Forbidden Tournaments and the Orientation (Completion) Problem

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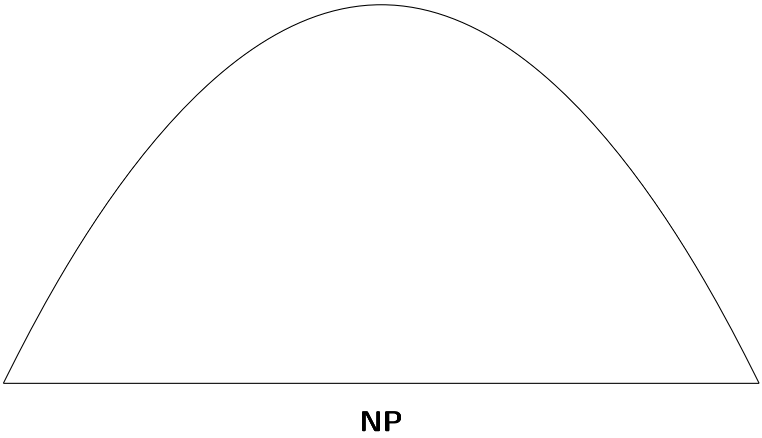
joint work with Manuel Bodirsky

Institute of Algebra
TU Dresden

28th SEG workshop on Combinatorics, Graph Theory, and Algorithms

ERC Synergy Grant POCOCOP (GA 101071674)

Decision problems



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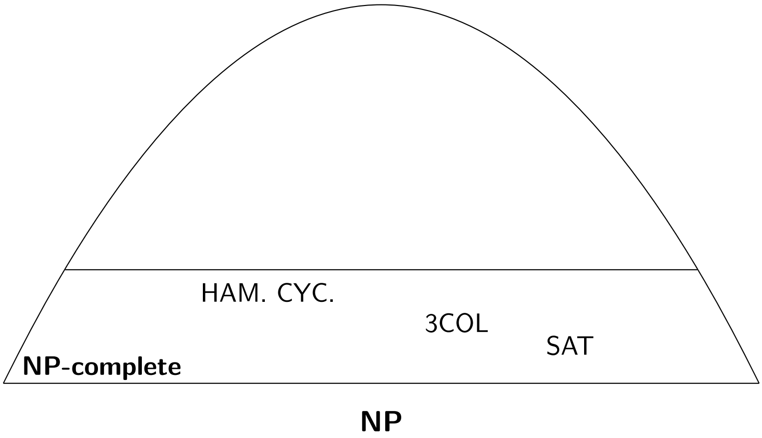

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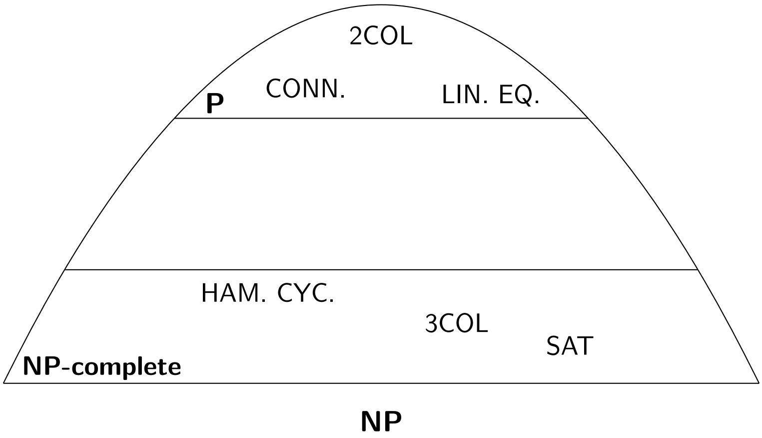

