Extending partial automorphisms

Matěj Konečný

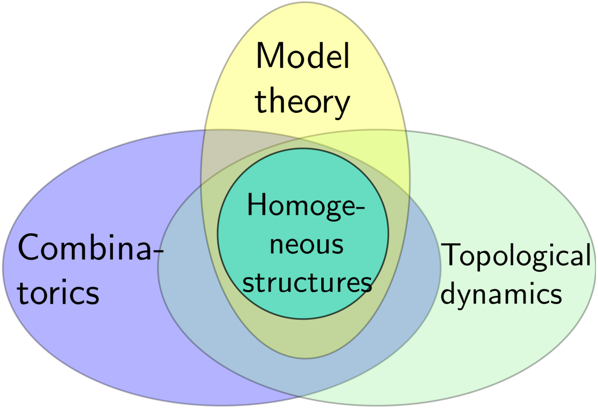
TU Dresden

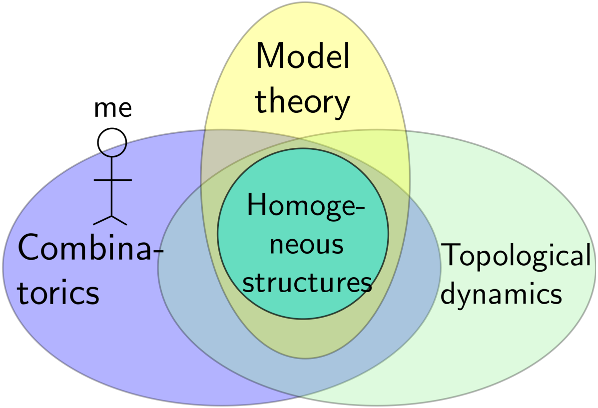
AGK, 4. 1. 2024

Funded by the European Union (project POCOCOP, ERC Synergy grant No. 101071674). Views and opinions expressed are however those of the author only and do not necessarily reflect those of the European Union or the European Research Council Executive Agency. Neither the European Union nor the granting authority can be held responsible for them.

Funded by the European Union

European Research Council
Established by the European Commission


Model theory
Combinatorics
Homogeneous structures
Topological dynamics


Model
theory
me
Homoge-
neous
structures
Combina-
torics
Topological
dynamics

ω

$\mathrm{Sym}(\omega)$

$\mathrm{Sym}(\omega) \subseteq_{\mathrm{cl}} \omega^\omega$

$\mathrm{Sym}(\omega) \subseteq_{\mathrm{cl}} \omega^\omega$

$\mathrm{Sym}(\omega)$ with composition is a Polish group.

$\mathrm{Sym}(\omega) \subseteq_{\mathrm{cl}} \omega^\omega$

$\mathrm{Sym}(\omega)$ with composition is a Polish group.

Pointwise stabilisers of finite sets form a system of neighbourhoods of the identity.

Theorem (Truss, 1992)

$\mathrm{Sym}(\omega)$ *has a comeagre (complement of a ctable union of nowhere dense sets) conjugacy class (orbit of the action* $\mathrm{Sym}(\omega) \curvearrowright \mathrm{Sym}(\omega)$ *with* $g \cdot x = g^{-1}xg$*).*

The elements of this class have infinitely many cycles of every finite length and no infinite cycles.

Theorem (Truss, 1992)

$\mathrm{Sym}(\omega)$ *has a comeagre (complement of a ctable union of nowhere dense sets) conjugacy class (orbit of the action* $\mathrm{Sym}(\omega) \curvearrowright \mathrm{Sym}(\omega)$ *with* $g \cdot x = g^{-1}xg$*).*

The elements of this class have infinitely many cycles of every finite length and no infinite cycles.

We call these the generic elements of $\mathrm{Sym}(\omega)$.

Theorem (Truss, 1992)

$\mathrm{Sym}(\omega)$ *has a comeagre (complement of a ctable union of nowhere dense sets) conjugacy class (orbit of the action* $\mathrm{Sym}(\omega) \curvearrowright \mathrm{Sym}(\omega)$ *with* $g \cdot x = g^{-1}xg$*).*
The elements of this class have infinitely many cycles of every finite length and no infinite cycles.

We call these the generic elements of $\mathrm{Sym}(\omega)$.

Fact

If a Polish group G has a comeagre conjugacy class D then $G = D^2$, and every element of G is a commutator. If G is uncountable then it has no proper normal subgroups of countable index.

Definition (Hodges, Hodkinson, Lascar, Shelah, 1993)

A Polish group G has *n*-generic elements if $G \curvearrowright G^n$ has a comeagre orbit for

$$g \cdot (x_1, \ldots, x_n) = (g^{-1} x_1 g, \ldots, g^{-1} x_n g).$$

Definition (Hodges, Hodkinson, Lascar, Shelah, 1993)

A Polish group G has *n-generic elements* if $G \curvearrowright G^n$ has a comeagre orbit for

$$g \cdot (x_1, \dots, x_n) = (g^{-1} x_1 g, \dots, g^{-1} x_n g).$$

It has *ample generics* if it has n-generic elements for every $n \geq 1$.

Definition (Hodges, Hodkinson, Lascar, Shelah, 1993)

A Polish group G has n-generic elements if $G \curvearrowright G^n$ has a comeagre orbit for

$$g \cdot (x_1, \ldots, x_n) = (g^{-1} x_1 g, \ldots, g^{-1} x_n g).$$

It has ample generics if it has n-generic elements for every $n \geq 1$.

Theorem (Kechris, Rosendal, 2006)

Let G be a Polish group with ample generics. Then G has the small index property (i.e. all subgroups of index $< 2^{\aleph_0}$ are open).

