CSPs on Symmetric Sets

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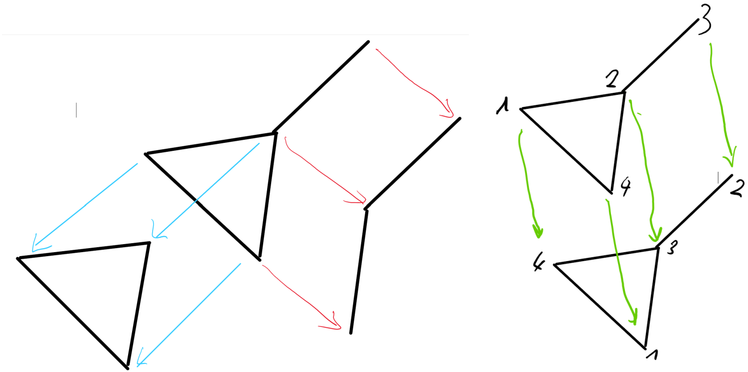
Symmetric Set Example: Graphs

- $G_n :=$ set of graphs on vertex set $[n] = \{1, 2, \ldots, n\}$
- injective maps $[k] \to [n]$ give maps $G_n \to G_k$



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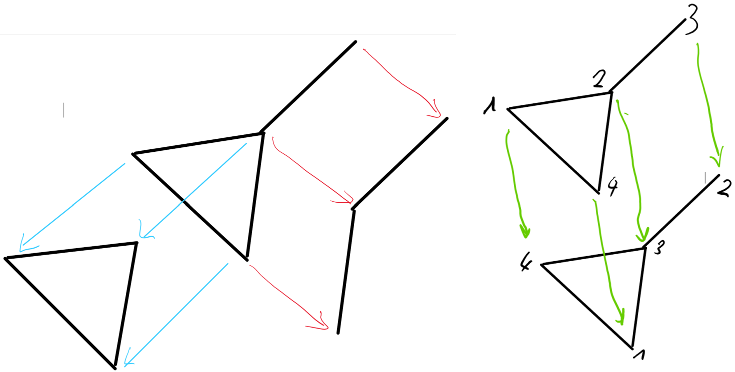
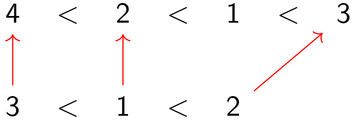


- $G :=$ all G_n together maps $G_n \to G_k$ for all $[k] \to [n]$

Symmetric Set Example: Linear Orders

- $L_n :=$ set of linear orders on $[n]$
- $[k] \to [n]$ again gives map $L_n \to L_k$



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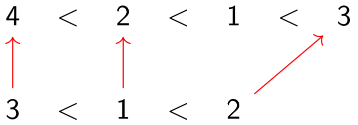


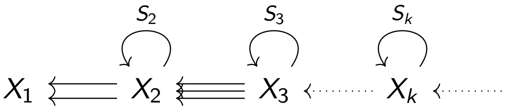
- $L :=$ all L_n together maps $L_n \to L_k$ for all $[k] \to [n]$

Symmetric Sets Formalism

Definition

A *symmetric set* is a sequence of sets $X_1, X_2, \ldots$ together with maps $X(f) : X_n \to X_k$ for every injective map $[k] \to [n]$ such that

$$X(f \circ g) = X(g) \circ X(f)$$
$$X(id) = id$$



Relations on Symmetric Sets

X symmetric set, $R \subseteq X_n$ is called n-ary relation on X.

- $\{\text{single edge}\} \subseteq G_2$
- $\{\text{triangle-free graphs}\} \subseteq G_{37}$
- $\{(1 < 2)\} \subseteq L_2$
- $Betw = \{(1 < 2 < 3), (3 < 2 < 1)\} \subseteq L_3$

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Question: Is there a linear order on $\{1, 2, 3, 4\}$ with

$$Betw(1, 2, 3) \wedge Betw(2, 3, 4) \wedge Betw(2, 1, 4)$$

$$1 \quad < \quad 2 \quad < \quad 3 \quad < \quad 4$$

Constraint Satisfaction Problems

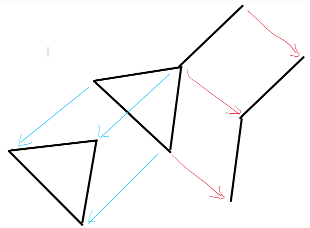
Let X symmetric set, $R \subseteq X_k$. $CSP(X, R)$ is the following problem:

INPUT:

number n and conjunction of terms $R(\bar{v})$, where $\bar{v} \in [n]^{[k]}$

QUESTION:

Is there an element of $a \in X_n$ such that $X(\bar{v})(a) \in R$



Canonical Polymorphisms

Definition

Natural transformations $f : X^n \to X$ are called *canonical polymorphisms* of $R \subseteq X_k$ if

$$\bar{r} \in R^n \implies f_k(\bar{r}) \in R$$

$$\begin{array}{ccccccccc}
(X_1)^n & \leftleftarrows & (X_2)^n & \Lleftarrow & (X_3)^n & \leftarrow\cdots & (X_k)^n & \supseteq & R^n \\
\downarrow f_1 & & \downarrow f_2 & & \downarrow f_3 & & \downarrow f_k & & \\
X_1 & \leftleftarrows & X_2 & \Lleftarrow & X_3 & \leftarrow\cdots & X_k & \supseteq & R
\end{array}$$

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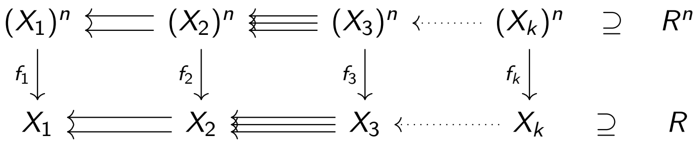

Corollary (of Bulatov, Zhuk 2017)

If R has a "nice" canonical polymorphism, then $CSP(X, R) \in \mathbf{P}$

Diagonal Polymorphisms

Definition

Natural transformations $f : X_{(n\times -)} \to X$ are called *diagonal polymorphisms* of $R \subseteq X_k$ if for $\bar{R} \in X_{n\times k}$

$$\forall i = 1 \dots n \quad \pi_i(\bar{r}) \in R \implies f_k(\bar{r}) \in R$$

$$\begin{array}{ccccccccc}
X_n & \Longleftarrow & X_{n\times 2} & \Lleftarrow & X_{n\times 3} & \leftarrow\cdots & X_{n\times k} & & \\
\downarrow f_1 & & \downarrow f_2 & & \downarrow f_3 & & \downarrow f_k & & \\
X_1 & \Longleftarrow & X_2 & \Lleftarrow & X_3 & \leftarrow\cdots & X_k & \supseteq & R
\end{array}$$

polymorphisms of Fraisse limits	$\nexists$ "nice" $\implies$ **NP**-hard	Barto, Pinsker 2021
$\cup$		
diagonal polymorphisms	$\nexists$ "nice" $\implies$ **NP**-hard	Conjecture (with some assumptions)
$\cup$		
canonical polymorphisms	$\exists$ "nice" $\implies$ **P**	Bulatov, Zhuk 2017