EPPA numbers of graphs

Matěj Konečný

TU Dresden

MCW 2024

David Bradley-Williams, Peter J. Cameron, Jan Hubička, and MK: EPPA numbers of graphs (arXiv:2311.07995)

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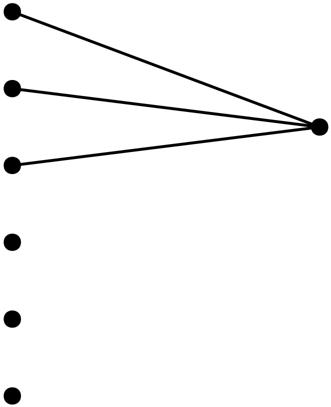

Midsummer Combinatorial Workshop XXV

August 3 - August 7, 2020, Prague

Programme (tentative)

All lectures take place in room S5 at the building of Charles University, Malostranske namesti 25.

		Monday
Morning		Registration
9:00-9:20		Welcome, brief information about program of the workshop
9:25-9:55	Peter Cameron	Lockdown theorem
10:00-10:30		Coffee break
10:30-11:00	Pavel Patak	Better bounds for abstract Radon theorem
11:10-11:40	Vaclav Rozhon	Deterministic network decomposition
11:50-12:20	Jan Hubicka	Big Ramsey degrees using parameter spaces
		Thursday
9:00-9:30	Antoine Mottet	Cores over Ramsey structures
9:30-10:00	Carl Feghali	Revisiting a theorem of Talbot
10:00-10:30		Coffee break
10:30-11:00	Denys Bulavka	Optimal bounds for the colorful fractional Helly theorem
11:10-11:40	Johanna Wiehe	The chromatic polynomial of a digraph
11:50-12:20	Matej Konecny	On EPPA numbers of graphs
Evening		Picnic?
		Friday
9:30-10:00	Peter Zeman	Testing isomorphism of circular-arc graphs

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Definition

A graph **G** is homogeneous if every partial automorphism of **G** with finite domain extends to an automorphism of **G**.

Finite homogeneous graphs [Gardiner, 1976]

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Definition (EPPA, extension property for partial automorphisms)

Let **B** be a graph and let **A** be its **induced** subgraph. **B** is an EPPA-witness for **A** if every partial automorphism of **A** extends to an automorphism of **B**.

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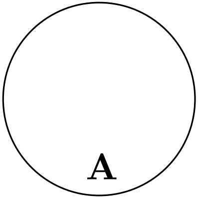

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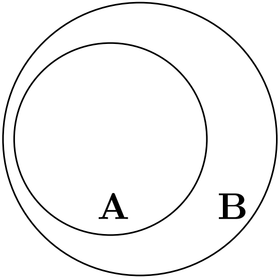

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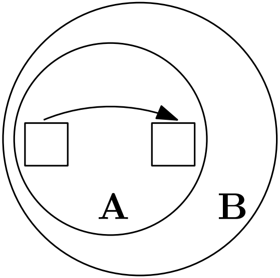

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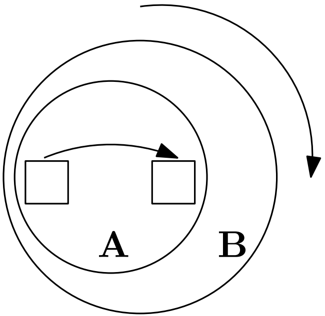

EPPA numbers

Definition

$$\mathrm{eppa}(\mathbf{A}) = \min\{|\mathbf{B}| : \mathbf{B} \text{ is an EPPA-witness for } \mathbf{A}\}.$$

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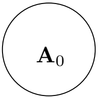
Problem (Hrushovski, 1992)

Improve the bounds.

A connection to model theory

Pick a graph $\mathbf{A}_0$.



