# EPPA numbers of graphs

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TU Dresden

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David Bradley-Williams, Peter J. Cameron, Jan Hubička, and MK: EPPA numbers of graphs (arXiv:2311.07995)

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### Definition

A structure **A** is homogeneous if every partial automorphism of **A** with finite domain extends to an automorphism of **A**.

## Definition (EPPA, extension property for partial automorphisms)

Let **B** be a structure and let **A** be its **induced** substructure. **B** is an EPPA-witness for **A** if every partial automorphism of **A** extends to an automorphism of **B**.

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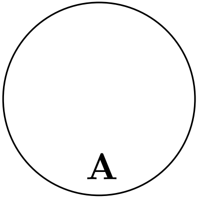

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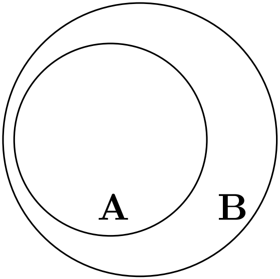

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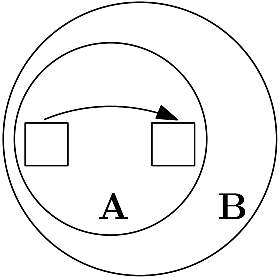

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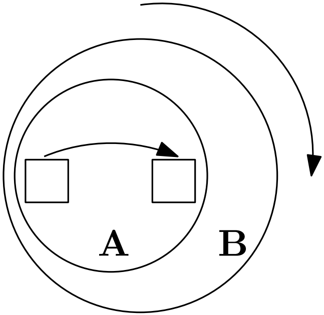

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### Theorem (Hrushovski, 1992)

*The class of all finite graphs has EPPA.*

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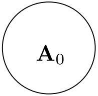

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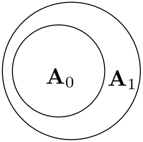

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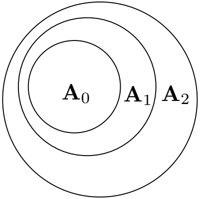

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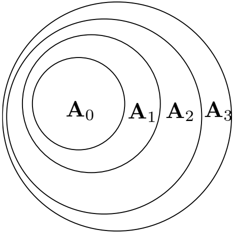

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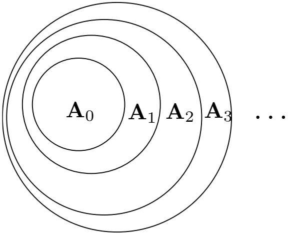

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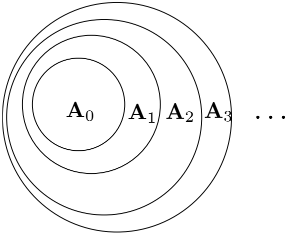


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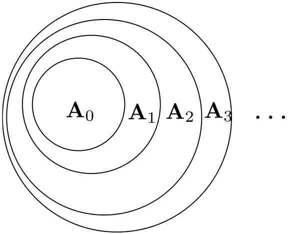


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**Theorem [Kechris, Rosendal, 2007]:** *The class of all finite substructures of a homogeneous structure* **M** *has EPPA if and only if* $\mathrm{Aut}(\mathbf{M})$ *can be written as the closure of a chain of compact subgroups.*

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- complements thereof,
- subgraphs of the finite homogeneous graphs [Gardiner, 1976].

# Examples

- Graphs [Hrushovski, 1992], $K_n$-free graphs [Herwig, 1998]
- Relational structures (with forbidden cliques) [Herwig, 1995], [Hodkinson, Otto, 2003]
- Metric spaces [Solecki, 2005; Vershik, 2008], also [Conant, 2019]
- Two-graphs [Evans, Hubička, K, Nešetřil, 2018]
- Metrically homogeneous graphs [AB-WHKKKP, 2017], [K, 2019]
- Generalised metric spaces [Hubička, K, Nešetřil, 2019+]
- $n$-partite and semigeneric tournaments [Hubička, Jahel, K, Sabok, 2024]
- Groups [Siniora, 2017]
- ...