Definition (Hodges, Hodkinson, Lascar, Shelah, 1993)

A Polish group G has *n-generic elements* if $G \curvearrowright G^n$ has a comeagre orbit for

$$g \cdot (x_1, \ldots, x_n) = (g^{-1}x_1g, \ldots, g^{-1}x_ng).$$

It has ample generics if it has n-generic elements for every $n \geq 1$.

Theorem (Kechris, Rosendal, 2006)

Let G be a Polish group with ample generics. Then G has the small index property (i.e. all subgroups of index $< 2^{\aleph_0}$ are open).

If moreover G is an oligomorphic closed subgroup of $\mathrm{Sym}(\omega)$ then it has uncountable cofinality (i.e. G cannot be written as the union of a countable chain of its proper subgroups), 21-Bergman property and property (FE).

Definition (Hodges, Hodkinson, Lascar, Shelah, 1993)

A Polish group G has *n-generic elements* if $G \curvearrowright G^n$ has a comeagre orbit for

$$g \cdot (x_1, \ldots, x_n) = (g^{-1} x_1 g, \ldots, g^{-1} x_n g).$$

It has ample generics if it has n-generic elements for every $n \geq 1$.

Theorem (Kechris, Rosendal, 2006)

Let G be a Polish group with ample generics. Then G has the small index property (i.e. all subgroups of index $< 2^{\aleph_0}$ are open).

If moreover G is an oligomorphic closed subgroup of $\mathrm{Sym}(\omega)$ then it has uncountable cofinality (i.e. G cannot be written as the union of a countable chain of its proper subgroups), 21-Bergman property and property (FE).

Theorem (Maybe HHLS, 1993?)

$\mathrm{Sym}(\omega)$ *has ample generics.*

What about other topological groups?

What about other topological groups?

Fact

Let **M** *be a relational structure with vertex set* ω. *Then* $\mathrm{Aut}(\mathbf{M})$ *is a closed subgroup of* $\mathrm{Sym}(\omega)$.

What about other topological groups?

Fact

Let **M** *be a relational structure with vertex set* ω*. Then* $\mathrm{Aut}(\mathbf{M})$ *is a closed subgroup of* $\mathrm{Sym}(\omega)$*.*

Conversely, let G *be a closed subgroup of* $\mathrm{Sym}(\omega)$*. Then there is a relational structure* **M** *with vertex set* ω *such that* $G = \mathrm{Aut}(\mathbf{M})$*.*

What about other topological groups?

Fact

Let **M** *be a relational structure with vertex set* ω*. Then* $\mathrm{Aut}(\mathbf{M})$ *is a closed subgroup of* $\mathrm{Sym}(\omega)$*.*
Conversely, let G *be a closed subgroup of* $\mathrm{Sym}(\omega)$*. Then there is a relational structure* **M** *with vertex set* ω *such that* $G = \mathrm{Aut}(\mathbf{M})$*.*
In fact, **M** *can be chosen to be homogeneous.*

Homogeneous structures

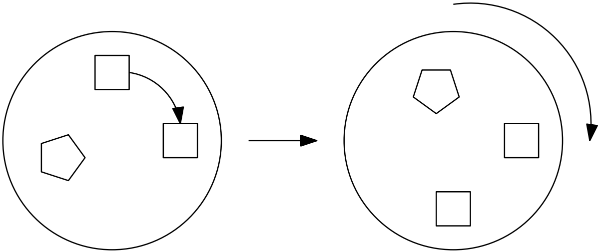
Let **A** be a structure. A partial function $f\colon A \to A$ is a partial automorphism of **A** if it is an isomorphism of $\mathrm{Dom}(f)$ and $\mathrm{Range}(f)$.

Homogeneous structures

Let $\mathbf{A}$ be a structure. A partial function $f: A \to A$ is a partial automorphism of $\mathbf{A}$ if it is an isomorphism of $\mathrm{Dom}(f)$ and $\mathrm{Range}(f)$.
We say that f extends to $\alpha \in \mathrm{Aut}(\mathbf{A})$ if $f \subseteq \alpha$.

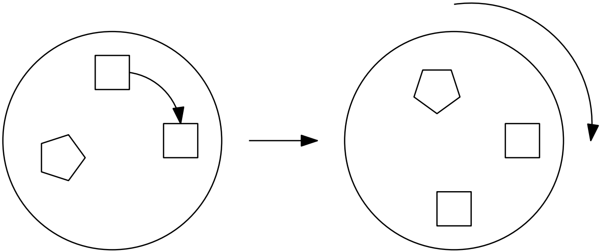
Homogeneous structures

Let **A** be a structure. A partial function $f: A \to A$ is a partial automorphism of **A** if it is an isomorphism of $\mathrm{Dom}(f)$ and $\mathrm{Range}(f)$.
We say that f extends to $\alpha \in \mathrm{Aut}(\mathbf{A})$ if $f \subseteq \alpha$.
A is homogeneous if every partial automorphism of **A** with finite domain can be extended to an automorphism of **A**.



Homogeneous structures

Let **A** be a structure. A partial function $f : A \to A$ is a partial automorphism of **A** if it is an isomorphism of $\mathrm{Dom}(f)$ and $\mathrm{Range}(f)$.
We say that f extends to $\alpha \in \mathrm{Aut}(\mathbf{A})$ if $f \subseteq \alpha$.
A is homogeneous if every partial automorphism of **A** with finite domain can be extended to an automorphism of **A**.



Observation
The set of all partial automorphisms of a fixed structure forms an inverse monoid.

Fraïssé's theorem (1950's)

Let $\mathbf{M}$ be a countable homogeneous relational structure. Let $\mathrm{Age}(\mathbf{M})$ be the class of all finite structures which embed into $\mathbf{M}$.

Fraïssé's theorem (1950's)

Let $\mathbf{M}$ be a countable homogeneous relational structure. Let $\mathrm{Age}(\mathbf{M})$ be the class of all finite structures which embed into $\mathbf{M}$. Then $\mathrm{Age}(\mathbf{M})$ has the joint embedding property (JEP) and the amalgamation property (AP).

