* \boxtimes \mathbb{Z}_p$



$\sigma$  is called a **perfect linear congruence**.

## Nice bridges

### Theorem

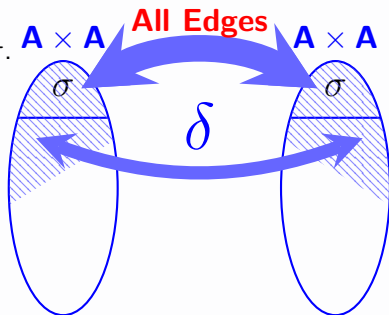
Suppose

- ▶  $\mathbf{A} = (A; w)$ , where  $w$  is a WNU operation.
- ▶  $\sigma \in \text{Con}(\mathbf{A})$  is  $\wedge$ -irreducible.
- ▶  $\delta \leq \mathbf{A}^4$  is a bridge from  $\sigma$  to  $\sigma$ .
- ▶  $\delta(x, x, y, y)$  defines  $\mathbf{A} \times \mathbf{A}$ .

Then  $\mathbf{A}/\sigma \in \mathbf{A}/\sigma^* \boxtimes \mathbb{Z}_p$

$\mathbf{C} \in \mathbf{B} \boxtimes \mathbb{Z}_p$ :

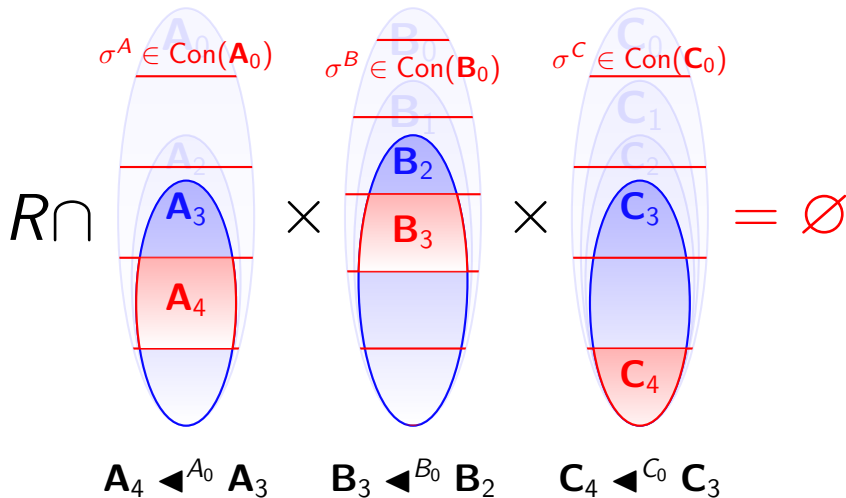
- domain  $C = B \times \mathbb{Z}_p$
- $f^{\mathbf{C}}(x_1, \dots, x_n) = (f^{\mathbf{B}}(x_1^{(1)}, \dots, x_n^{(1)}),$   
 $g f(x_1^{(1)}, \dots, x_n^{(1)}) + a_1 x_1^{(2)} + a_2 x_2^{(2)} + \dots + a_n x_n^{(2)})$



$\sigma$  is called a **perfect linear congruence**.

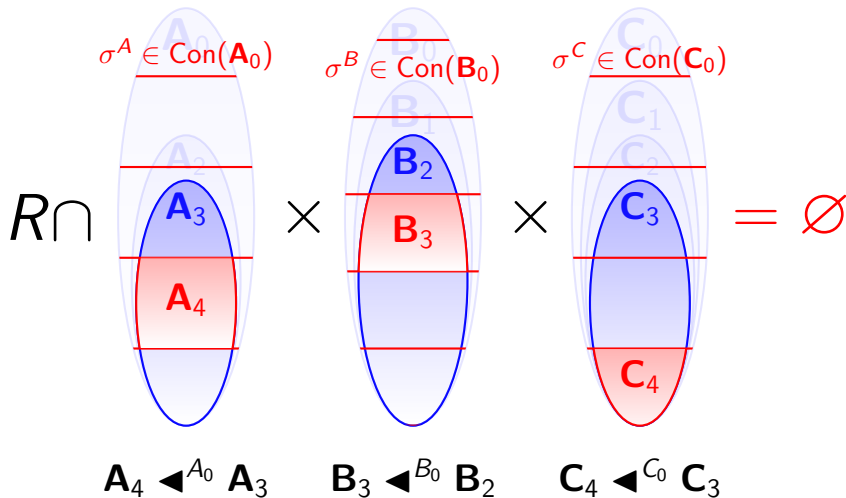
## Helly property

$$R \leq_{sd} \mathbf{A}_0 \times \mathbf{B}_0 \times \mathbf{C}_0, \mathbf{A}_k \lll^{A_0} \mathbf{A}_0, \mathbf{B}_l \lll^{B_0} \mathbf{B}_0, \mathbf{C}_m \lll^{C_0} \mathbf{C}_0.$$



