Polymorphisms in CSPs, Topology and Social Choices

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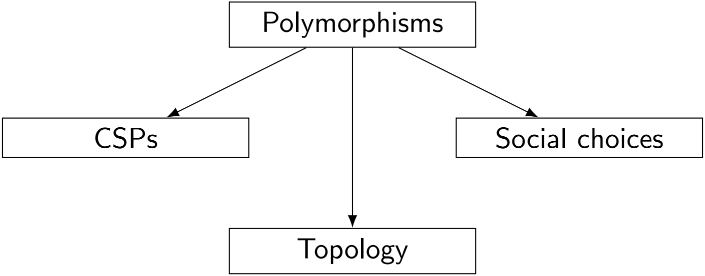
30th May 2024

ERC Synergy Grant POCOCOP (GA 101071674)

Some Application of Polymorphisms



Polymorphisms ...

Definition

Let $\mathcal{C}$ be a category with finite products. Let A and B be objects in $\mathcal{C}$. Then,

$$\mathrm{Pol}(A, B) = (\mathrm{Hom}_{\mathcal{C}}(A^n, B) \mid n \in \mathbb{N})$$

The polymorphisms $\mathrm{Pol}(A, B)$ define a minion.
The polymorphisms $\mathrm{Pol}(A) \coloneqq \mathrm{Pol}(A, A)$ define a clone and a minion.

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Example

Polymorphisms of structures, clones, minions, topological spaces, Coalgebras over a field (with $\otimes$), ...

Polymorphisms in CSPs

Theorem

Let A and B be finite strucutres.

- $\mathrm{Pol}(A, B)$ *determines complexity of the* $PCSP(A, B)$
- *Minionhomomorphisms induce logspace reductions of PCSPs*

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Questions

- Which minions correspond to problems in P? (Solved for $A = B$ by Bulatov 2017 and Zhuk 2017)
- Which minions correspond to problems in L, NL, $\mathrm{Mod}_2\, L$, ... ?
- Find good definitions for a generalization to infinite domain (P)CPS.

Polymorphisms in Topology

Consider $A = B$ topological spaces with continuous maps.

Theorem (multiple contributors)

Let A be a connected compact simplicial complex (a nice topological space). The following are equivalent:

1. *A is contractible.*

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3. *A has an idempotent majority polymorphism. (Taylor 1977)*

Definition

idempotent	$\forall x \in A : x = f(x, \ldots, x)$
majority	$\forall x, y \in A : x = f(x, x, y) = f(x, y, x) = f(y, x, x)$

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4. *A has idempotent fully symmetric polymorphisms of all arities. (Eckmann, Ganea, Hilton 1962 and Weinberger 2004)*

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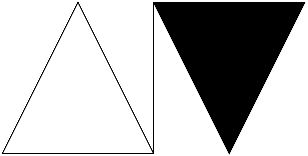
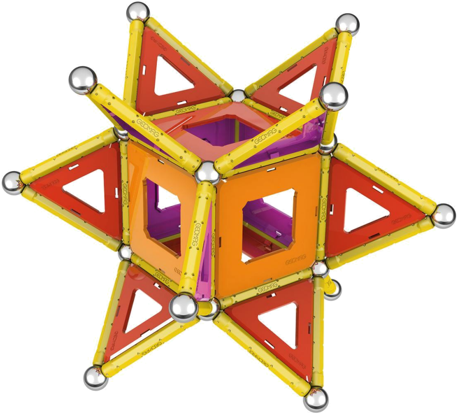
majority	$\forall x, y \in A : x = f(x, x, y) = f(x, y, x) = f(y, x, x)$
fully symmetric	$\forall x_1 \ldots x_n \in A, \forall \sigma : f(x_1, \ldots, x_n) = f(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$

Polymorphisms in Topology





Two connected compact abstract simplicial complexes[1]

[1] right picture from https://www.kinderkram-direkt.de/GEOMAG-Magnet-Spiel-Panels-114-Teile.htm

Polymorphisms in Topology

Consider $A = B$ abstract simplicial complexes with simplicial maps.

Theorem (Larose, Zádori 2005 and Meyer unpublished)

Let A be a connected compact abstract simplicial complex that has any idempotent Taylor polymorphism. Then, every connected component of A is contractible.

Questions

- Which polymorphisms classify that a map is contractible (homotopic to a constant map)?
- What about infinite simplicial complexes?

Polymorphisms in Social Choices

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Theorem

Consider a tournament over multiple rounds with $n \geq 3$ participants:

Round №	1	2	3	4	result
Adam	1st	3rd	4th	1st	3rd
Bertalan	2nd	4th	3rd	3rd	4th
Celestin	3rd	2nd	2nd	4th	2nd
Dmitriy	4th	1st	1st	2nd	1st

The result of a person should only depend on the rankings of that person (not on the name or the ranking of the others).

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Proof.

