A non-finitely related minimal Taylor algebra

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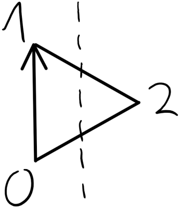

Definition: minimal Taylor

Definition

A finite idempotent algebra $\mathbb{A}$ is called *Taylor*, if

- $\mathbb{A}$ satisfies a non-trivial Maltsev condition
- (Maroti, McKenzie) $\mathrm{Clo}(\mathbb{A})$ contains a weak-NU operation

$$w(y, x, \ldots, x) \approx w(x, y, x, \ldots, x) \approx \cdots \approx w(x, \ldots, x, y)$$

- (Barto, Kozik) $\mathrm{Clo}(\mathbb{A})$ contains a cyclic operations c_p for all primes $p > |\mathbb{A}|$

$$c_p(x_1, x_2, \ldots, x_p) \approx c_p(x_p, x_1, \ldots, x_{p-1})$$

- $\mathrm{HS(P)}(\mathbb{A})$ contains a naked set

Definition

A Taylor algebra $\mathbb{A}$ is called *minimal*, if any other algebra $\mathbb{B}$ with $\mathrm{Clo}(\mathbb{B}) \subsetneq \mathrm{Clo}(\mathbb{A})$ is not Taylor.

Definition: finitely related

Definition

An algebra $\mathbb{A}$ is called *finitely related* if there is a finite set of relations $\{R_1, \ldots R_n\}$ on the domain of $\mathbb{A}$, such that

$$f \in \mathrm{Clo}(\mathbb{A}) \iff f \text{ preserves all } R_i$$

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Conjecture (Brady)

Every minimal Taylor algebra is finitely related.

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Conjecture (Brady)

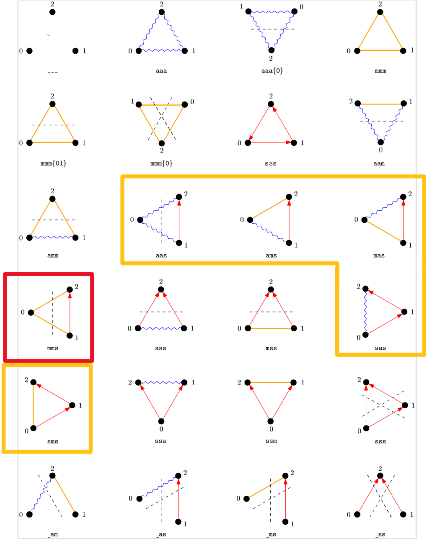
Every minimal Taylor algebra is finitely related.

Theorem (H.)

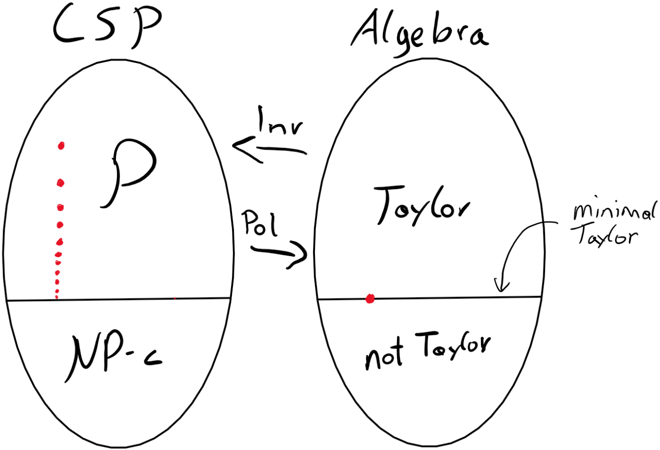
There is a minimal Taylor algebra that is not finitely related.