Question (Herwig, Lascar, 2000)

*Do finite tournaments have EPPA?*

# EPPA numbers of graphs

Given a graph $\mathbf{G}$, let $\mathrm{eppa}(\mathbf{G})$ be the least number of vertices of an EPPA-witness for $\mathbf{G}$. Put $\mathrm{eppa}(n) = \max\{\mathrm{eppa}(\mathbf{G}) : |\mathbf{G}| = n\}$.

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Problem (Hrushovski, 1992)

*Improve the bounds.*

### Theorem (Herwig, Lascar, 2000)

*For every* $\mathbf{G}$ *with* $n$ *vertices and maximum degree* $\Delta$ *we have that* $\mathrm{eppa}(\mathbf{G}) \leq \binom{\Delta n}{\Delta} \in n^{\mathcal{O}(n)}$.
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Independently proved also by Andréka and Németi.

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Proof.

1. Let $\mathbf{G} = (V, E)$ be a graph. Assume that $\mathbf{G}$ is $\Delta$-regular.
2. Define $\mathbf{H}$ so that $V(\mathbf{H}) = \binom{E}{\Delta}$ and $XY \in E(\mathbf{H})$ if $X \cap Y \neq \emptyset$.
3. Embed $\psi\colon \mathbf{G} \to \mathbf{H}$ sending $v \mapsto \{e \in E : v \in e\}$.
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For non-regular graphs, add "half-edges" to make them regular.

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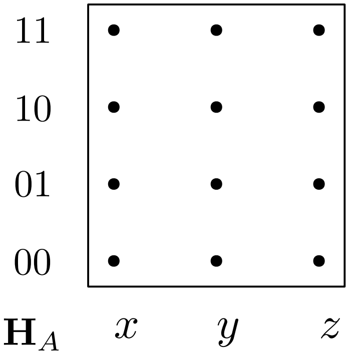

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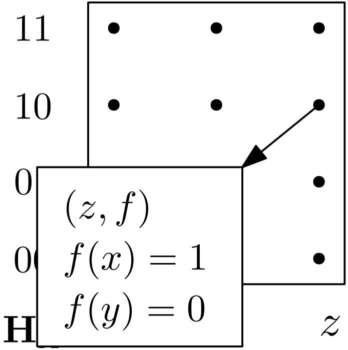

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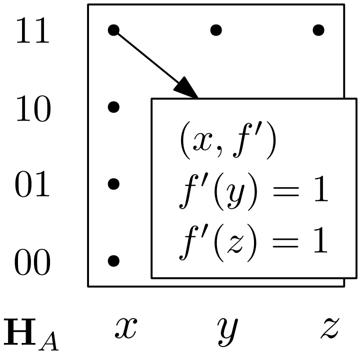

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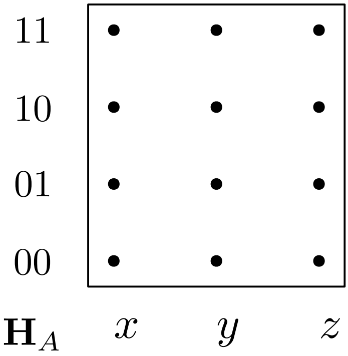

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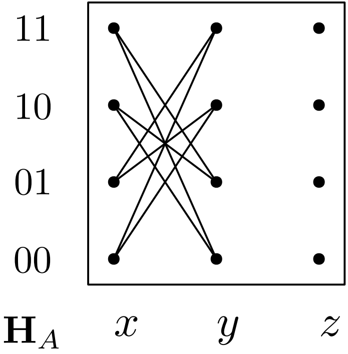

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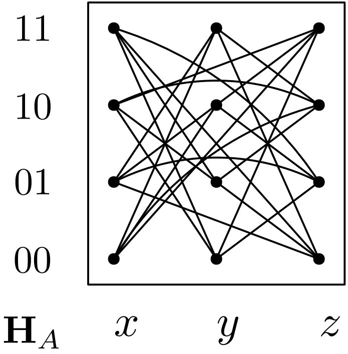

