Minimal operations over permutation groups

Paolo Marimon Michael Pinsker

TU Wien

PALS, November 12, 2024

European Research Council
Established by the European Commission

TECHNISCHE UNIVERSITÄT WIEN

POCOCOP ERC Synergy Grant No. 101071674

Outline

Clones

Definition 1 (Clone)

Let B be a (possibly infinite) set.
Let $\mathcal{O}^{(n)} = B^{B^n}$ be the set of functions $f : B^n \to B$, and
$\mathcal{O} := \bigcup_{n \in \mathbb{N}} \mathcal{O}^{(n)}$.
We call $\mathcal{C} \subseteq \mathcal{O}$ a **clone** over B if

- $\mathcal{C}$ contains all projections;
- $\mathcal{C}$ is closed under composition.

For $\mathcal{S} \subseteq \mathcal{O}$, $\langle \mathcal{S} \rangle$ is the smallest clone containing $\mathcal{S}$.

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Closed clones

Interested in clones which are **closed** in the **pointwise convergence topology**:[1] For $\mathcal{S} \subseteq \mathcal{O}$,
$f \in \overline{\mathcal{S}} \Leftrightarrow$ for all $A \subseteq B$ finite there is $g \in \mathcal{S}$ such that $g_{\restriction A} = f_{\restriction A}$.

For B finite, topology trivialises (i.e. closed clone=clone).

$\overline{\langle \mathcal{S} \rangle}$ denotes the smallest closed clone containing $\mathcal{S}$.

There is a correspondence between:

- closed subclones of $\mathcal{O}$;
- polymorphism clones of relational structures on B.

Definition of polymorphism clone

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Monoidal intervals

Let $\mathcal{T}$ be a **transformation monoid** on B (i.e. unary operations containing Id, and closed under composition).

Closed clones whose unary operations are $\overline{\mathcal{T}}$ form an interval in the lattice of closed clones on B, known as the **monoidal interval of** $\mathcal{T}$.

Studying the structure and size of monoidal intervals has a long history in universal algebra (Szendrei 1986, Chapter 3):

- for $\mathcal{O}_B^{(1)}$ (Burle 1967);
- for $G \curvearrowright B$ a permutation group (Pálfy and Szendrei 1982; Kearnes and Szendrei 2001) (with focus on collapse);
- for other monoids (Krokhin 1995);
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Minimal operations

What is the minimal amount of structure in a clone containing $\mathcal{T}$?

Definition 2 (Minimal clone)

Let $\mathcal{D} \supsetneq \mathcal{C}$ be closed subclones of $\mathcal{O}$.
$\mathcal{D}$ is **minimal above** $\mathcal{C}$ if there is no closed clone $\mathcal{E}$ such that $\mathcal{C} \subsetneq \mathcal{E} \subsetneq \mathcal{D}$.

Definition 3 (almost minimal and minimal operations)

The k-ary operation $f \in \mathcal{O} \setminus \mathcal{C}$ is **almost minimal above** $\mathcal{C}$ if for each $r < k$,

$$\overline{\langle \mathcal{C} \cup \{f\} \rangle} \cap \mathcal{O}^{(r)} = \mathcal{C} \cap \mathcal{O}^{(r)}.$$

If f is almost minimal above $\mathcal{C}$ and $\overline{\langle \mathcal{C} \cup \{f\} \rangle}$ is minimal above $\mathcal{C}$, then f is **minimal above** $\mathcal{C}$.

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Basic facts on minimality and almost minimality

- $\mathcal{D}$ is minimal above $\mathcal{C}$ if and only if $\mathcal{D} = \overline{\langle \mathcal{C} \cup \{f\} \rangle}$ for f minimal above $\mathcal{C}$;
- minimal elements in the interval of $\mathcal{T}$ above $\overline{\langle \mathcal{T} \rangle}$ correspond to minimal clones above $\overline{\langle \mathcal{T} \rangle}$ which are not essentially unary[2];
- For B finite, if $\mathcal{E} \supsetneq \mathcal{C}$, there is $\mathcal{E} \supseteq \mathcal{D} \supsetneq \mathcal{C}$ minimal above $\mathcal{C}$;
- this can fail over an infinite set, but holds in the settings that interest us (see next slide);
- ALWAYS, if $\mathcal{E} \supsetneq \mathcal{C}$, there is $f \in \mathcal{E} \setminus \mathcal{C}$ almost minimal above $\mathcal{C}$;

We will study minimal operations above $\overline{\langle G \rangle}$ for $G \curvearrowright B$ a non-trivial permutation group.

