Minimal operations over permutation groups

Paolo Marimon   Michael Pinsker

TU Wien

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Outline

Operation clones and the pointwise convergence topology

Definition 1 (Operation clone)

Let B be a (possibly infinite) set, $\mathcal{O}^{(n)} = B^{B^n}$ and $\mathcal{O} := \bigcup_{n\in\mathbb{N}} \mathcal{O}^{(n)}$.
We call $\mathcal{C} \subseteq \mathcal{O}$ an **operation clone** over B if

- $\mathcal{C}$ contains all projections;
- $\mathcal{C}$ is closed under composition.

We equip $\mathcal{O}$ with the **pointwise convergence topology**: For $\mathcal{S} \subseteq \mathcal{O}$, $f \in \overline{\mathcal{S}} \Leftrightarrow$ for all $A \subseteq B$ finite there is $g \in \mathcal{S}$ such that $g_{\restriction A} = f_{\restriction A}$.

$\overline{\langle \mathcal{S} \rangle}$ denotes the smallest closed clone containing $\mathcal{S}$.

There is a correspondence between:

- closed subclones of $\mathcal{O}$;
- polymorphism clones of relational structures on B.

Minimal clones and operations I

Let $\mathcal{D} \supsetneq \mathcal{C}$ be closed subclones of $\mathcal{O}$.

Definition 2 (almost minimal)

The k-ary operation $f \in \mathcal{D} \setminus \mathcal{C}$ is **almost minimal above** $\mathcal{C}$ if for each $r < k$,

$$\overline{\langle \mathcal{C} \cup \{f\} \rangle} \cap \mathcal{O}^{(r)} = \mathcal{C} \cap \mathcal{O}^{(r)}.$$

Definition 3 (Minimal operation)

$f \in \mathcal{D} \setminus \mathcal{C}$ is **minimal above** $\mathcal{C}$ if it almost minimal above $\mathcal{C}$, and everything new that it generates with $\mathcal{C}$ also generates f with $\mathcal{C}$:

$$h \in \overline{\langle \mathcal{C} \cup \{f\} \rangle} \setminus \mathcal{C} \Rightarrow f \in \overline{\langle \mathcal{C} \cup \{h\} \rangle}\ .$$

Minimal clones and operations II

Definition 4

$\mathcal{D}$ is **minimal above** $\mathcal{C}$ if there is no closed clone $\mathcal{E}$ such that $\mathcal{C} \subsetneq \mathcal{E} \subsetneq \mathcal{D}$.

- There is a correspondence between
  - minimal closed clones above $\mathcal{C}$;
  - clones $\overline{\langle \mathcal{C} \cup \{f\} \rangle}$ for f minimal above $\mathcal{C}$.
- For $\mathcal{C} \subsetneq \mathcal{D}$, there is always $f \in \mathcal{D}$ **almost minimal** above $\mathcal{C}$;
- For the structures B we are most interested in[1], $\mathrm{Pol}(B)$ always has a **minimal** operation above $\overline{\langle \mathrm{Aut}(B) \rangle}$;

[1] i.e. B is finite or ω-categorical in a finite relational language.

Some terminology for operations

We define some operations in virtue of the identities they satisfy:

- **ternary quasi-majority:**

$$m(x,x,y) \approx m(x,y,x) \approx m(y,x,x) \approx m(x,x,x);$$

- **quasi-Malcev:**

$$M(y,y,x) \approx M(x,y,y) \approx M(x,x,x);$$

- For f **idempotent**, i.e. $f(x,\ldots,x) \approx x$, remove the 'quasi';
- **ternary minority:**

$$\mathfrak{m}(y,y,x) \approx \mathfrak{m}(y,x,y) \approx \mathfrak{m}(x,y,y) \approx \mathfrak{m}(x,x,x) \approx x;$$

- **semiprojection**: a k-ary f such that there is an $i \in \{1,\ldots,k\}$ such that whenever $|\{a_1,\ldots,a_k\}| < k$,

$$f(a_1,\ldots,a_k) = a_i.$$

Rosenberg's five types theorem

Theorem 5 (Five types theorem, Rosenberg 1986)

Let B be finite and f be minimal above $\langle 1 \rangle$. Then, f is of one of the following five types:

1. *a unary function;*
2. *a binary function;*
3. *a ternary majority operation;*
4. *a ternary minority operation;*
5. *a k-ary semiprojection for some $k \geq 3$.*

Moreover, the ternary minority corresponds to $x + y + z$ in some Boolean group.

A group is **Boolean** if every element has order 2.

Our project

For $G \curvearrowright B$ a non-trivial group G acting faithfully on B,
we want to classify the minimal operations lying above $\overline{\langle G \rangle}$.

STRATEGY: Classify almost minimal operations first.

The classification of almost minimal operations splits into three cases:

- G is not a Boolean group acting freely on B;
- G is a Boolean group acting freely on B and $\neq \mathbb{Z}_2$;
- $\mathbb{Z}_2$ acting freely on B.

For minimal operations, the last two cases can be treated together.