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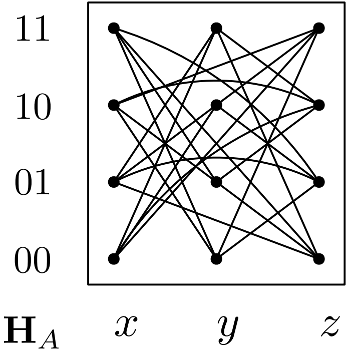

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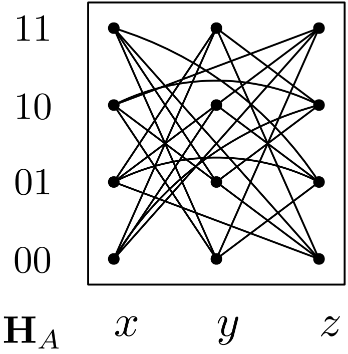

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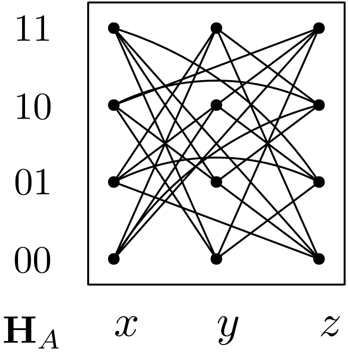

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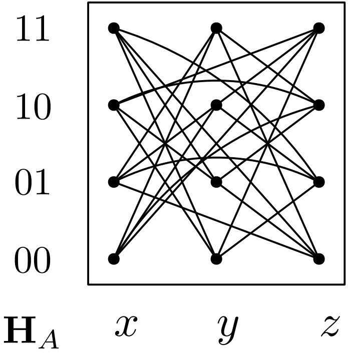

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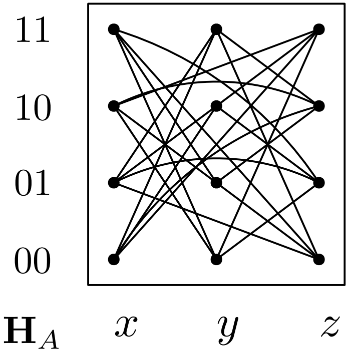

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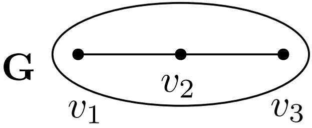

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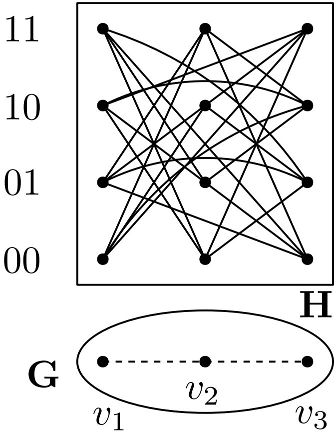

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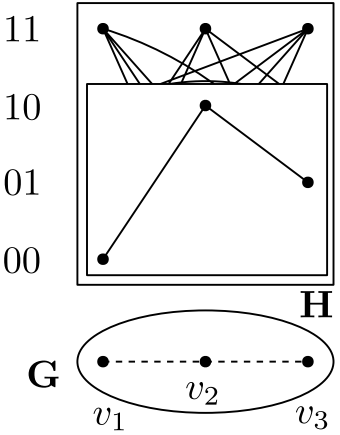

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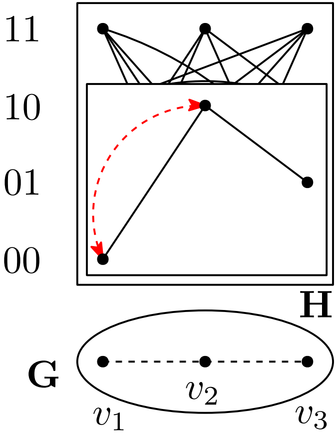