[2] f is **essentially unary** if it depends on only one variable. Otherwise, it is **essential**.

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Motivation: infinite domain CSPs

Motivation from infinite-domain CSPs ‣ More on CSPs:

- $\mathrm{Aut}(B) \curvearrowright B$ is **oligomorphic**: it has finitely many orbits on B^n for each $n \in \mathbb{N}$ (B countable).We say B is ω-**categorical**.
  Examples of ω-categorical structures:
  - $(\mathbb{N}, =)$;
  - $(\mathbb{Q}, <)$;
  - The Random graph;
- can reduce to **core** case where $\overline{\mathrm{Aut}(B)} = \mathrm{End}(B)$;
- if $\mathrm{Pol}(B)$ is essentially unary, $\mathrm{CSP}(B)$ is hard.
  So can assume $\mathrm{Pol}(B) \supsetneq \overline{\langle \mathrm{Aut}(B) \rangle}$ and $\mathrm{Pol}(B)^{(1)} = \overline{\mathrm{Aut}(B)}$;
- there is $f \in \mathrm{Pol}(B)$ minimal above $\overline{\langle \mathrm{Aut}(B) \rangle}$ (finite language);
- understanding these minimal operations is very helpful: many arguments[3] rely on finding low arity (binary) essential polymorphism given the existence of an essential one;

[3] Bodirsky and Kára 2010; Bodirsky and Pinsker 2014; Mottet and Pinsker 2022.

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Some terminology for operations

We define some operations in virtue of the identities they satisfy:

- **ternary quasi-majority:**

$$m(x,x,y) \approx m(x,y,x) \approx m(y,x,x) \approx m(x,x,x);$$

- **quasi-Malcev:**

$$M(y,y,x) \approx M(x,y,y) \approx M(x,x,x);$$

- For f **idempotent**, i.e. $f(x,\dots,x) \approx x$, remove the 'quasi';
- **ternary minority:**

$$\mathfrak{m}(y,y,x) \approx \mathfrak{m}(y,x,y) \approx \mathfrak{m}(x,y,y) \approx \mathfrak{m}(x,x,x) \approx x;$$

- **quasi-semiprojection**: k-ary f such that there is an $i \in \{1,\dots,k\}$ and a unary operation g such that whenever $(a_1,\dots,a_k)$ is a non-injective tuple from B,

$$f(a_1,\dots,a_k) = g(a_i).$$

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Some terminology for operations

We define some operations in virtue of the identities they satisfy:

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Rosenberg's five types theorem

Theorem 4 (Five types theorem, Rosenberg 1986)

Let B be finite and f be minimal above $\langle \mathrm{Id} \rangle$. Then, f is one of:

1. *a unary operation;*
2. *a binary operation;*
3. *a ternary majority operation;*
4. *a minority of the form $x + y + z$ in some Boolean group $(B, +)$;*
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A group is **Boolean** if every element has order 2.

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Bodirsky and Chen 2007 classify minimal operations above oligomorphic[4] permutation groups.

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Note: No minority type!

We obtain better results even in this case!

[4] $G \curvearrowright B$ is **oligomorphic** if it has finitely many orbits on B^n for each $n \in \mathbb{N}$.

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In the oligomorphic case we improve on Bodirsky and Chen 2007:

- We reduced from four to three types ($G \curvearrowright B$ is not free);
- Stronger characterisation of the quasi-semiprojection case.

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It is nice to have a non-trivial theorem true for every non-trivial group and false for the trivial group.

G-invariant Boolean Steiner 3-quasigroups

Definition 7

A G-invariant Boolean **Steiner 3-quasigroup**[5] is a symmetric ternary minority operation m satisfying:

$$m(x,y,m(x,y,z)) \approx z\;; \qquad \text{(SQS)}$$

$$m(x,y,m(z,y,w)) \approx m(x,z,w)\;; \qquad \text{(Bool)}$$

$$\text{for all } \alpha,\beta,\gamma \in G, m(\alpha x,\beta y,\gamma z) \approx \alpha\beta\gamma m(x,y,z). \qquad \text{(Inv)}$$

[5] Studied in universal algebra (Quackenbush 1975; Ganter and Werner 1975) and design theory (Lindner and Rosa 1978).

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(SQS) yields that $m(x_1, x_2, x_3) = x_4 \wedge \bigwedge_{i<j\leq 4} x_i \neq x_j$ is a **Steiner quadruple system** on B: a 4-hypergraph on B such that every three vertices are in a unique 4-hyperedge.

Steiner 3-quasigroups correspond to Steiner quadruple systems on B.

