Finite Simple Groups in the Primitive Positive Constructability Poset

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Structures

Definition

A *(relational) structure* $\underline{A}$ over a signature σ is a set A together with subsets of powers of A for each element in σ.

Examples

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Examples of structures

- $(\mathbb{N};+,\cdot,1)$ where $+$ and $\cdot$ are considered as subset of $\mathbb{N}^3$ and 1 is considered as subset of $\mathbb{N}^1$.
- All groups, rings, modules,... in the usual way.
- A group action $G \curvearrowright X$ defines a structure on X, which we call $S(G \curvearrowright X)$. The signature is G and the relation corresponding to $g \in G$ is $\{(x, g.x) \mid x \in X\}$.
- graphs (with a binary relation)
- 3-SAT $= (\{\top,\bot\};\top,\bot,\wedge,\vee,\neg)$

Problems

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Examples

- $\mathrm{CSP}(3\text{-SAT}) = \mathrm{CSP}(\{\top, \bot\}; \top, \bot, \wedge, \vee, \neg)$ is the usual 3-SAT problem (NP-complete)
- $\mathrm{CSP}(\mathbb{N}; +, \cdot, 1)$ decides whether a system of equations can be solved in $\mathbb{N}$. (Turing Complete)
- The CSP of a finite (undirected) graph is to decide whether another finite graph can be mapped to this one. (If the graph is bipartite, this is in P, else it is NP-complete. Hell, Nešetřil 1990)
- The CSP of a finite structure is in P or NP complete. (Bulatov 2017; Zhuk 2017)

Reductions

Definition

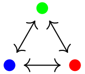
A *primitive positive construction* of a σ-structure $\underline{A}$ in a τ-structure $\underline{B}$ consists of

1. a positive integer n
2. a σ-structure $\underline{\tilde{B}}$ with base set B^n, where the k-ary relations of $\underline{\tilde{B}}$ are pp-definable as kn-ary relations in $\underline{B}$
3. σ-homomorphisms $f\colon \underline{\tilde{B}} \to \underline{A}$ and $g\colon \underline{A} \to \underline{\tilde{B}}$.

A primitive positive construction gives a logspace reduction from CSP($\underline{A}$) to CSP($\underline{B}$).

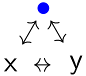
Example

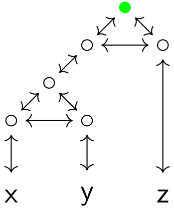
Graph 3-coloring (with colors ${\color{blue}\bullet}$, ${\color{red}\bullet}$, ${\color{green}\bullet}$) is NP-hard, because one can reduce

3-SAT to  by $n = 1$ and

$\top = {\color{green}\bullet}$

$\bot = {\color{red}\bullet}$

unequal(x, y) = 

$x \vee y \vee z$ = 

with identification maps

$$f({\color{red}\bullet}) = \bot \qquad g(\bot) = {\color{red}\bullet}$$

$$f({\color{blue}\bullet}) = f({\color{green}\bullet}) = \top \qquad g(\top) = {\color{green}\bullet}$$

Algebraically

Definition

A *polymorphism* of a σ-structure $\underline{A}$ is a homomorphism $\underline{A}^n \to \underline{A}$ for $n \in \mathbb{N}$.

$$\begin{array}{cccccc}
x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} & \overset{f}{\mapsto} & y_1 \\
x_{2,1} & x_{2,2} & x_{2,3} & x_{2,4} & \mapsto & y_2 \\
x_{3,1} & x_{3,2} & x_{3,3} & x_{3,4} & \mapsto & y_3 \\
\pitchfork & \pitchfork & \pitchfork & \pitchfork & \Longrightarrow & \pitchfork \\
R & R & R & R & & R
\end{array}$$

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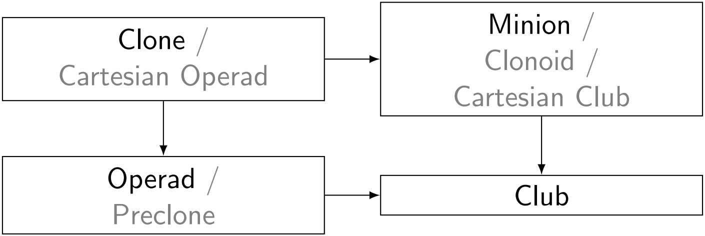
The set $\mathrm{Pol}(\underline{A})$ of all polymorphisms has the structure of a



Minions

The *minor* of $f\colon A^n \to A$ along $\alpha\colon [n] \to [m]$ is the map $f_\alpha\colon A^m \to A, (x_1, \ldots, x_m) \mapsto f(x_{\alpha(1)}, \ldots, x_{\alpha(n)})$.