Minimal Taylor algebras on 3 elements

- (Brady) there are 24
- (Barto, Brady, Jankovec, Vucaj, Zhuk) 18 are "well behaved": "nice" finite relational description
- (H.) one is evil
- 5 are unstudied
- Picture by (Vucaj)



Connection to constraint satisfaction problems

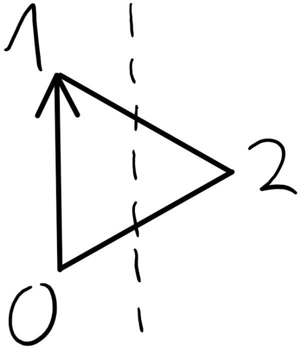


The algebra

$\mathbb{A} = (\{0,1,2\}, m)$ $\quad m$ ternary

- $m|_{\{0,2\}} = m|_{\{1,2\}} = \text{maj}$
- $m|_{\{0,1\}}(x,y,z) = x \vee y \vee z$
- $m(0,1,2) = \cdots$

  $\cdots = m(2,1,0) = 1$

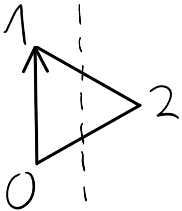


Theorem (H.)

The algebra $\mathbb{A}$ is minimal Taylor, but not finitely related.

Understanding the algebra: first observations

- m is symmetric $\implies$ $\mathbb{A}$ Taylor
- $\theta = (\,0, 1 \mid 2\,)$ is congruence
- $m/\theta =$ maj



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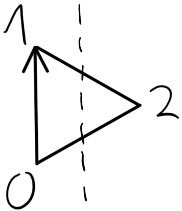
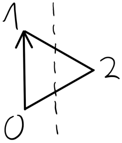


For all operations $f \in \mathrm{Clo}(\mathbb{A})$ we have

- $f|_{\{0,1\}} = \bigvee$ essential variables
- $f|_{\{0,2\}} \sim f|_{\{1,2\}} \sim f/\theta$ under bijections $\{0,2\} \sim \{1,2\} \sim \{(0,1),(2)\}$
- $f(2,\ldots,2,1,\ldots,1) = 1 \implies f(2,\ldots,2,0/1,\ldots,0/1) \in \{0,1\}$ and

  $f(2,\ldots,2,y_1,\ldots,y_k)|_{\{0,1\}^k} = \bigvee$ some y_i

$\mathbb{A}$ is minimal Taylor



Proposition

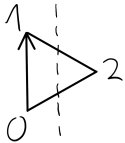
$\mathbb{A} = (\{0,1,2\}, m)$ is minimal Taylor.

Proof

Let $\mathcal{C} \leq \mathrm{Clo}(\mathbb{A})$ be Taylor $\implies$ $\exists c \in \mathcal{C}_5$ cyclic. Strategy: build m from c

- $c|_{\{0,1\}}(x_1, \dots, x_5) \approx x_1 \vee \cdots \vee x_5$
- $c|_{\{0,2\}} = c|_{\{1,2\}} =$ 5-ary majority
- $t(x,y,z) := c(x,x,y,y,z)$ agrees with m on 2-element subsets

$\mathbb{A}$ is minimal Taylor



Proposition

$\mathbb{A} = (\{0,1,2\}, m)$ is minimal Taylor.

Proof.

t and m agree on 2-element subalgebras. What about $t(0,1,2)$?

- $r(x,y,z) := t\big(t(x,x,y), t(x,x,y), z\big)$
- $t'(x,y,z) := t\big(r(x,y,z), r(y,z,x), r(z,x,y)\big)$

Check two things:

- t and t' agree on 2-element subalgebras
- $t'(0,1,2) = \cdots = t'(2,1,0) = 1$

So $t' = m$ and $\mathcal{C} = \mathrm{Clo}(\mathbb{A})$. □

Deciding finite relatedness: does theory help?

Theorem (Aichinger, Mayr, McKenzie)

If an algebra has a cube term, then it is finitely related.

Theorem (Markovic, Maroti, McKenzie)

$\mathbb{A}$ has no cube term if and only if there exist $C < B \leq \mathbb{A}$ such that all operations preserve $B^n \setminus (C \setminus B)^n$.