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Correspond to $x + y + z$ on a Boolean group on B.

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$m(x, y, z)$ also induces a Boolean Steiner 3-quasigroup on the G-orbits.

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Lemma 8 (Marimon and Pinsker 2024)

For $G \curvearrowright B$, there are G-invariant Boolean Steiner 3-quasigroup if and only if $G \curvearrowright B$ is a Boolean group acting freely on B with either 2^n or infinitely many orbits.

Orbit-semiprojections

f is an **orbit-semiprojection** if there is an $i \in \{1, \ldots, k\}$ and $g \in \overline{G}$ such that whenever at least two of the a_j lie in the same orbit,

$$f(a_1, \ldots, a_k) = g(a_i).$$

Following a similar idea to Pálfy 1986 on semiprojections,

Lemma 9 (Marimon and Pinsker 2024)

Let $G \curvearrowright B$ with s-many orbits and B finite. Then, for all $2 \leq k \leq s$ there is a k-ary orbit-semiprojection minimal above $\overline{\langle G \rangle}$.

- also holds for $\mathrm{Aut}(B) \curvearrowright B$ oligomorphic with B in a finite language;
- always holds for minimality in the lattice of *all* clones (rather than closed clones);
- almost minimal k-ary orbit-semiprojections exist for all $2 \leq k \leq s$.

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Problem: Find $G \curvearrowright B$ with three orbits such that there is no ternary orbit-semiprojection minimal above $\overline{\langle G \rangle}$.

Methods: almost minimality

To classify minimal operations, we first classify almost minimal operations above $\overline{\langle G \rangle}$;

This splits into three cases:

- $G \curvearrowright B$ not a Boolean group acting freely on B;
- $G \curvearrowright B$ a Boolean group acting freely on B with $|G| > 2$;
- $\mathbb{Z}_2 \curvearrowright B$ acting freely.

Case1: The Three types theorem

Theorem 11 (Three types theorem, Marimon and Pinsker 2024)

Let $G \curvearrowright B$ with s-many orbits be such that G is not a Boolean group acting freely on B. Let f be almost minimal above $\overline{\langle G \rangle}$. Then, f is one of:

1. *a unary operation;*
2. *a binary operation;*
3. *a k-ary orbit-semiprojection for $3 \leq k \leq s$.*

Case 2: the Boolean case

Theorem 12 (Boolean case, Marimon and Pinsker 2024)

Let $G \curvearrowright B$ be a Boolean group acting freely on B with s-many orbits and $|G| > 2$. Let f be an almost minimal operation above $\langle G \rangle$. Then, f is one of:

1. *a unary operation;*
2. *a binary operation;*
3. *a G-quasi-minority;*
4. *a k-ary orbit-semiprojection for $3 \leq k \leq s$.*

A ***G*-quasi-minority** is a ternary operation such that for all $\beta \in G$,

$$\mathfrak{m}(y, x, \beta x) \approx \mathfrak{m}(x, \beta x, y) \approx \mathfrak{m}(x, y, \beta x) \approx \mathfrak{m}(\beta y, \beta y, \beta y).$$

Case 3: the $\mathbb{Z}_2$ case

Theorem 13 ($\mathbb{Z}_2$ case, Marimon and Pinsker 2024)

Let $\mathbb{Z}_2$ act freely on B with s-many orbits. Let f be an almost minimal operation above $\langle \mathbb{Z}_2 \rangle$. Then, f is one of

1. *a unary operation;*
2. *a G-quasi-minority;*
3. *an odd majority;*
4. *an odd Malcev, up to permuting variables;*
5. *a k-ary orbit-semiprojection for $2 \leq k \leq s$.*

An **odd majority** m is a quasi-majority such that for $\gamma \neq \mathrm{Id}$ in $\mathbb{Z}_2$,

$$m(y, x, \gamma x) \approx m(x, \gamma x, y) \approx m(x, y, \gamma x) \approx m(y, y, y).$$

An **odd Malcev** is a quasi-Malcev such that $M(x, \gamma y, z)$ is an odd majority.
Odd majorities and odd Malcev cannot be minimal!

A question of Bodirsky

As mentioned earlier, finding low arity essential polymorphisms is often helpful for arguments in infinite-domain CSPs.

Question 1 (Question 24 in Bodirsky 2021)

Suppose B is ω-categorical, $\overline{\mathrm{Aut}(B)} = \mathrm{End}(B)$, and $\mathrm{Pol}(B)$ has an essential polymorphism. Does $\mathrm{Pol}(B)$ have a binary essential polymorphism?