Fraïssé's theorem (1950's)

Let **M** be a countable homogeneous relational structure. Let $\mathrm{Age}(\mathbf{M})$ be the class of all finite structures which embed into **M**. Then $\mathrm{Age}(\mathbf{M})$ has the joint embedding property (JEP) and the amalgamation property (AP).

Conversely, if $\mathcal{C}$ is a hereditary isomorphism-closed class of finite structures with JEP and AP such that it has only countably many members up to isomorphism then there is a homogeneous structure **M** such that $\mathcal{C} = \mathrm{Age}(\mathbf{M})$.

Fraïssé's theorem (1950's)

Let **M** be a countable homogeneous relational structure. Let $\mathrm{Age}(\mathbf{M})$ be the class of all finite structures which embed into **M**. Then $\mathrm{Age}(\mathbf{M})$ has the joint embedding property (JEP) and the amalgamation property (AP).

Conversely, if $\mathcal{C}$ is a hereditary isomorphism-closed class of finite structures with JEP and AP such that it has only countably many members up to isomorphism then there is a homogeneous structure **M** such that $\mathcal{C} = \mathrm{Age}(\mathbf{M})$. We call this **M** the Fraïssé limit of $\mathcal{C}$ and it is unique up to isomorphism.

Examples

By [Gardiner, 1976] and [Lachlan, Woodrow, 1980], the countable homogeneous graphs are the following:

1. C_5
2. $L(K_{3,3})$
3. Disjoint unions of cliques of the same size (finite or infinite)
4. The countable random graph ($\iff$ all finite graphs)
5. The K_n-free Henson graphs ($\iff$ all finite K_n-free graphs)
6. Complements thereof

Examples II.

- Finite linear orders $\iff$ $(\mathbb{Q}, \leq)$
- Finite sets $\iff$ the countable set
- Finite k-uniform hypergraphs $\iff$ the countable random k-uniform hypergraph
- Finite boolean algebras $\iff$ the countable atomless BA
- Finite tournaments $\iff$ the countable homogeneous tournament
- Finite metric spaces $\iff$ the Urysohn space
- Finite groups $\iff$ Hall's universal locally finite group

Proving ample generics

Proving ample generics

Theorem (Kechris, Rosendal, 2006)

Let $\mathbf{M}$ *be a homogeneous structure. If* $\mathrm{Age}(\mathbf{M})$ *has APA and EPPA then* $\mathrm{Aut}(\mathbf{M})$ *has ample generics.*

EPPA

Definition (EPPA, extension property for partial automorphisms)

Let $\mathbf{A} \subseteq \mathbf{B}$ be finite structures. $\mathbf{B}$ is an EPPA-witness for $\mathbf{A}$ if every partial automorphism of $\mathbf{A}$ extends to an automorphism of $\mathbf{B}$.

EPPA

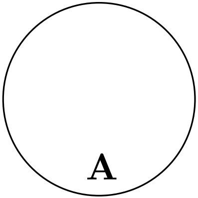
Definition (EPPA, extension property for partial automorphisms)

Let $\mathbf{A} \subseteq \mathbf{B}$ be finite structures. $\mathbf{B}$ is an EPPA-witness for $\mathbf{A}$ if every partial automorphism of $\mathbf{A}$ extends to an automorphism of $\mathbf{B}$.
A class $\mathcal{C}$ of finite structures has EPPA if for every $\mathbf{A} \in \mathcal{C}$ there is $\mathbf{B} \in \mathcal{C}$, which is an EPPA-witness for $\mathbf{A}$.

EPPA

Definition (EPPA, extension property for partial automorphisms)

Let $\mathbf{A} \subseteq \mathbf{B}$ be finite structures. $\mathbf{B}$ is an EPPA-witness for $\mathbf{A}$ if every partial automorphism of $\mathbf{A}$ extends to an automorphism of $\mathbf{B}$. A class $\mathcal{C}$ of finite structures has EPPA if for every $\mathbf{A} \in \mathcal{C}$ there is $\mathbf{B} \in \mathcal{C}$, which is an EPPA-witness for $\mathbf{A}$.

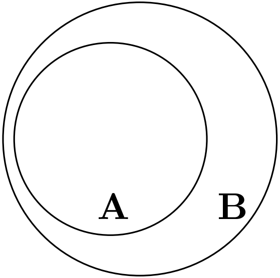


EPPA

Definition (EPPA, extension property for partial automorphisms)

Let $\mathbf{A} \subseteq \mathbf{B}$ be finite structures. $\mathbf{B}$ is an EPPA-witness for $\mathbf{A}$ if every partial automorphism of $\mathbf{A}$ extends to an automorphism of $\mathbf{B}$.

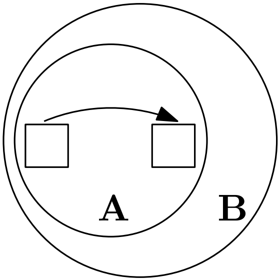
A class $\mathcal{C}$ of finite structures has EPPA if for every $\mathbf{A} \in \mathcal{C}$ there is $\mathbf{B} \in \mathcal{C}$, which is an EPPA-witness for $\mathbf{A}$.



EPPA

Definition (EPPA, extension property for partial automorphisms)

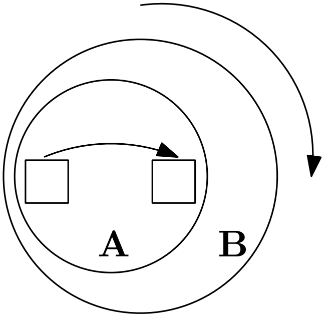
Let $\mathbf{A} \subseteq \mathbf{B}$ be finite structures. $\mathbf{B}$ is an EPPA-witness for $\mathbf{A}$ if every partial automorphism of $\mathbf{A}$ extends to an automorphism of $\mathbf{B}$.
A class $\mathcal{C}$ of finite structures has EPPA if for every $\mathbf{A} \in \mathcal{C}$ there is $\mathbf{B} \in \mathcal{C}$, which is an EPPA-witness for $\mathbf{A}$.