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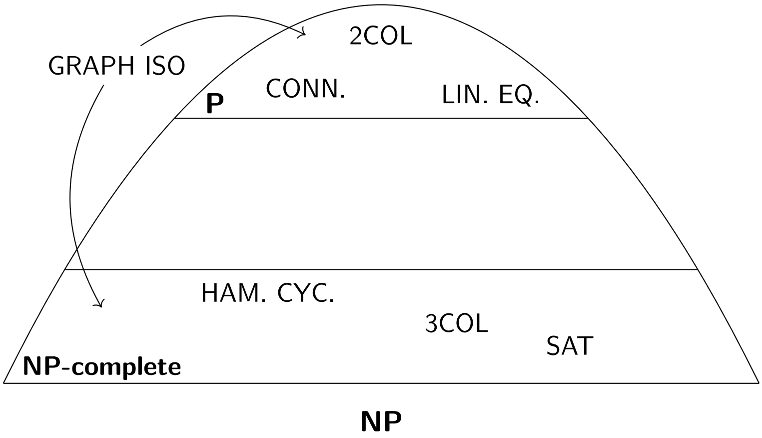

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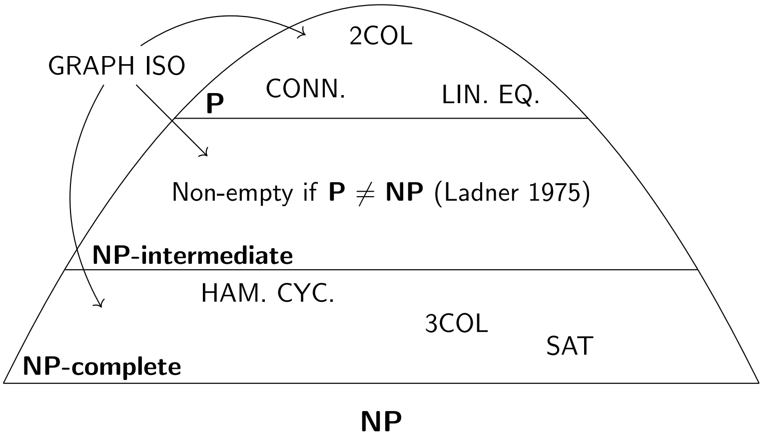

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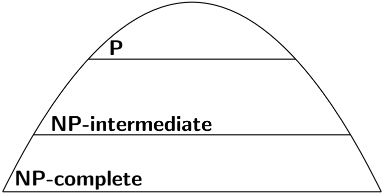
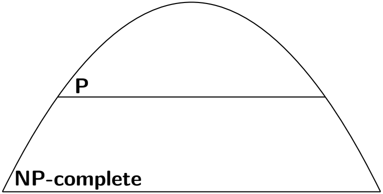


Subclass of **NP**

Full expressive power of **NP**
(up to P-time eq.)



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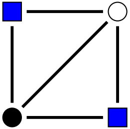
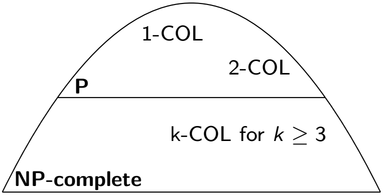
P vs. **NP-complete** dichotomy

Decision problems

k-COL: On input graph G decide if there is a k-vertex colouring without monochromatic edges



G



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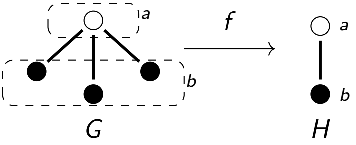
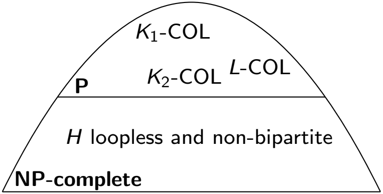
Decision problems

H-COL: On input graph G decide if there is a homomorphism $f\colon G \to H$

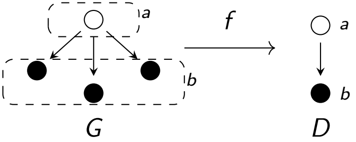


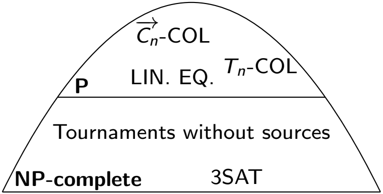


P vs. **NP-complete** dichotomy (Hell-Nešetřil Theorem)