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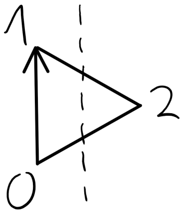
$\{1\} < \{0,1\}$ is cube term blocker:

- $m|_{\{0,1\}}$ is join semilattice
- m preserves $\{0,1\}^n \setminus \{0\}^n$

$\Longrightarrow$ $\mathbb{A}$ does not have cube term



Deciding finite relatedness: does theory help?

Theorem (Barto, Bulin)

If $\exists B : B \triangleleft_J \mathbb{A}$ but $B \not\triangleleft \mathbb{A}$, then $\mathbb{A}$ is not finitely related.

- $B \triangleleft_J \mathbb{A}$ if $B \leq \mathbb{A}$ and
  $p_0(x, y, z) \approx x$
  $p_n(x, y, z) \approx z$
  $p_i(x, y, y) \approx p_{i+1}(x, x, y)$
  $p_i(B, A, B) \subseteq B$
- $B \triangleleft \mathbb{A}$ if $B \leq \mathbb{A}$ and
  $t(B, \ldots, B, A, B, \ldots, B) \subseteq B$

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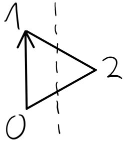

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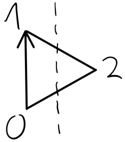

$\{1\}, \{2\}, \{0, 1\}, \{1, 2\} \lhd_m \mathbb{A}$

assume $\{0\}, \{0, 2\} \lhd_J \mathbb{A}$:

$\implies p_i(0, 1, 0) = 0$

$\implies y$ dummy

$\implies p_i(x, y, z) \approx p_{i+1}(x, y, z)$ ↯

How to prove non-finitely related?

Observation

An algebra $\mathbb{A}$ is not finitely related iff $\forall n \exists f_n$ of arity $> n$ such that all n-ary minors of f_n are in $\mathrm{Clo}(\mathbb{A})$, but $f_n \notin \mathrm{Clo}(\mathbb{A})$.

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Proof.

Show ($\Longleftarrow$): Let $\{R_1, R_2, \dots\}$ be a relational basis.
Find $f_n \notin \mathrm{Clo}(\mathbb{A})$ where f_n does not preserve some R_i, but all its n-ary minors do.

$$f_n(\underbrace{\bar{x}_1, \bar{x}_2, \bar{x}_3, \dots, \bar{x}_{ar}}_{\in R_i,\text{ more than } n \text{ many}}) \notin R_i$$

Then $|R_i| > n$, hence $|\{R_1, R_2, \dots\}| = \infty$. $\square$

Proof strategy

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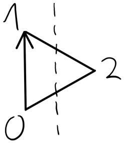
An algebra $\mathbb{A}$ is not finitely related iff $\forall n \exists f_n$ of arity $> n$ such that all n-ary minors of f_n are in $\mathrm{Clo}(\mathbb{A})$, but $f_n \notin \mathrm{Clo}(\mathbb{A})$.

Strategy

- Find construction method for functions "almost in $\mathrm{Clo}(\mathbb{A})$"
- Construct f_n for every n (arity $\approx n^2/2$)
- Find relations R_n of arity n
- Show f_n does not preserve R_n, i.e $f_n \notin \mathrm{Clo}(\mathbb{A})$
- Show that n-ary minors of f_n are in $\mathrm{Clo}(\mathbb{A})$

Understanding functions: mod congruence

- $\theta = (\, 0, 1 \mid 2 \,)$ is congruence
- $m/\theta = \mathrm{maj}$



Definition

$X \subseteq [n]$ *big* (w.r.t. f) if $f(\underbrace{2, \ldots, 2,}_{X}$ something$) = 2$

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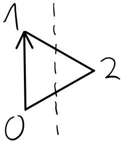


Definition

$X \subseteq [n]$ *big* (w.r.t. f) if $f(\underbrace{2, \ldots, 2,}_{X}$ something$) = 2$

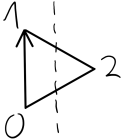
- X big and $X \subseteq X' \implies X'$ big
- big sets pairwise intersect
- exactly one of X and $[n] \setminus X$ is big