EPPA

Definition (EPPA, extension property for partial automorphisms)

Let $\mathbf{A} \subseteq \mathbf{B}$ be finite structures. $\mathbf{B}$ is an EPPA-witness for $\mathbf{A}$ if every partial automorphism of $\mathbf{A}$ extends to an automorphism of $\mathbf{B}$.
A class $\mathcal{C}$ of finite structures has EPPA if for every $\mathbf{A} \in \mathcal{C}$ there is $\mathbf{B} \in \mathcal{C}$, which is an EPPA-witness for $\mathbf{A}$.



EPPA

Definition (EPPA, extension property for partial automorphisms)

Let $\mathbf{A} \subseteq \mathbf{B}$ be finite structures. $\mathbf{B}$ is an EPPA-witness for $\mathbf{A}$ if every partial automorphism of $\mathbf{A}$ extends to an automorphism of $\mathbf{B}$. A class $\mathcal{C}$ of finite structures has EPPA if for every $\mathbf{A} \in \mathcal{C}$ there is $\mathbf{B} \in \mathcal{C}$, which is an EPPA-witness for $\mathbf{A}$.

Theorem (Hrushovski, 1992)

The class of all finite graphs has EPPA.

(Non-)examples

(Non-)examples

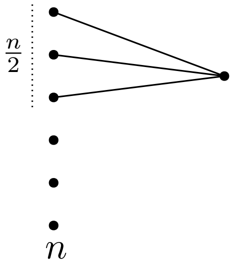
- Every finite set is an EPPA-witness for itself.

(Non-)examples

- Every finite set is an EPPA-witness for itself.
- Finite linear orders do not have EPPA.

(Non-)examples

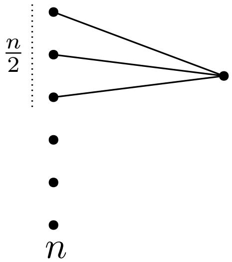
- Every finite set is an EPPA-witness for itself.
- Finite linear orders do not have EPPA.
- [Bradley-Williams, Cameron, Hubička, K, 2023] Every EPPA-witness of this graph **G** has at least $\Omega(2^n/\sqrt{n})$ vertices:



(Non-)examples

- Every finite set is an EPPA-witness for itself.
- Finite linear orders do not have EPPA.
- [Bradley-Williams, Cameron, Hubička, K, 2023] Every EPPA-witness of this graph **G** has at least $\Omega(2^n/\sqrt{n})$ vertices:

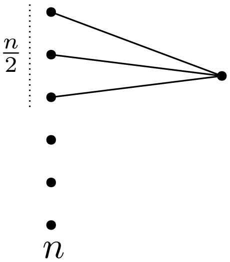
Every permutation of the left part is a partial automorphism of **G**.



(Non-)examples

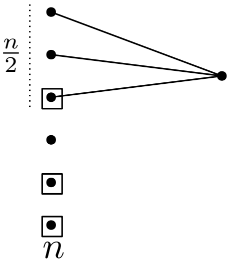
- Every finite set is an EPPA-witness for itself.
- Finite linear orders do not have EPPA.
- [Bradley-Williams, Cameron, Hubička, K, 2023] Every EPPA-witness of this graph **G** has at least $\Omega(2^n/\sqrt{n})$ vertices:

- Every permutation of the left part is a partial automorphism of **G**.
- **Claim:** In every EPPA-witness, for every $S \in \binom{[n]}{n/2}$, there is a vertex connected to S and not to $[n] \setminus S$.



(Non-)examples

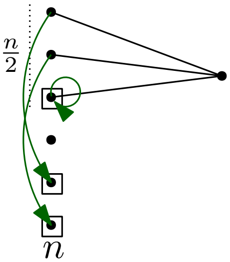
- Every finite set is an EPPA-witness for itself.
- Finite linear orders do not have EPPA.
- [Bradley-Williams, Cameron, Hubička, K, 2023] Every EPPA-witness of this graph **G** has at least $\Omega(2^n/\sqrt{n})$ vertices:



- Every permutation of the left part is a partial automorphism of **G**.
- **Claim:** In every EPPA-witness, for every $S \in \binom{[n]}{n/2}$, there is a vertex connected to S and not to $[n] \setminus S$.
- Pick arbitrary $S \in \binom{[n]}{n/2}$.

(Non-)examples

- Every finite set is an EPPA-witness for itself.
- Finite linear orders do not have EPPA.
- [Bradley-Williams, Cameron, Hubička, K, 2023] Every EPPA-witness of this graph **G** has at least $\Omega(2^n/\sqrt{n})$ vertices:

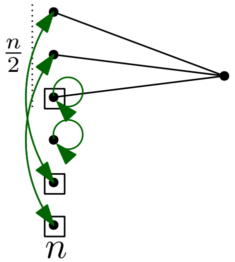


- Every permutation of the left part is a partial automorphism of **G**.
- **Claim:** In every EPPA-witness, for every $S \in \binom{[n]}{n/2}$, there is a vertex connected to S and not to $[n] \setminus S$.
- Pick arbitrary $S \in \binom{[n]}{n/2}$.

(Non-)examples

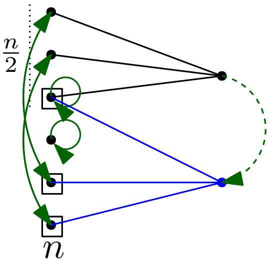
- Every finite set is an EPPA-witness for itself.
- Finite linear orders do not have EPPA.
- [Bradley-Williams, Cameron, Hubička, K, 2023] Every EPPA-witness of this graph **G** has at least $\Omega(2^n/\sqrt{n})$ vertices:

- Every permutation of the left part is a partial automorphism of **G**.
- **Claim:** In every EPPA-witness, for every $S \in \binom{[n]}{n/2}$, there is a vertex connected to S and not to $[n] \setminus S$.
- Pick arbitrary $S \in \binom{[n]}{n/2}$.