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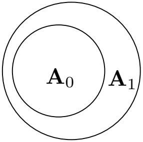

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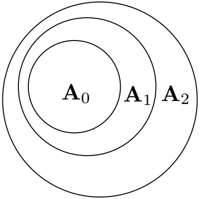

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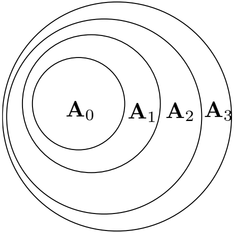

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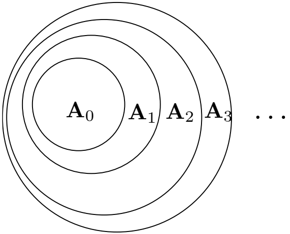

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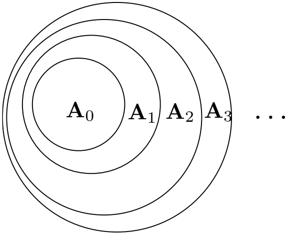

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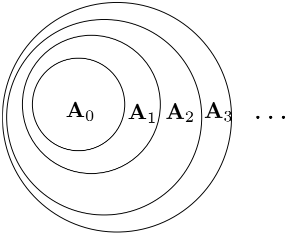

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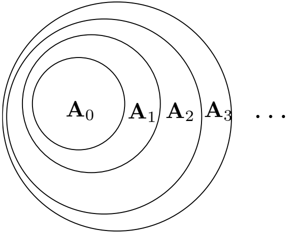

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Theorem [Hodges, Hodkinson, Lascar, Shelah, 1993]: *The countable random graph has the small index property.*
Theorem [Kechris, Rosendal, 2007]: *The class of all finite substructures of a homogeneous structure* $\mathbf{M}$ *has EPPA if and only if* $\mathrm{Aut}(\mathbf{M})$ *can be written as the closure of a chain of compact subgroups.*

Theorem (Herwig, Lascar, 2000)

For every $\mathbf{G}$ *with* n *vertices,* m *edges and maximum degree* Δ *we have that* $\mathrm{eppa}(\mathbf{G}) \leq \binom{\Delta n - m}{\Delta} \in n^{\mathcal{O}(n)}$.
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Independently proved also by Andréka and Németi.

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Proof.

1. Let $\mathbf{G} = (V, E)$ be a graph. Assume that $\mathbf{G}$ is Δ-regular.
2. Define $\mathbf{H}$ so that $V(\mathbf{H}) = \binom{E}{\Delta}$ and $XY \in E(\mathbf{H})$ if $X \cap Y \neq \emptyset$.
3. Embed $\psi\colon \mathbf{G} \to \mathbf{H}$ sending $v \mapsto \{e \in E : v \in e\}$.
4. A partial automorphism of $\mathbf{G}$ gives a partial permutation of E.
5. Extend it to a permutation of E respecting the partial automorphism.
6. Every permutation of E induces an automorphism of $\mathbf{H}$. $\square$

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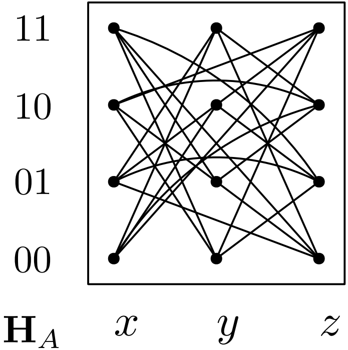
For non-regular graphs, add "half-edges" to make them regular.

An upper bound [Evans, Hubička, K, Nešetřil, 2021]

Given set A, define graph $\mathbf{H}_A$.
$H_A = \{(x, f) : x \in A, f : A \setminus \{x\} \to \{0, 1\}\}$.
$\{(x, f), (y, g)\} \in E \iff x \neq y$ and $f(y) \neq g(x)$.

1. For a permutation $\pi : A \to A$ define $\alpha_\pi : H_n \to H_n$ by $\alpha_\pi((x, f)) = (\pi(x), g)$, where $g(y) = f(\pi^{-1}(y))$.
2. $\alpha_\pi \in \mathrm{Aut}(\mathbf{H}_A)$.
3. For $x \neq y \in A$ define α_{xy} by $\alpha_{xy}((z, f)) = (z, g)$ where $g(w) = 1 - f(w)$ if $\{x, y\} = \{z, w\}$ and $g(w) = f(w)$ otherwise.
4. $\alpha_{xy} \in \mathrm{Aut}(\mathbf{H}_A)$.