Proposition

Functions in $\mathrm{Clo}(\mathbb{A})$ modulo θ are determined by their big sets, where

Describing functions: Beating

$$f(2,\dots,2,1,\dots,1)=1 \implies$$
$$f(2,\dots,2,y_1,\dots,y_k)|_{\{0,1\}^k} = \bigvee \text{some } y_i$$



Definition

$X, Y \subseteq [n]$ disjoint, say X *beats* Y (w.r.t. f) if

$$f(\underbrace{2,\dots,2}_{X},\underbrace{1,\dots,1}_{Y},0,\dots 0) \in \{0,2\}$$

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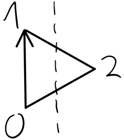

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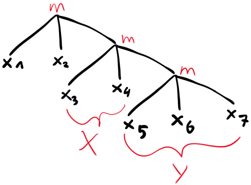
$$f(\underbrace{2,\dots,2}_{X},\underbrace{1,\dots,1}_{Y},0,\dots 0) \in \{0,2\}$$

- X big iff X beats $[n] \setminus X$
- Y dummy iff $\emptyset$ beats Y
- X beats Y iff Y not part of disjunction



How do beatings interact?

Proposition

Functions in $\mathrm{Clo}(\mathbb{A})$ are determined by beatings.

Question

Given a set of statements "X_i beats Y_i", is there a function $f \in \mathrm{Clo}(\mathbb{A})$ where they hold?

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Functions in $\mathrm{Clo}(\mathbb{A})$ are determined by beatings.

Question

Given a set of statements "X_i beats Y_i", is there a function $f \in \mathrm{Clo}(\mathbb{A})$ where they hold?

- Rules for big sets must hold (supersets are big, big sets intersect,...)
- X beats Y, $Y' \subseteq Y$ and $X \subseteq X' \implies X'$ beats Y' (*)
- X beats Y and Y beats $Z \implies X$ beats Z
- X beats Y and Y beats $X \implies X$ and Y dummy
- ...?

But: (*) can be used to define functions

High arity relations

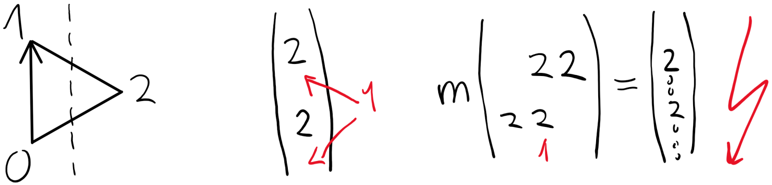
Definition

Let $R_n \subseteq \{0,1,2\}^n$, $\bar{x} \in R_n$ if and only if

$$\forall i < j : \quad x_i = x_j = 2 \implies \exists k > i : \quad x_k = 1$$

Show that R_n is preserved by m.

Proof.



□

What do the relations say?

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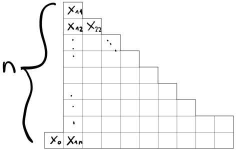
"variables cannot beat each other"

There is no function in $\mathrm{Clo}(\mathbb{A})$ where X beats Y, Y beats X and $X \cup Y$ big.

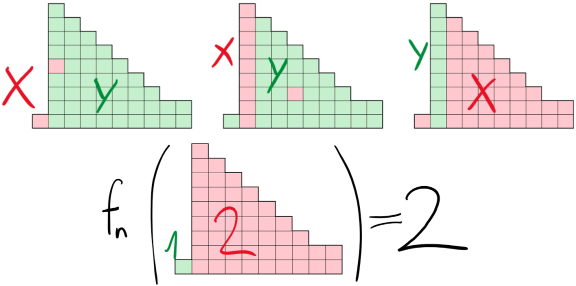
$$f\begin{pmatrix} 2\dots2, 2\dots2, 2\dots2 \\ 1\dots1, 2\dots2, 2\dots2 \\ 0\dots0, 2\dots2, 1\dots1 \\ 0\dots0, \underbrace{1\dots1}_{X}, \underbrace{2\dots2}_{Y} \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 0 \\ 0 \end{pmatrix} \notin R_4$$