(Non-)examples

- Every finite set is an EPPA-witness for itself.
- Finite linear orders do not have EPPA.
- [Bradley-Williams, Cameron, Hubička, K, 2023] Every EPPA-witness of this graph **G** has at least $\Omega(2^n/\sqrt{n})$ vertices:



- Every permutation of the left part is a partial automorphism of **G**.
- **Claim:** In every EPPA-witness, for every $S \in \binom{[n]}{n/2}$, there is a vertex connected to S and not to $[n] \setminus S$.
- Pick arbitrary $S \in \binom{[n]}{n/2}$.

Theorem (Herwig, Lascar, 2000)

If the maximum degree of $\mathbf{G}$ *is* Δ*, then it has an EPPA-witness on at most* $\binom{\Delta n}{\Delta}$ *vertices.*

Theorem (Herwig, Lascar, 2000)

If the maximum degree of $\mathbf{G}$ *is* Δ*, then it has an EPPA-witness on at most* $\binom{\Delta n}{\Delta}$ *vertices.*

Proof.

1. Let $\mathbf{G} = (V, E)$ be a graph. Assume that $\mathbf{G}$ is Δ-regular.
2. Define $\mathbf{H}$ so that $V(\mathbf{H}) = \binom{E}{\Delta}$ and $XY \in E(\mathbf{H})$ if $X \cap Y \neq \emptyset$.
3. Embed $\psi\colon \mathbf{G} \to \mathbf{H}$ sending $v \mapsto \{e \in E : v \in e\}$.
4. A partial automorphism of $\mathbf{G}$ gives a partial permutation of E.
5. Extend it to a permutation of E respecting the partial automorphism.
6. Every permutation of E induces an automorphism of $\mathbf{H}$. $\square$

Theorem (Herwig, Lascar, 2000)

If the maximum degree of $\mathbf{G}$ *is* Δ*, then it has an EPPA-witness on at most* $\binom{\Delta n}{\Delta}$ *vertices.*

Proof.

1. Let $\mathbf{G} = (V, E)$ be a graph. Assume that $\mathbf{G}$ is Δ-regular.
2. Define $\mathbf{H}$ so that $V(\mathbf{H}) = \binom{E}{\Delta}$ and $XY \in E(\mathbf{H})$ if $X \cap Y \neq \emptyset$.
3. Embed $\psi\colon \mathbf{G} \to \mathbf{H}$ sending $v \mapsto \{e \in E : v \in e\}$.
4. A partial automorphism of $\mathbf{G}$ gives a partial permutation of E.
5. Extend it to a permutation of E respecting the partial automorphism.
6. Every permutation of E induces an automorphism of $\mathbf{H}$. $\square$

For non-regular graphs, add "half-edges" to make them regular.

Theorem (Herwig, Lascar, 2000)

If the maximum degree of $\mathbf{G}$ *is* Δ*, then it has an EPPA-witness on at most* $\binom{\Delta n}{\Delta}$ *vertices.*

Proof.

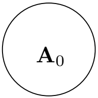
1. Let $\mathbf{G} = (V, E)$ be a graph. Assume that $\mathbf{G}$ is Δ-regular.
2. Define $\mathbf{H}$ so that $V(\mathbf{H}) = \binom{E}{\Delta}$ and $XY \in E(\mathbf{H})$ if $X \cap Y \neq \emptyset$.
3. Embed $\psi\colon \mathbf{G} \to \mathbf{H}$ sending $v \mapsto \{e \in E : v \in e\}$.
4. A partial automorphism of $\mathbf{G}$ gives a partial permutation of E.
5. Extend it to a permutation of E respecting the partial automorphism.
6. Every permutation of E induces an automorphism of $\mathbf{H}$. $\square$

For non-regular graphs, add "half-edges" to make them regular.

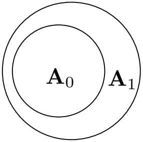
[Evans, Hubička, K, Nešetřil, 2021]: Every graph on n vertices has an EPPA-witness on $n2^{n-1}$ vertices.

Suppose that a class $\mathcal{C}$ of L-structures has EPPA.

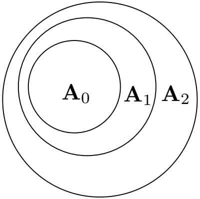
Suppose that a class $\mathcal{C}$ of L-structures has EPPA. Pick $\mathbf{A}_0 \in \mathcal{C}$



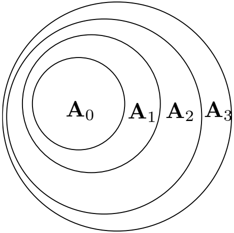
Suppose that a class $\mathcal{C}$ of L-structures has EPPA. Pick $\mathbf{A}_0 \in \mathcal{C}$



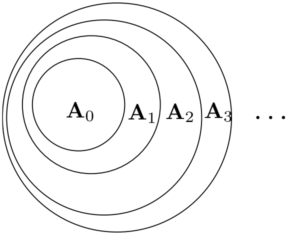
Suppose that a class $\mathcal{C}$ of L-structures has EPPA. Pick $\mathbf{A}_0 \in \mathcal{C}$



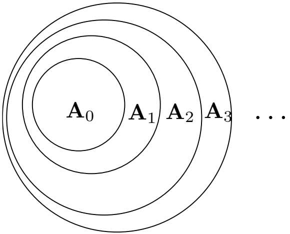
Suppose that a class $\mathcal{C}$ of L-structures has EPPA. Pick $\mathbf{A}_0 \in \mathcal{C}$



Suppose that a class $\mathcal{C}$ of L-structures has EPPA. Pick $\mathbf{A}_0 \in \mathcal{C}$

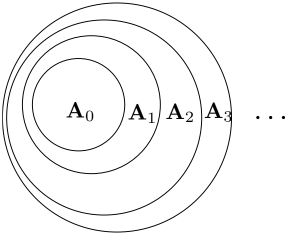


Suppose that a class $\mathcal{C}$ of L-structures has EPPA. Pick $\mathbf{A}_0 \in \mathcal{C}$



Let **M** be the union of the chain. **M** is homogeneous.