Classifying almost minimal operations I

Theorem 6 (Three Types Theorem, Marimon and Pinsker 2024)

Let B be a ~~finite~~ set. Let $G \curvearrowright B$ be such that G is not a Boolean group acting freely on B. Let f be almost minimal above $\overline{\langle G \rangle}$. Then, f is one of:

1. *a unary function;*
2. *a binary function;*
3. *a k-ary orbit-semiprojection for $3 \leq k \leq s$, where $s = |\mathrm{Orb}(G)|$.*

f is an **orbit-semiprojection** if there is an $i \in \{1, \ldots, k\}$ and a unary operation $g \in \overline{G}$ such that **whenever at least two of the a_j lie in the same orbit,**

$$f(a_1, \ldots, a_k) = g(a_i).$$

Classifying almost minimal operations II

Theorem 7 (Boolean case, Marimon and Pinsker 2024)

Let $G \curvearrowright B$ be a Boolean group acting freely on B with s-many orbits and $|G| > 2$. Let f be an almost minimal operation above $\langle G \rangle$. Then, f is of one of the following types:

1. *f is unary;*
2. *f is binary;*
3. *f is a ternary twisted minority;*
4. *f is a k-ary orbit-semiprojection for $3 \leq k \leq s$.*

A **twisted minority** is a ternary operation such that for all $\beta \in G$,

$$\mathfrak{m}(y, x, \beta x) \approx \mathfrak{m}(x, \beta x, y) \approx \mathfrak{m}(x, y, \beta x) \approx \mathfrak{m}(\beta y, \beta y, \beta y).$$

Classifying almost minimal operations III

Theorem 8 ($\mathbb{Z}_2$ case, Marimon and Pinsker 2024)

Let $\mathbb{Z}_2$ act freely on B with s-many orbits. Let f be an almost minimal operation above $\langle \mathbb{Z}_2 \rangle$. Then, f is of one of the following types:

1. *f is unary;*
2. *f is a ternary twisted minority;*
3. *f is an odd majority;*
4. *f is, up to permuting its variables, an odd Malcev;*
5. *f is a k-ary orbit-semiprojection for $2 \leq k \leq s$.*

An **odd majority** m is a quasi-majority such that for γ the non-identity element in $\mathbb{Z}_2$,

$$m(y, x, \gamma x) \approx m(x, \gamma x, y) \approx m(x, y, \gamma x) \approx m(y, y, y).$$

An **odd Malcev** is a quasi-Malcev such that $M(x, \gamma y, z)$ is an odd majority.

Minimal operations

- **Orbit semiprojections** only generate other orbit semiprojections, so for $|\mathrm{Orb}(G)| \geq 3$, they always exist as minimal;[2]
- **Odd majorities** and **odd Malcevs** are **NEVER** minimal;
- **Minimal twisted minorities** are particularly well-behaved:
  - $\mathfrak{m}(x, y, z) = w$ is a symmetric 4-ary relation;
  - For $\alpha, \beta, \gamma \in G, \mathfrak{m}(\alpha x, \beta y, \gamma z) \approx \alpha\beta\gamma\mathfrak{m}(x, y, z)$;
  - $\mathfrak{m}$ induces on the orbits of G a minority of the form $\mathfrak{m}^\star(x, y, z) = x + y + z$ in a Boolean group;
  - They exist above $\langle G \rangle$ if and only if $|\mathrm{Orb}(G)| = 2^n$ or is infinite.

[2] At least in the contexts that interest us.

Applications: finding low arity essential polymorphisms I

We want to study $\mathrm{CSP}(B)$ for B countably infinite and $\mathrm{Aut}(B) \curvearrowright B$ **oligomorphic**: $\mathrm{Aut}(B) \curvearrowright B^n$ has finitely many orbits for all n.
We call these structures **ω-categorical**.

Sufficient to work with a **model complete core**: $\overline{\mathrm{Aut}(B)} = \mathrm{End}(B)$.

It is often helpful to find low arity essential polymorphisms:

Definition 9 (Essentially unary and essential operations)

f is **essentially unary** if there is unary g and $1 \leq i \leq k$ such that

$$f(x_1, \ldots, x_k) \approx g(x_i).$$

Otherwise, f is **essential**.

Applications: finding low arity essential polymorphisms II

For B ω-categorical, people developed techniques to find binary essential polymorphisms given an essential one (cf. Bodirsky and Kára 2008, and Mottet and Pinsker 2022).

Question 1 (Question 24 in Bodirsky 2021)

Does an ω-categorical model complete core with an essential polymorphism also have a binary essential polymorphism?

Answer:

- **NO**: whenever $\mathrm{Aut}(B)$ has ≥ 3 orbits, $\overline{\langle \mathrm{Aut}(B) \cup \{f\} \rangle}$ for f an orbit semiprojection is a counterexample;
- However, this is true on ≤ 2 orbits;
- Moreover, this is true when $\mathrm{CSP}(B)$ is not hard...

Applications: finding low arity essential polymorphisms III

Theorem 10 (Marimon and Pinsker 2024)

Suppose B *is a finite or* ω*-categorical model complete core and* $\mathrm{Aut}(B) \curvearrowright B$ *is not the free action of a Boolean group on* B *(always the case if* B *is* ω*-categorical).*

($\star$) *Suppose* $\mathrm{Pol}(B)$ *does not have a uniformly continuous* $h1$*-homomorphism to* Proj *(the clone of projections on* $\{0, 1\}$*);*

Then, $\mathrm{Pol}(B)$ *contains a binary essential polymorphism.*

- $\neg(\star) \Rightarrow \mathrm{CSP}(B)$ is NP hard;
- Proof uses the Three Types Theorem (for almost minimal operations).

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