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For any oligomorphic $\mathrm{Aut}(B) \curvearrowright B$ with s orbits for $s \geq 3$, there is an ω-categorical structure B' such that

$$\mathrm{Pol}(B') := \overline{\langle \mathrm{Aut}(B) \cup \{f\} \rangle},$$

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However... there is a kernel of truth in the question...

Why easy problems lie above binary operations

Theorem 14 (Marimon and Pinsker 2024)

Suppose B is finite or ω-categorical, $\overline{\mathrm{Aut}(B)} = \mathrm{End}(B)$, and $\mathrm{Aut}(B) \curvearrowright B$ is not the free action of a Boolean group on B (always the case if B is ω-categorical). Suppose

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Then, $\mathrm{Pol}(B)$ contains a binary essential polymorphism.

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[5] $\xi : \mathcal{C} \to \mathcal{D}$ is **uniformly continuous** if there is some finite $B' \subseteq B$ such that $f_{\restriction B'} = g_{\restriction B'}$ implies $\xi(f) = \xi(g)$.

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- A uniformly continuous homomorphism to $\mathcal{P}_{\{0,1\}}$ implies $\mathrm{CSP}(B)$ is NP-hard;
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Proof idea.

Suppose by contrapositive that $\mathrm{Pol}(B) \cap \mathcal{O}^{(2)} = \overline{\langle \mathrm{Aut}(B) \rangle} \cap \mathcal{O}^{(2)}$.

Then, all ternary operations in $\mathcal{C}$ are almost minimal.

So $\mathrm{Pol}(B) \cap \mathcal{O}^{(3)}$ consists entirely of essentially unary operations and orbit-semiprojections.

These will only satisfy trivial identities, which is sufficient to build a uniformly continuous homomorphism to $\mathcal{P}_{\{0,1\}}$. □

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Indeed, we also have

Lemma 15 (Marimon and Pinsker 2024)

Let $G \curvearrowright B$.
Consider $\mathcal{OS} := \overline{\langle G \cup \{f | f \text{ is an orbit-semiprojection for } G \curvearrowright B\} \rangle}$.
There is a uniformly continuous homomorphism $\xi : \mathcal{OS} \to \mathcal{P}_{\{0,1\}}$.

Thank you!

Funding Statement: Funded by the European Union (ERC, POCOCOP, 101071674). Views and opinions expressed are however those of the authors only and do not necessarily reflect those of the European Union or the European Research Council Executive Agency. Neither the European Union nor the granting authority can be held responsible for them.

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Polymorphisms

Let B be a relational structure.

Definition 16 (Polymorphisms)

$f : B^n \to B$ is a **polymorphism** if it **preserves all relations** of B:

$$\begin{pmatrix} a_1^1 \\ \vdots \\ a_k^1 \end{pmatrix}, \ldots, \begin{pmatrix} a_1^n \\ \vdots \\ a_k^n \end{pmatrix} \in R^B \Rightarrow \begin{pmatrix} f(a_1^1, \ldots, a_1^n) \\ \vdots \\ f(a_k^1, \ldots, a_k^n) \end{pmatrix} \in R^B.$$

We call $\mathbf{Pol}(\boldsymbol{B})$ the set of polymorphisms of B.
The **polymorphism clone of** B.

- Unary polymorphism = homomorphism;
- **Projections** to one coordinate **are always polymorphisms**.

▸ Back to main presentation

Constraint Satisfaction Problems

$\tau =$ finite relational language.

Definition 17 ($\mathrm{CSP}(B)$)

Let B be a **fixed** structure.
$\mathrm{CSP}(B)$ is the following computational problem:

- **INPUT**: A **finite** τ-structure A;
- **OUTPUT**: Is there a homomorphism $A \to B$?

- B is finite $\Rightarrow \mathrm{CSP}(B)$ is in NP;
- The **computational complexity** of $\mathrm{CSP}(B)$ in a finite or ω-categorical setting is determined by identities satisfied by $\mathrm{Pol}(B)$.

Examples of CSPs

Example 18 (n-colorability for graphs)

Let K_n be the complete graph on n verteces. Then,

- $\mathrm{CSP}(K_n) = n$-colorability problem for graphs;
- NP-complete for $n > 2$ and in P for $n = 2$ (Karp 1972).

Example 19 (digraph acyclicity)

Consider $(\mathbb{Q}, <)$. Then,

- $\mathrm{CSP}(\mathbb{Q}, <) =$ digraph acyclicity, i.e.
  INPUT: a finite directed graph D;
  OUTPUT Does D contain a finite directed cycle?
- In P (Kahn 1962).

Back to main presentation