Definition

A *minion homomorphism* from $\mathrm{Pol}(\underline{A})$ to $\mathrm{Pol}(\underline{B})$ is a map of sets F, that

- preserves arities and
- preserves minors, i.e. $F(f_\alpha) = (Ff)_\alpha$

Picture from https://www.pngwing.com/id/free-png-svred, at 7.Oct.2024

Minor Condition

A *height-1-condition* or *minor condition* of $\underline{A}$ is a condition of the form

$$\exists f \in \mathrm{Pol}(\underline{A}) : \bigwedge f_\alpha = f_\beta$$

Examples

$f(x) = f(y)$	constant
$f(x,x,x) = f(x,y,y) = f(y,y,x)$	quasi Maltsev
$f(x,x,x) = f(x,x,y) = f(x,y,x) = f(y,x,x)$	quasi majority
$f(x,y,z) = f(y,z,x) = f(y,x,z)$	(fully) symmetric of arity 3
$f(x,x,y) = f(x,y,y)$ and symmetric	totally symmetric of arity 3
$f(x,x,y) = f(z,z,y)$ and symmetric	general. minority of arity 3

Three Definitions

Theorem (Barto, Opršal, Pinsker 2018)

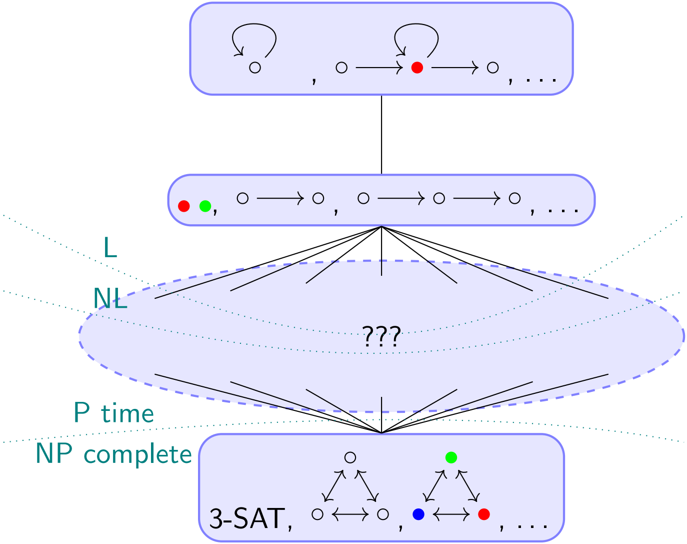
For two structures $\underline{A}$ and $\underline{B}$, the following is equivalent:

1. *$\underline{A}$ pp-constructs $\underline{B}$.*
2. *There is a minion-homomorphism* $\mathrm{Pol}(\underline{A}) \to \mathrm{Pol}(\underline{B})$.
3. *Every minor condition valid in* $\mathrm{Pol}(\underline{A})$ *is valid in* $\mathrm{Pol}(\underline{B})$.

In this case, $\mathrm{CSP}(\underline{B})$ reduces to $\mathrm{CSP}(\underline{A})$ in logspace (L).

The Primitive Positive Constructability Poset

The PP-Constructability Poset on Finite Structures



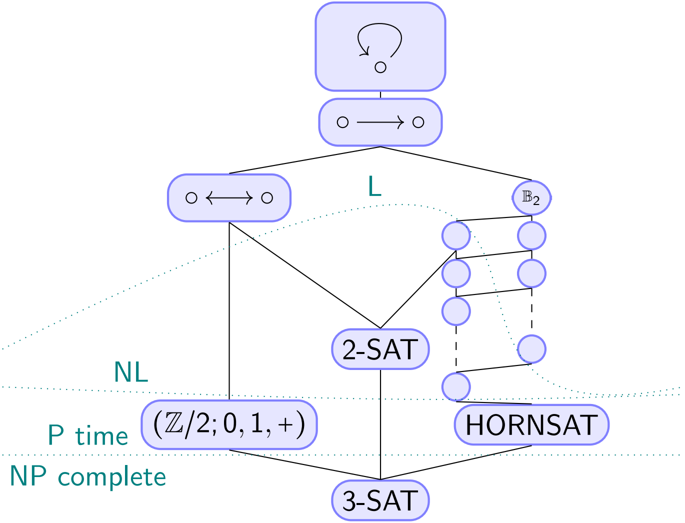
The PP-Constructability Poset on Finite Structures

1. Every equivalence class contains an idempotent structure $\underline{A}$.

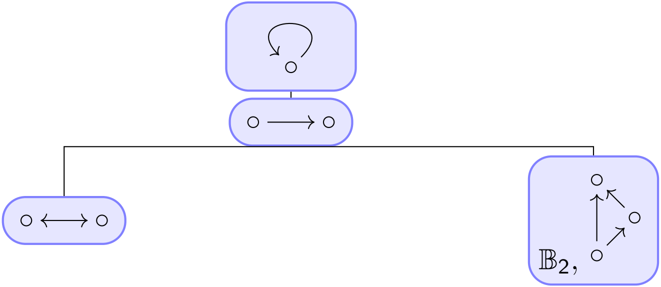
$$\mathrm{End}(\underline{A}) = \{\mathrm{id}_A\}$$