Every polymorphism of K_n is essentially unary. □

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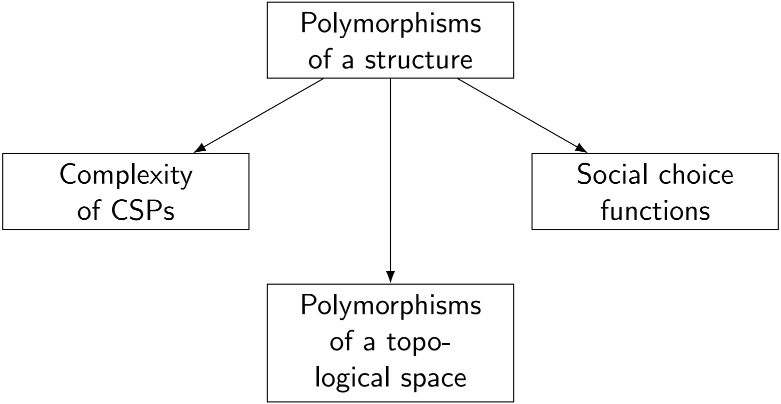

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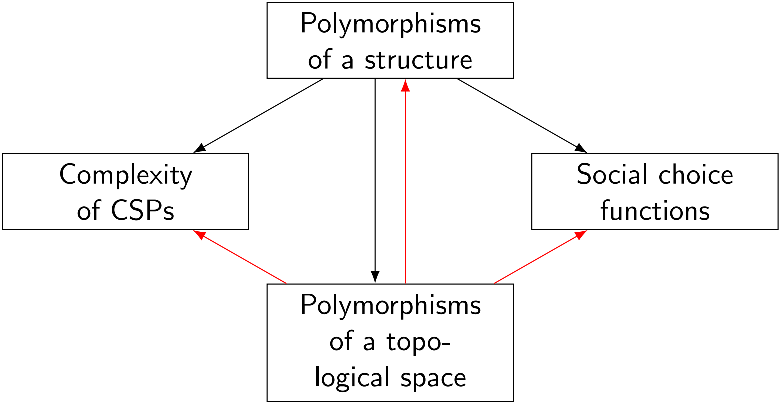


Topology in Social Choices

Theorem (Weinberger 2004)

Let A be a topological space that has for every n a continuous, unanimous, anonymous choice function. Then, A is contractible or the choice functions are not sober.

Example

Let A be a set with the structure of a simplicial complex.
If there is a choice function on this set satisfying any Taylor conditions, then each component is contractible.

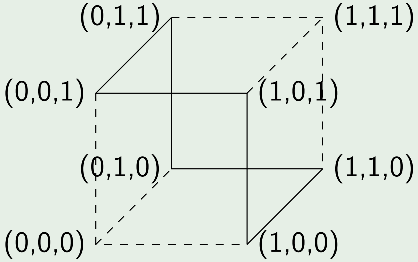
Topology in Structures

Definition

Let A be a structure and $B \subseteq A^n$ be a pp-definable subset. Then, B becomes a simplicial complex with faces

$$\left\{ (b_{1,1}, \ldots, b_{1,n}), \ldots, (b_{k,1}, \ldots, b_{k,n}) \in B^k \;\middle|\; \bigwedge_{i:[n]\to[k]} (b_{i_1,1}, \ldots, b_{i_n,n}) \in B \right\}$$

Example



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Theorem (Hell, Nešetřil 1990)

Every finite (undirected) simple non-bipartite graph G has no Taylor polymorphism.

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Proof.

The set of all tuples of two points connected by an edge in G has the structure of a simplicial complex C with an automorphism α representing the flip of the edge. Now, assume that G has a Taylor polymorphism.

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2. The core of (C, α) has a Taylor polymorphism.

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5. α has a fixed point in the core. (Brouwer fixed-point theorem)

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(Idea based on Krokhin, Opršal, Wrochna, Živný 2022) □

Topology in CSP

Theorem (Schnider, Weber at CG Week 2024)

Let A be an idempotent Boolean structure.

1. *If A has a Schaefer polymorphism, then every pp-definable set has trivial homology groups when considered as simplicial complex.*
2. *If A has no Schaefer polymorphism, then every compact simplicial complex can be obtained by a pp-definable set (up to a homeomorphism).*

Topology in CSP

Theorem (Meyer unpublished)

Let A be an idempotent structure.

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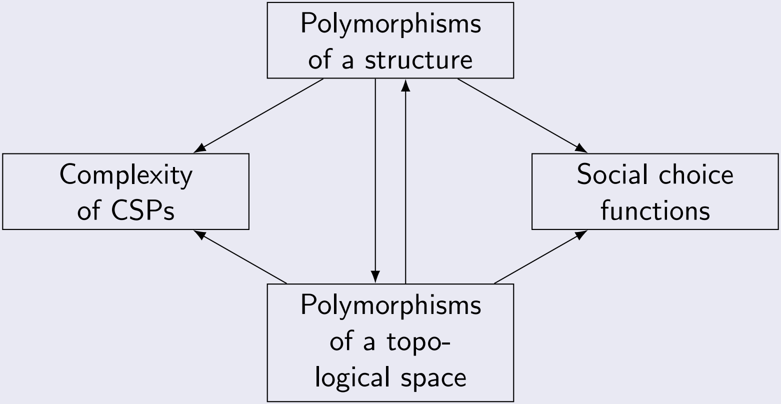
Questions

- Can we get new results for promise CSPs? (K.O.W.Ž. 2022)
- Does this result generalizes to 0-homotopic maps and PCSPs?
- What about infinite domain CSP and non-compact simplicial complexes?

Thank you for your attention



Funding statement: Funded by the European Union (ERC, POCOCOP, 101071674).
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