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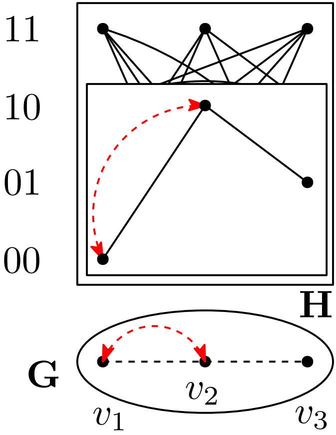

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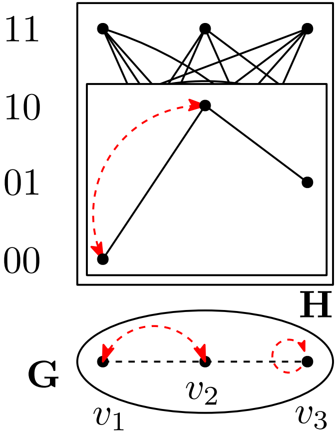

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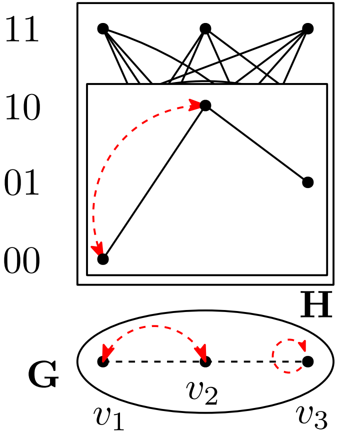

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*There is* **G** *such that every EPPA-witness for* **G** *has at least* $\Omega(2^n/\sqrt{n})$ *vertices. Consequently,* $\mathrm{eppa}(n) \geq \Omega(2^n/\sqrt{n})$.

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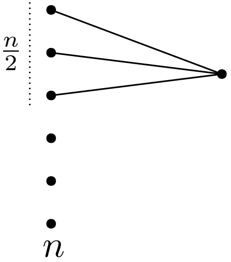


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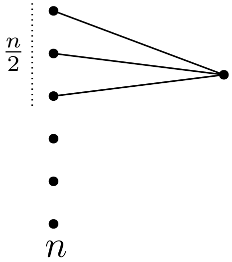


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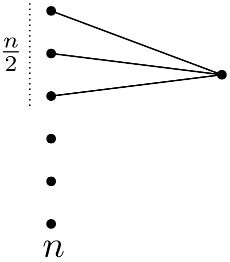


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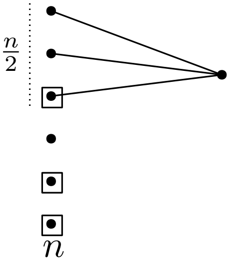


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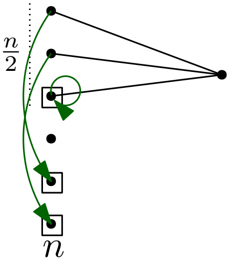


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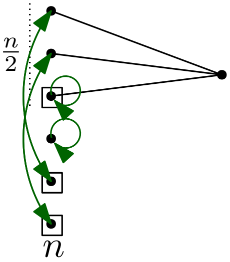


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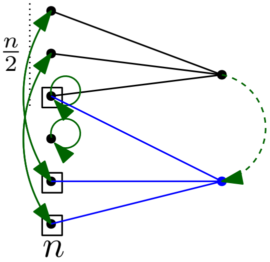


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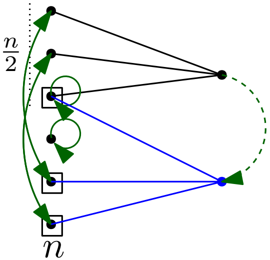


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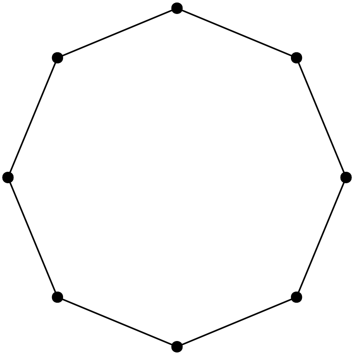