2. The poset of all smooth digraphs is classified (Bodirsky, Starke, Vucaj 2021)
3. The poset of all 2-Element structures is classified (Bodirsky, Vucaj 2020)

The PP-Constructability Poset on 2-Element Structures



The Third Layer of the PP-Constructability Poset



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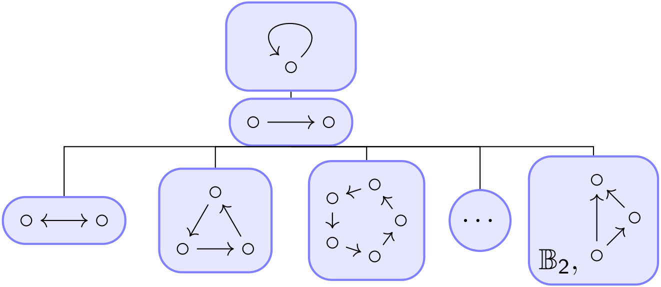

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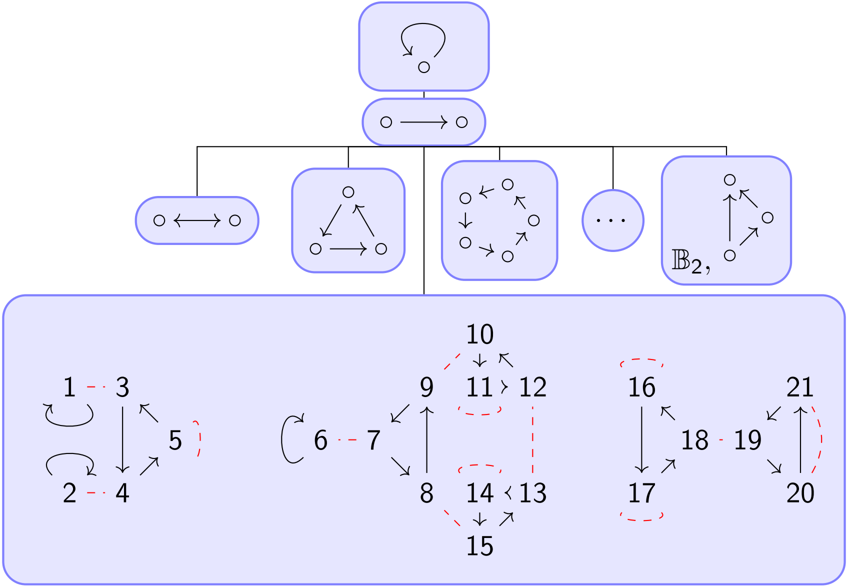

The Third Layer of the PP-Constructability Poset

Theorem

The pp-constructabillity poset has a third layer consisting of the equivalence classes of

1. $\mathbb{B}_2$ *and*
2. *for all finite simple groups* G*, the structure* $S(G \curvearrowright \mathbb{P}(G))$*, where* $\mathbb{P}(G)$ *is the disjoint union of all primitive group actions.*

Moreover,

$$\mathbb{P}(G) = \begin{cases} G & \text{(with multiplication)} \\ & \text{if } G \text{ is cyclic} \\ \{M \leq G \text{ maximal subgroup}\} & \text{(with conjugation)} \\ & \text{if } G \text{ is nonabelian simple} \end{cases}$$

Proof overview

Let $\underline{A}$ be a structure.

1. If $\underline{A}$ has a quasi Maltsev polymorphism and fully symmetric polymorphisms of all arities, then $\circ \longrightarrow \circ$ pp-constructs $\underline{A}$.
2. If $\underline{A}$ has no quasi Maltsev polymorphism, then $\underline{A}$ pp-constructs $\mathbb{B}_2$. (Opršal 2018)
3. If $\underline{A}$ has not fully symmetric polymorphism of an arity n, then $\underline{A}$ pp-constructs $S(G \curvearrowright \mathbb{P}(G))$ for G finite simple group.
4. $S(G \curvearrowright \mathbb{P}(G))$ does not pp-construct $S(G' \curvearrowright \mathbb{P}(G'))$ for $G \neq G'$ different, finite simple

Part 1

Let $\underline{A}$ be a structure with $\mathrm{End}(\underline{A}) = \mathrm{id}_A$, quasi Maltsev, symetric of all arities.

$$f(x,x,x) = f(x,y,y) = f(y,y,x) = x \qquad \text{~~quasi~~ Maltsev}$$

$$f(x,y,z) = f(y,z,x) = f(y,x,z) \qquad \text{(fully) symmetric of arity 3}$$

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- $\underline{A}$ has a majority polymorphism. (Vucaj 2023)