Decision problems

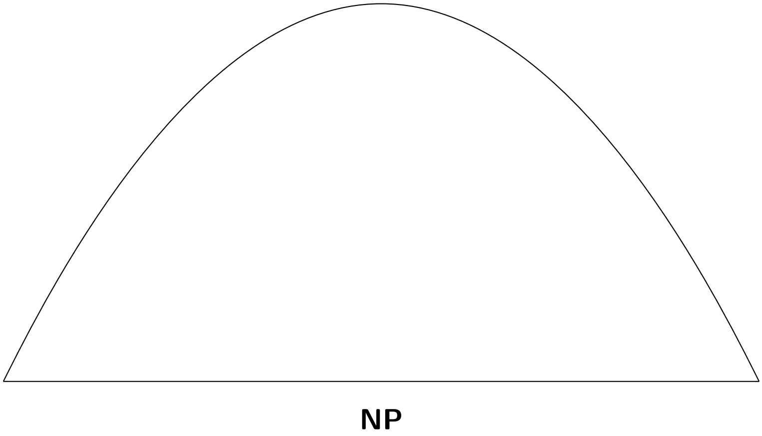
D-COL: On input digraph (hypergraph) G decide if there is a homomorphism $f\colon G \to D$





P vs. **NP-complete** dichotomy
Bulatov, Zhuk (2017)

Decision problems



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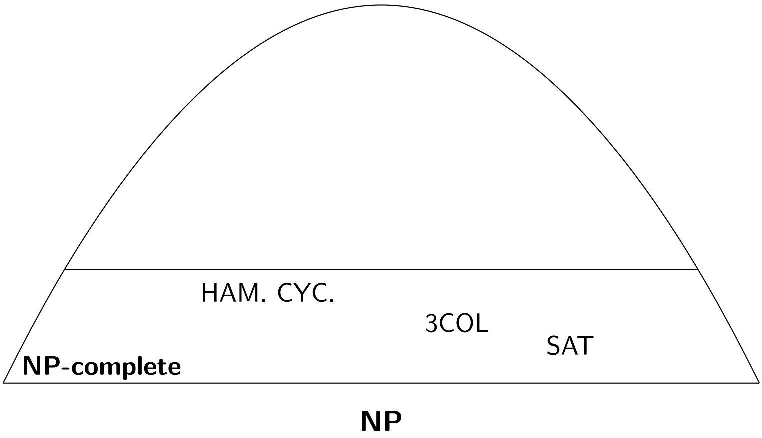

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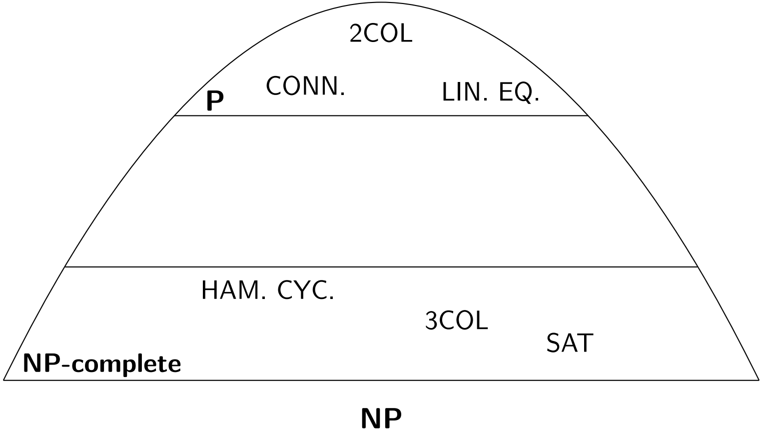

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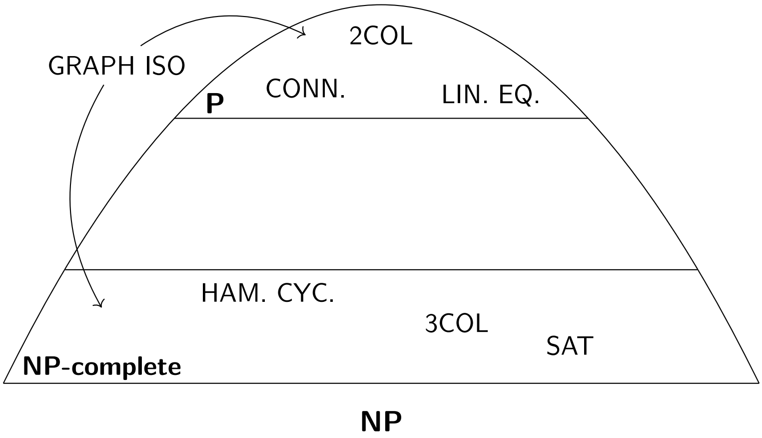