Suppose that a class $\mathcal{C}$ of L-structures has EPPA. Pick $\mathbf{A}_0 \in \mathcal{C}$



Let $\mathbf{M}$ be the union of the chain. $\mathbf{M}$ is homogeneous.

Theorem (Kechris–Rosendal, 2007)

The class of all substructures of a homogeneous structure $\mathbf{M}$ *has EPPA if and only if* $\mathrm{Aut}(\mathbf{M})$ *can be written as the closure of a chain of compact subgroups.*

Which classes have EPPA'?

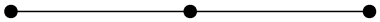
- Graphs [Hrushovski, 1992], K_n-free graphs [Herwig, 1998]
- Relational structures (with forbidden cliques) [Herwig, 2000], [Hodkinson–Otto, 2003]
- Metric spaces [Solecki, 2005; Vershik, 2008], also [Hubička–K–Nešetřil, 2019]
- Two-graphs [Evans–Hubička–K–Nešetřil, 2020]
- Metrically homogeneous graphs [AB-WHKKKP, 2017], [K, 2020]
- Generalised metric spaces [Hubička–K–Nešetřil, 2019+]
- n-partite and semigeneric tournaments [Hubička–Jahel–K–Sabok, 2024+]
- Groups [Siniora, 2017]
- ...

Which classes have EPPA'?

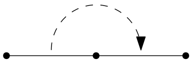
- Graphs [Hrushovski, 1992], K_n-free graphs [Herwig, 1998]
- Relational structures (with forbidden cliques) [Herwig, 2000], [Hodkinson–Otto, 2003]
- Metric spaces [Solecki, 2005; Vershik, 2008], also [Hubička–K–Nešetřil, 2019]
- Two-graphs [Evans–Hubička–K–Nešetřil, 2020]
- Metrically homogeneous graphs [AB-WHKKKP, 2017], [K, 2020]
- Generalised metric spaces [Hubička–K–Nešetřil, 2019+]
- n-partite and semigeneric tournaments [Hubička–Jahel–K–Sabok, 2024+]
- Groups [Siniora, 2017]
- ...

Except for two-graphs, all these examples admit ample generics.

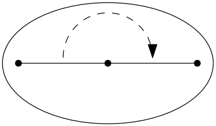
Proving EPPA?



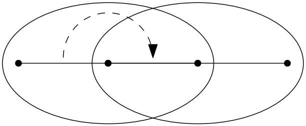
Proving EPPA?



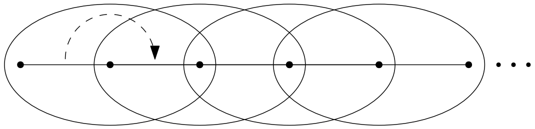
Proving EPPA?



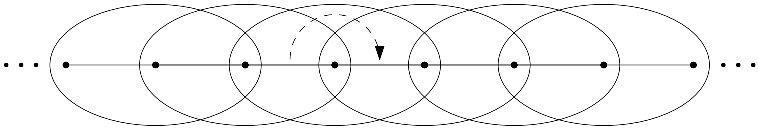
Proving EPPA?



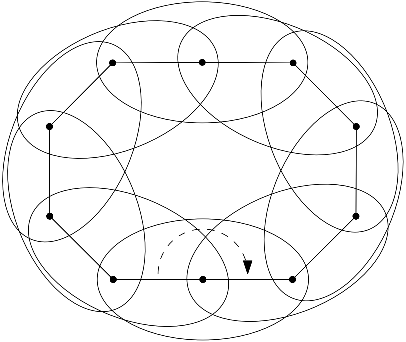
Proving EPPA?



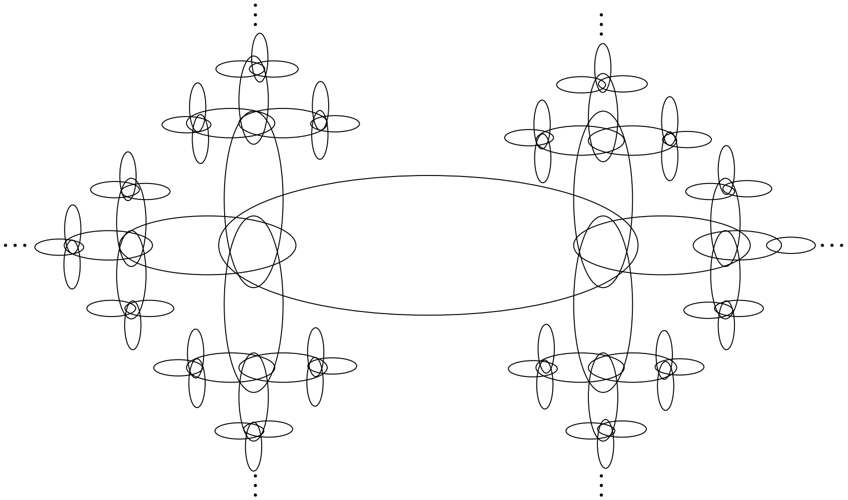
Proving EPPA?



Proving EPPA?