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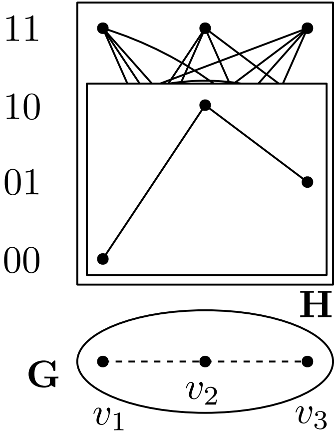
5. Fix a graph $\mathbf{G}$ and consider $\mathbf{H}_G$.
6. Embed $\mathbf{G}$ to $\mathbf{H}_G$ vertex-by-vertex, preserving projections.
7. Pick a partial automorphism f of $\mathbf{G}$, project it to G, and extend it to a permutation π of G.
8. Consider α_π. There is a canonical choice of $\alpha_{x_i y_i}$'s such that $\alpha_\pi \circ \alpha_{x_1 y_1} \circ \cdots \circ \alpha_{x_k y_k}$ extends f.



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3. Valuation graphs ($n2^{n-1}$).

A lower bound

Observation (Bradley-Williams, Cameron, Hubička, K, 2023)

$$\mathrm{eppa}(n) \geq \Omega(2^n/\sqrt{n}).$$

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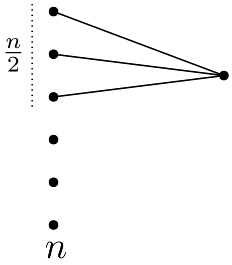

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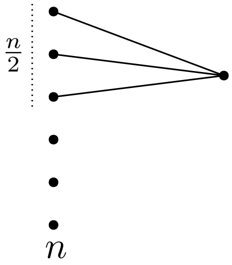

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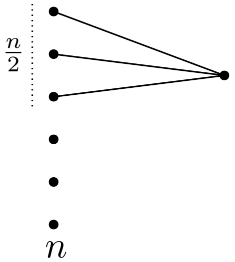
- **Claim:** In every EPPA-witness, for every $S \in \binom{[n]}{n/2}$, there is a vertex connected to S and not to $[n] \setminus S$.



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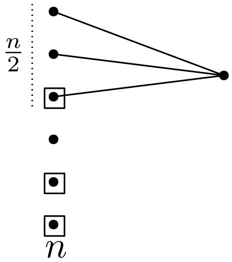

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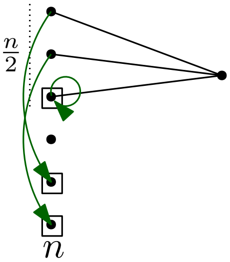

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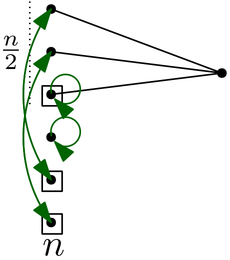

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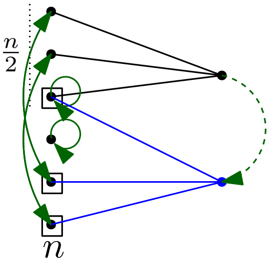

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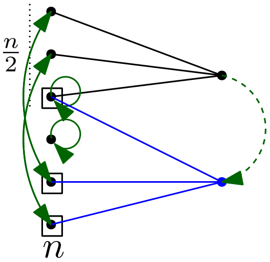
- $\mathrm{eppa}(\mathbf{G}) \geq \binom{n}{n/2} \in \Omega(2^n/\sqrt{n})$.



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If $\mathbf{G}$ *contains an independent set* I *and a vertex connected to exactly* k *members of* I *then* $\mathrm{eppa}(\mathbf{G}) \geq \binom{|I|}{k}$.

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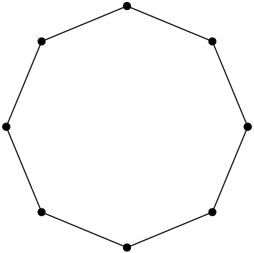
Cycles have quadratic EPPA numbers.