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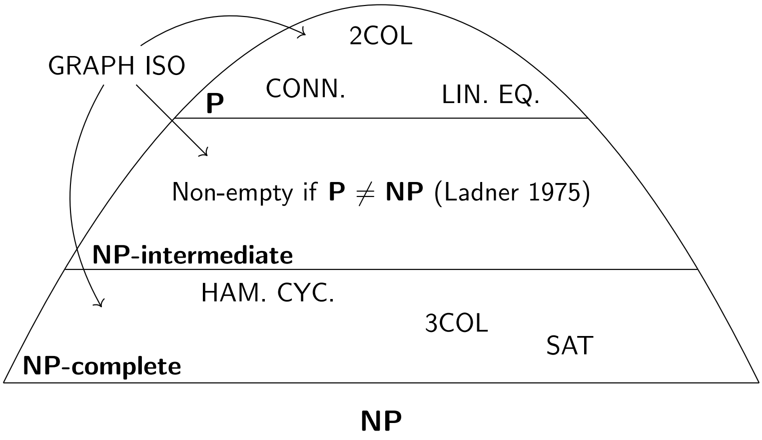

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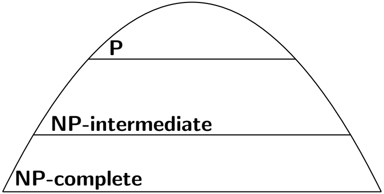

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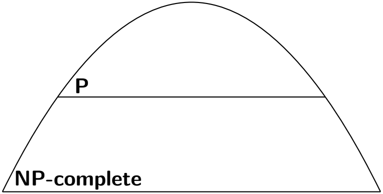

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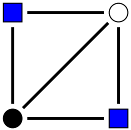

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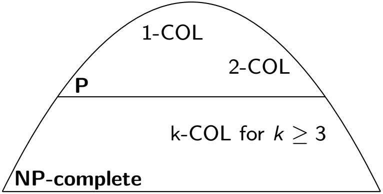

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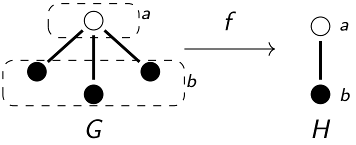

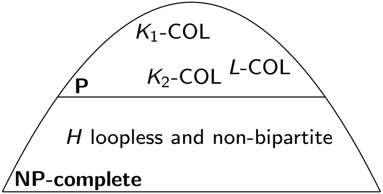

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(Hell-Nešetřil Theorem)

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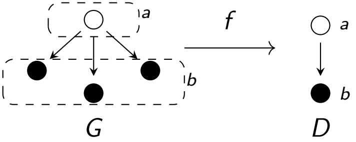

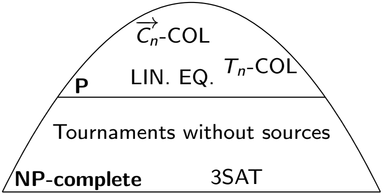

P vs. **NP-complete** dichotomy
Bulatov, Zhuk (2017)

Infinite H-colouring problems

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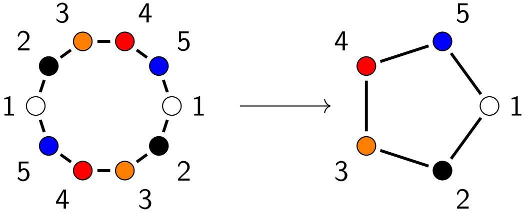