Multiple partial automorphisms are a different beast



Profinite topology

For a group G, the profinite topology on G is given by the following basis of open sets:

$$\{gH : g \in G, H \leq G, [G : H] < \omega\}.$$

Profinite topology

For a group G, the profinite topology on G is given by the following basis of open sets:

$$\{gH : g \in G, H \leq G, [G : H] < \omega\}.$$

Cf. Hall's theorem, Ribes–Zalesskii theorem, Mackey's construction.

Profinite topology

For a group G, the profinite topology on G is given by the following basis of open sets:

$$\{gH : g \in G, H \leq G, [G : H] < \omega\}.$$

Cf. Hall's theorem, Ribes–Zalesskii theorem, Mackey's construction.
The pro-odd (or odd-adic) topology on G:

$$\{gH : g \in G, H \leq G, [G : H] \text{ is odd}\}.$$

Profinite topology

For a group G, the profinite topology on G is given by the following basis of open sets:

$$\{gH : g \in G, H \leq G, [G : H] < \omega\}.$$

Cf. Hall's theorem, Ribes–Zalesskii theorem, Mackey's construction.
The pro-odd (or odd-adic) topology on G:

$$\{gH : g \in G, H \leq G, [G : H] \text{ is odd}\}.$$

Theorem (Herwig, Lascar, 2000)

The class of finite tournaments has EPPA $\iff$ for every $n \geq 2$, a finitely generated $H \leq F_n$ is pro-odd-closed if and only if

$$H = \sqrt{H} = \{a \in F_n : a^2 \in H\}.$$

Profinite topology

For a group G, the profinite topology on G is given by the following basis of open sets:

$$\{gH : g \in G, H \leq G, [G : H] < \omega\}.$$

Cf. Hall's theorem, Ribes–Zalesskii theorem, Mackey's construction.
The pro-odd (or odd-adic) topology on G:

$$\{gH : g \in G, H \leq G, [G : H] \text{ is odd}\}.$$

Theorem (Herwig, Lascar, 2000)

The class of finite tournaments has EPPA $\iff$ for every $n \geq 2$, a finitely generated $H \leq F_n$ is pro-odd-closed if and only if

$$H = \sqrt{H} = \{a \in F_n : a^2 \in H\}.$$

Question (Herwig, Lascar, 2000)

Do finite tournaments have EPPA?

Theorem (Hubička–K–Nešetřil 2022)

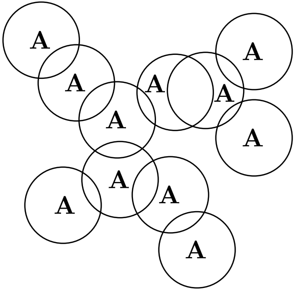
Let L be a language where all functions are unary. Given a finite L-structure $\mathbf{A}$ and $n \geq 1$, there is a finite L-structure $\mathbf{B}$ satisfying the following:

1. $\mathbf{B}$ *is an EPPA-witness for* $\mathbf{A}$.
2. *Every irreducible substructure of* $\mathbf{B}$ *embeds into* $\mathbf{A}$.
3. *Every substructure of* $\mathbf{B}$ *on at most n vertices is a substructure of* _{a blowup of} *a tree amalgamation of copies of* $\mathbf{A}$.

Theorem (Hubička–K–Nešetřil 2022)

Let L be a language where all functions are unary. Given a finite L-structure $\mathbf{A}$ and $n \geq 1$, there is a finite L-structure $\mathbf{B}$ satisfying the following:

1. $\mathbf{B}$ *is an EPPA-witness for* $\mathbf{A}$.
2. *Every irreducible substructure of* $\mathbf{B}$ *embeds into* $\mathbf{A}$.
3. *Every substructure of* $\mathbf{B}$ *on at most n vertices is a substructure of* a blowup of *a tree amalgamation of copies of* $\mathbf{A}$.



Corollary (Solecki, 2005; Vershik, 2008)

Finite metric spaces have EPPA.

Corollary (Solecki, 2005; Vershik, 2008)

Finite metric spaces have EPPA.

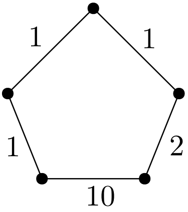
Proof (sketch).

Corollary (Solecki, 2005; Vershik, 2008)

Finite metric spaces have EPPA.

Proof (sketch).

- A finite edge-labelled graph **A** has a homomorphism to a metric space if and only if no non-metric cycle has a homomorphism to **A**.



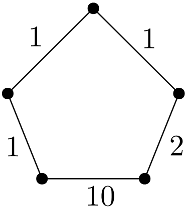
Corollary (Solecki, 2005; Vershik, 2008)

Finite metric spaces have EPPA.

Proof (sketch).

- A finite edge-labelled graph **A** has a homomorphism to a metric space if and only if no non-metric cycle has a homomorphism to **A**.

CSP(Urysohn)!!!!



Corollary (Solecki, 2005; Vershik, 2008)

Finite metric spaces have EPPA.

Proof (sketch).

- A finite edge-labelled graph $\mathbf{A}$ has a homomorphism to a metric space if and only if no non-metric cycle has a homomorphism to $\mathbf{A}$.
- Let $S \subseteq \mathbb{R}^+$ be the (finite) set of distances in $\mathbf{A}$. There are finitely many non-metric S-labelled cycles.

Corollary (Solecki, 2005; Vershik, 2008)

Finite metric spaces have EPPA.

Proof (sketch).

- A finite edge-labelled graph $\mathbf{A}$ has a homomorphism to a metric space if and only if no non-metric cycle has a homomorphism to $\mathbf{A}$.
- Let $S \subseteq \mathbb{R}^+$ be the (finite) set of distances in $\mathbf{A}$. There are finitely many non-metric S-labelled cycles. Let n be the size of the largest one.

Corollary (Solecki, 2005; Vershik, 2008)

Finite metric spaces have EPPA.

Proof (sketch).