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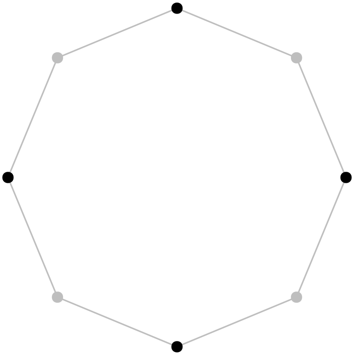

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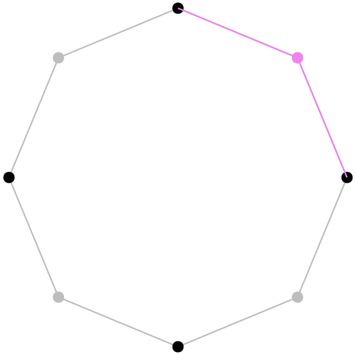

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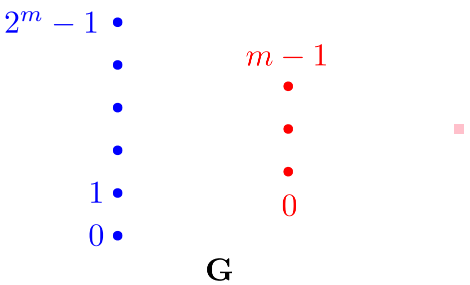

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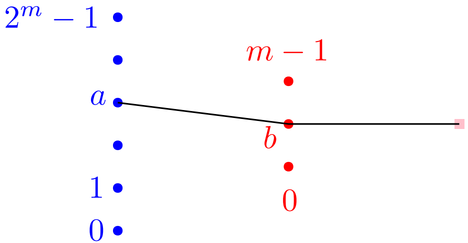


**G**

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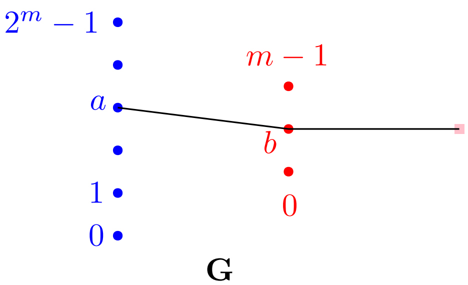


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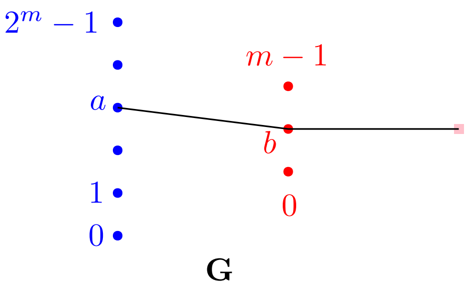


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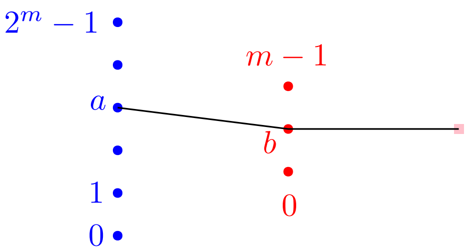


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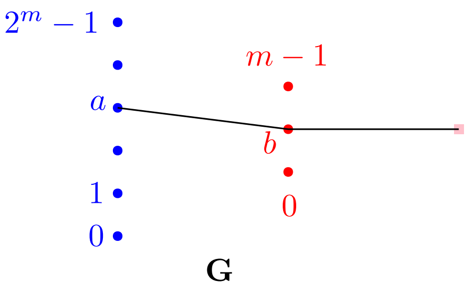


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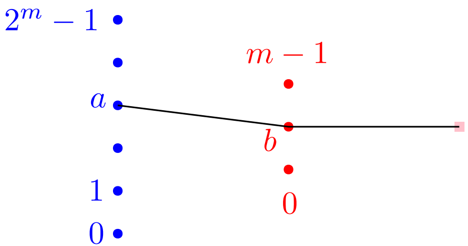


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**(Answers?)**

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