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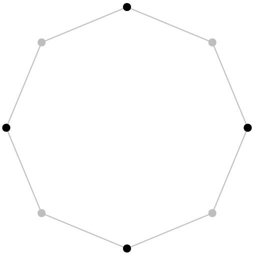

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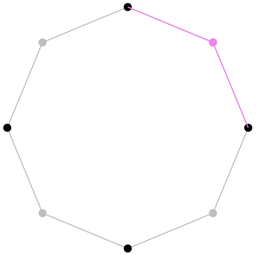

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(Note that there are only $2^{O(n \log n)}$ partial automorphisms of any n-vertex structure.)

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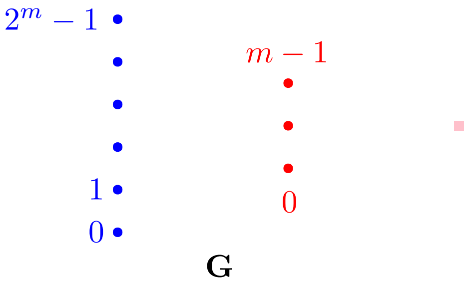

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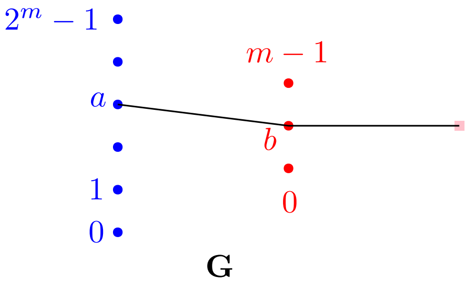

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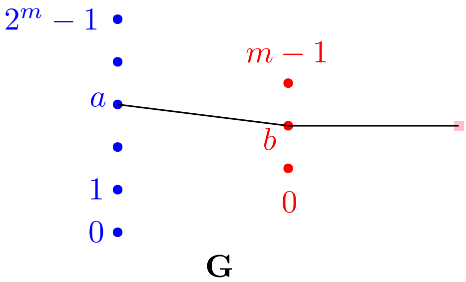

- $ab\square$ is a hyperedge $\iff$ the b-th bit of a is 1.
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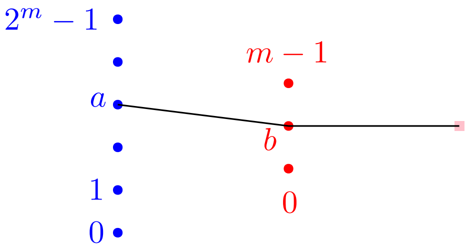

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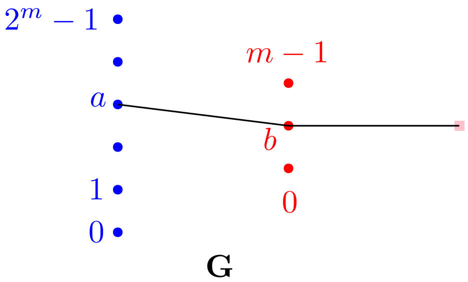

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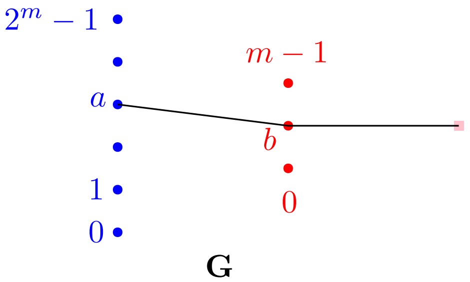

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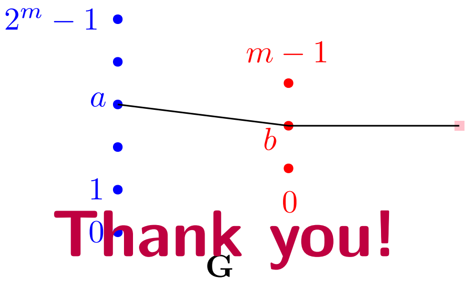

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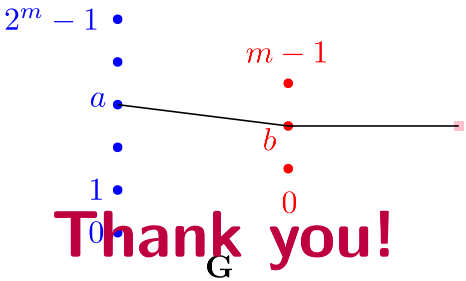

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