## Helly property

$$R \leq_{sd} \mathbf{A}_0 \times \mathbf{B}_0 \times \mathbf{C}_0, \mathbf{A}_k \lll^{A_0} \mathbf{A}_0, \mathbf{B}_l \lll^{B_0} \mathbf{B}_0, \mathbf{C}_m \lll^{C_0} \mathbf{C}_0.$$



### Theorem

$\text{pr}_{1,2}(R)$  is linked  $\Rightarrow \sigma^A$  and  $\sigma^B$  are **perfect linear congruences**.

## Results

## Results

- Strong Subalgebras and Bridges are now connected!

## Results

- Strong Subalgebras and Bridges are now connected!
- PC and Linear Cases are joined!



## Results

- Strong Subalgebras and Bridges are now connected!
- PC and Linear Cases are joined!
- Congruences are global!

## Results

- Strong Subalgebras and Bridges are now connected!
- PC and Linear Cases are joined!
- Congruences are global!
- Subalgebras are stronger!

## Results

- Strong Subalgebras and Bridges are now connected!
- PC and Linear Cases are joined!
- Congruences are global!
- Subalgebras are stronger!

### Existence of an XY-symmetric operation

Every finite Taylor algebra has an XY-symmetric term operation,

## Results

- Strong Subalgebras and Bridges are now connected!
- PC and Linear Cases are joined!
- Congruences are global!
- Subalgebras are stronger!

### Existence of an XY-symmetric operation

Every finite Taylor algebra has an XY-symmetric term operation, where  $f$  is **XY-symmetric** if for every permutation  $\sigma$

$$|\{a_1, \dots, a_n\}| = 2 \Rightarrow f(a_1, \dots, a_n) = f(a_{\sigma(1)}, \dots, a_{\sigma(n)})$$

## Results

- Strong Subalgebras and Bridges are now connected!
- PC and Linear Cases are joined!
- Congruences are global!
- Subalgebras are stronger!

### Existence of an XY-symmetric operation

Every finite Taylor algebra has an XY-symmetric term operation, where  $f$  is **XY-symmetric** if for every permutation  $\sigma$

$$|\{a_1, \dots, a_n\}| = 2 \Rightarrow f(a_1, \dots, a_n) = f(a_{\sigma(1)}, \dots, a_{\sigma(n)})$$

### A simplified proof of the CSP Dichotomy Conjecture

12 pages modulo properties of strong subalgebras.

## Results

- Strong Subalgebras and Bridges are now connected!
- PC and Linear Cases are joined!
- Congruences are global!
- Subalgebras are stronger!

### Existence of an XY-symmetric operation

Every finite Taylor algebra has an XY-symmetric term operation, where  $f$  is **XY-symmetric** if for every permutation  $\sigma$

$$|\{a_1, \dots, a_n\}| = 2 \Rightarrow f(a_1, \dots, a_n) = f(a_{\sigma(1)}, \dots, a_{\sigma(n)})$$

### A simplified proof of the CSP Dichotomy Conjecture

12 pages modulo properties of strong subalgebras.

### New CSP algorithms based on Linear Relaxation

## Results

- Strong Subalgebras and Bridges are now connected!
- PC and Linear Cases are joined!
- Congruences are global!
- Subalgebras are stronger!

### Existence of an XY-symmetric operation

Every finite Taylor algebra has an XY-symmetric term operation, where  $f$  is **XY-symmetric** if for every permutation  $\sigma$

$$|\{a_1, \dots, a_n\}| = 2 \Rightarrow f(a_1, \dots, a_n) = f(a_{\sigma(1)}, \dots, a_{\sigma(n)})$$

### A simplified proof of the CSP Dichotomy Conjecture

12 pages modulo properties of strong subalgebras.

### New CSP algorithms based on Linear Relaxation

Coming soon!

## Results

- Strong Subalgebras and Bridges are now connected!
- PC and Linear Cases are joined!
- Congruences are global!
- Subalgebras are stronger!

### Existence of an XY-symmetric operation

Every finite Taylor algebra has an XY-symmetric term operation, where  $f$  is **XY-symmetric** if for every permutation  $\sigma$

$$|\{a_1, \dots, a_n\}| = 2 \Rightarrow f(a_1, \dots, a_n) = f(a_{\sigma(1)}, \dots, a_{\sigma(n)})$$

### A simplified proof of the CSP Dichotomy Conjecture

12 pages modulo properties of strong subalgebras.

### New CSP algorithms based on Linear Relaxation

Coming soon!



Thank you for your attention