High arity functions

- Building f_n: $n(n+1)/2+2$ variables



- f_n/θ: X' big iff it contains one of the following X

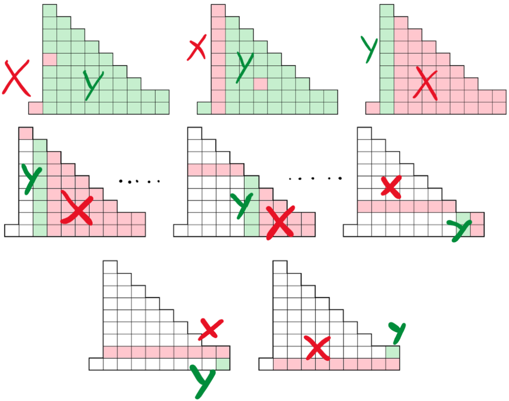


High arity functions

Definition

Let f_n be the function where X' beats Y' iff

- $X' \supseteq X$ and $Y' \subseteq Y$ with X, Y one of



Intuition

- Variables cannot beat each other
- f_n tries to hide this

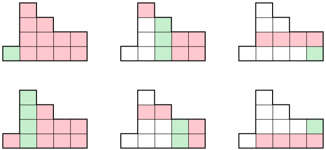
(Not) preserving relations

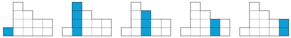
Definition

Let $R_n \subseteq \{0,1,2\}^n$, $\bar{x} \in R_n$ if and only if

$$\forall i < j : \quad x_i = x_j = 2 \implies \exists k > i : \quad x_k = 1$$

f_n does not preserve $R_{n+2} \implies f_n \notin \mathrm{Clo}(\mathbb{A})$





$$f_4\begin{pmatrix} 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 2 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 0 & 2 & 0 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 2 & 2 \\ 0 & 0 & 2 & 0 & 0 & 2 & 0 & 0 & 1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 2 & 0 & 0 & 2 & 0 & 2 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 2 & 0 & 2 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

What remains?

Strategy

- Find construction method for functions "almost in $\mathrm{Clo}(\mathbb{A})$" ✓
- Construct f_n for every n (arity $= n(n+1)/2+2$)✓
- Find relations R_n of arity n✓
- Show f_n does not preserve R_{n+2}, i.e $f_n \notin \mathrm{Clo}(\mathbb{A})$✓
- Show that $(n-1)$-ary minors of f_n are in $\mathrm{Clo}(\mathbb{A})$

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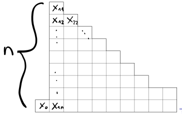
- Show that $(n-1)$-ary minors of f_n are in $\mathrm{Clo}(\mathbb{A})$

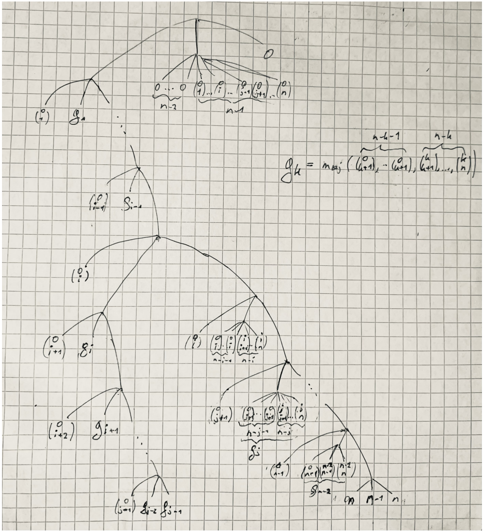
Last part:

- $(n-1)$-ary minors must identify two variables from first column
- Show that f_n is in $\mathrm{Clo}(\mathbb{A})$ if two variables from first column are identified i.e. $f_n^{(x_{1,i}=x_{1,j})} \in \mathrm{Clo}(\mathbb{A})$



Building minors

$$f_n^{(x_{1,i}=x_{1,j})} =$$



Thank you for listening!

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