- A finite edge-labelled graph **A** has a homomorphism to a metric space if and only if no non-metric cycle has a homomorphism to **A**.
- Let $S \subseteq \mathbb{R}^+$ be the (finite) set of distances in **A**. There are finitely many non-metric S-labelled cycles. Let n be the size of the largest one.
- Use [HKN2022] to get **B**. It is an S-edge-labelled graph with no non-metric cycles on $\leq n$ vertices.

Corollary (Solecki, 2005; Vershik, 2008)

Finite metric spaces have EPPA.

Proof (sketch).

- A finite edge-labelled graph **A** has a homomorphism to a metric space if and only if no non-metric cycle has a homomorphism to **A**.
- Let $S \subseteq \mathbb{R}^+$ be the (finite) set of distances in **A**. There are finitely many non-metric S-labelled cycles. Let n be the size of the largest one.
- Use [HKN2022] to get **B**. It is an S-edge-labelled graph with no non-metric cycles on $\leq n$ vertices.
- Compute shortest path distances in **B**.

□

Corollary (Solecki, 2005; Vershik, 2008)

Finite metric spaces have EPPA.

Proof (sketch).

- A finite edge-labelled graph **A** has a homomorphism to a metric space if and only if no non-metric cycle has a homomorphism to **A**.
- Let $S \subseteq \mathbb{R}^+$ be the (finite) set of distances in **A**. There are finitely many non-metric S-labelled cycles. Let n be the size of the largest one.
- Use [HKN2022] to get **B**. It is an S-edge-labelled graph with no non-metric cycles on $\leq n$ vertices.
- Compute shortest path distances in **B**.

□

Theorem (Rosendal, 2011)

EPPA for metric spaces $\iff$ Ribes–Zalesskii theorem (if G is a countable discrete group and $H_1, \ldots, H_n \leq G$ f. g. then $H_1 H_2 \cdots H_n$ is profinite-closed in G).

Do I have time?

Coherent EPPA

An EPPA-witness **B** for **A** is coherent if the map from partial automorphisms to their extensions respects composition.

Coherent EPPA

An EPPA-witness **B** for **A** is coherent if the map from partial automorphisms to their extensions respects composition.

Theorem (Bhattacharjee–Macpherson 2005, Solecki–Siniora 2019)

If $\mathrm{Age}(\mathbf{M})$ *has coherent EPPA then* $\mathrm{Aut}(\mathbf{M})$ *contains a dense locally finite subgroup.*

Coherent EPPA

An EPPA-witness **B** for **A** is coherent if the map from partial automorphisms to their extensions respects composition.

Theorem (Bhattacharjee–Macpherson 2005, Solecki–Siniora 2019)

If $\mathrm{Age}(\mathbf{M})$ *has coherent EPPA then* $\mathrm{Aut}(\mathbf{M})$ *contains a dense locally finite subgroup.*

Except for two-graphs, n-partite tournaments and semigeneric tournaments, whenever we have EPPA, we also have coherent EPPA.

Normal subgroups

Normal subgroups

- [Truss, 1985] The automorphism group of the random graph is simple.

Normal subgroups

- [Truss, 1985] The automorphism group of the random graph is simple.
- [Macpherson, Tent, 2011] Automorphism groups of Fraïssé limits of free amalgamation classes are simple.

Normal subgroups

- [Truss, 1985] The automorphism group of the random graph is simple.
- [Macpherson, Tent, 2011] Automorphism groups of Fraïssé limits of free amalgamation classes are simple.
- [Tent, Ziegler, 2013] The automorphism group of the Urysohn sphere is simple.

Normal subgroups

- [Truss, 1985] The automorphism group of the random graph is simple.
- [Macpherson, Tent, 2011] Automorphism groups of Fraïssé limits of free amalgamation classes are simple.
- [Tent, Ziegler, 2013] The automorphism group of the Urysohn sphere is simple.
- [Evans, Hubička, K, Li, Ziegler, 2021] Automorphism groups of various homogeneous metric-like structures are simple.

Ramsey classes

Ramsey classes

Theorem (Kechris, Pestov, Todorčević, 2005)

$\mathrm{Age}(\mathbf{M})$ *has the Ramsey property* $\iff$ $\mathrm{Aut}(\mathbf{M})$ *is extremely amenable.*

Ramsey classes

Theorem (Kechris, Pestov, Todorčević, 2005)

$\mathrm{Age}(\mathbf{M})$ *has the Ramsey property* $\iff$ $\mathrm{Aut}(\mathbf{M})$ *is extremely amenable.*

A topological group G is extremely amenable if every continuous action on a compact space has a fixed point. (Equivalently, the *universal minimal flow* of G is a singleton.)

Ramsey classes

Theorem (Kechris, Pestov, Todorčević, 2005)

$\mathrm{Age}(\mathbf{M})$ *has the Ramsey property* $\iff$ $\mathrm{Aut}(\mathbf{M})$ *is extremely amenable.*

A topological group G is extremely amenable if every continuous action on a compact space has a fixed point. (Equivalently, the *universal minimal flow* of G is a singleton.)
A class $\mathcal{C}$ of finite structures has the Ramsey property if for every $\mathbf{A}, \mathbf{B} \in \mathcal{C}$ there is $\mathbf{C} \in \mathcal{C}$ such that for every colouring of embeddings $\mathbf{A} \to \mathbf{C}$ by 2 colours there is an embedding $\mathbf{B} \to \mathbf{C}$ which is monochromatic.

Big Ramsey degrees

Cf. AGK talk of Honza Hubička from Dec 14. Connected to (topological) self-embedding monoids.

Thank you!

Cf. AGK talk of Honza Hubička from Dec 14. Connected to (topological) self-embedding monoids.

Thank you!

Cf. AGK talk of Honza Hubička from Dec 14. Connected to (topological) self-embedding monoids.

(Questions?)