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- $\underline{A}$ has generalised pairing polymorphisms: arity $2n+1$, mapping

$$\text{permutation of } (x, y_1, y_1, y_2, y_2, \dots, y_n, y_n) \mapsto x$$

Proof: Induction, Exercise. Hint:

$$\text{majority}\begin{pmatrix} \text{Maltsev}(x_1,x_3,x_2) \\ \text{Maltsev}(x_3,x_2,x_1) \\ \text{Maltsev}(x_2,x_1,x_3) \end{pmatrix}, \quad \text{Maltsev}(x_1, \text{pairing}(x_3,\dots,x_{2n+1}), x_2)$$

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$$\text{symmetric}_{\text{arity } n!}(\text{pairing}(\text{permutation of}(x_1, \ldots, x_n)) \mid \text{all permutations})$$

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- $\text{Pol}(\circ \longrightarrow \circ)$ maps to $\text{Pol}(\underline{A})$. (Vucaj, Zhuk 2024)
  Idea: Map the generators of $\text{Pol}(\circ \longrightarrow \circ)$ to generalized minority and totally symmetric polymorphism.

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$\underline{A}$ pp-constructs a group action without fixed point, namely $S(S_n \curvearrowright \mathrm{Pol}_n(\underline{A}))$.

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2. If $N \trianglelefteq G$, $N \curvearrowright X$ trivial, then $S(G \curvearrowright X)$ pp-constructs $S(G/N \curvearrowright X)$.

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3. If $N \trianglelefteq G$, $N \curvearrowright X$ with fixed points, then

   $$\mathrm{Fix}(N) = \{x \in X \mid N.x = x\}$$

   is closed under G action. Moreover, $S(G \curvearrowright X)$ pp-constructs $S(G \curvearrowright \mathrm{Fix}(N))$ and $S(G/N \curvearrowright \mathrm{Fix}(N))$.

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What is left?

G simple, every maximal subgroup of G has a fixed point

Part 4

$S(G \curvearrowright \mathbb{P}(G))$ does not pp-construct $S(G' \curvearrowright \mathbb{P}(G'))$ for $G \neq G'$ different, finite simple.

Definition

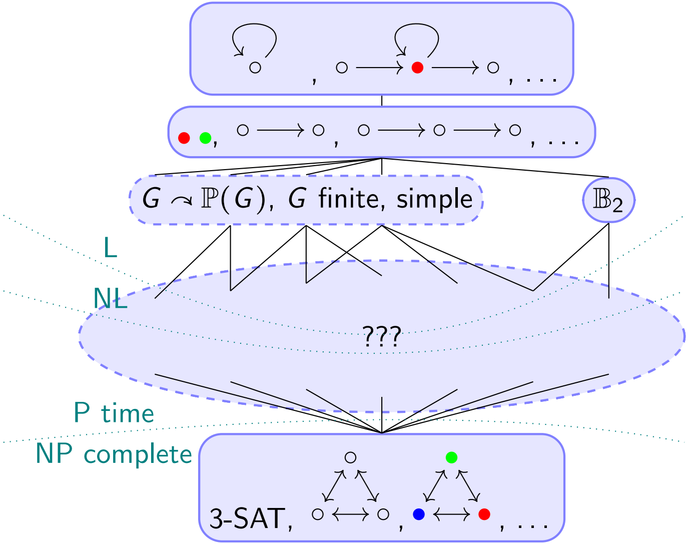
For $G \curvearrowright X$, define the minor condition $\Sigma(G \curvearrowright X)$ as $\exists f \in \mathrm{Pol}_{|X|}(\underline{A})$,

$$\forall g \in G : f(x_1, \ldots, x_{|X|}) = f(x_{g.1}, \ldots, x_{g.|X|})$$

- $S(G \curvearrowright X)$ does not satisfy $\Sigma(G \curvearrowright X)$.
- If $S(G \curvearrowright X)$ does not satisfy $\Sigma(H \curvearrowright Y)$, then
  - there is no appropriate map $X^Y \to X$,
  - there is a problem child m in $X^Y = \mathrm{map}(Y, X)$,
  - there are subgroups $G'_m \trianglelefteq G_m \leq G$, $H'_m \trianglelefteq H_m \leq H$ such that $G_m \curvearrowright X, H_m \curvearrowright Y$ nontrivial and $G_m/G'_m \cong H_m/H'_m \ncong \{1\}$.

$S(G \curvearrowright \mathbb{P}(G))$ satisfies $\Sigma(G' \curvearrowright \mathbb{P}(G'))$ but not $\Sigma(G \curvearrowright \mathbb{P}(G))$ □

The PP-Constructability Poset on Finite Structures



Thank you for your attention

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