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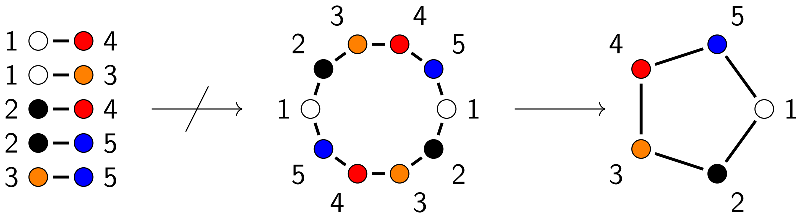
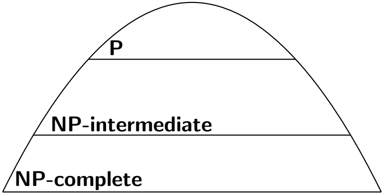


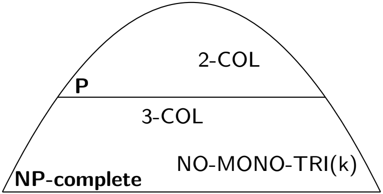
Infinite H-colouring problems

Forbidden vertex-coloured pattern problems



Forbid (induced) subgraphs

Full expressive power of **NP**
Feder-Vardi (1999)

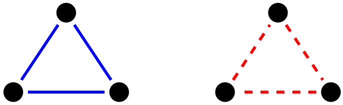


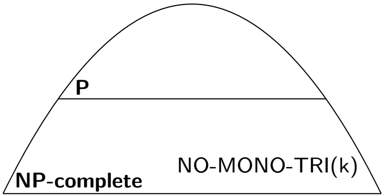
Forbid homomorphically

P vs. **NP-complete** dichotomy
Feder-Vardi (1999) + Bulatuv, Zhuk (2017)
Bodirsky-Madelain-Mottet (2021)

Infinite H-colouring problems

Forbidden edge-coloured patterns





Conjectured **P** vs. **NP-complete** dichotomy
Bodirsky-Pinsker (2012)

Oriented expressions of graph classes

Example 1 (Robbins 1939)

A graph G is 2-edge-connected if and only if it admits a strongly connected orientation.

Oriented expressions of graph classes

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Example 2 (definition)

A graph G is a comparability graph if and only if it admits a transitive orientation.

Oriented expressions of graph classes

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A graph G is a comparability graph if and only if it admits a transitive orientation.

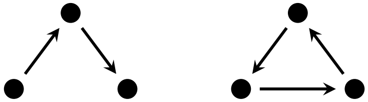
Example 3 (Roy-Gallai-Hasse-Vitaver Theorem)

A graph G is k-colourable if and only if it admits an orientation with no directed walk of length k.

Oriented expressions of graph classes

Example 2 (Definition)

A graph G is a comparability graph if and only if it admits an $\mathcal{F}$-free orientation.



Oriented expressions of graph classes

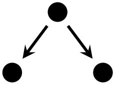
Further examples

- Proper circular-arc graphs

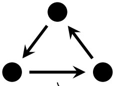
- Proper Helly circular-arc graphs
- 3-colourable comparability graphs
- Star forests
- Unicyclic graphs
- k-colourable graphs
- C_{2k+1}-colourable graphs

Oriented expressions of graph classes

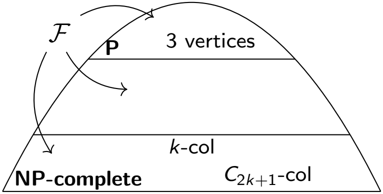
$\mathcal{F}$-**free orientation problem**:
On input graph G decide if there is an $\mathcal{F}$-free orientation of G



B_1

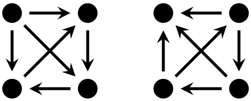


$\overrightarrow{C}_3$

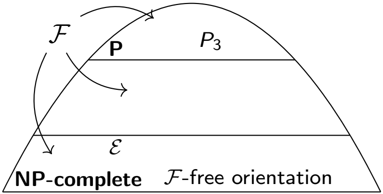


Oriented expressions of graph classes

$\mathcal{F}$-free orientation completion problem (Bang-Jensen, Huang, Zhu (2017)): On input partially oriented graph G decide if there is an $\mathcal{F}$-free orientation completion of G

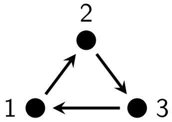


$\mathcal{E}$



The $\mathcal{F}$-free orientation problem

Example 1: Every tournament in $\mathcal{F}$ has a directed cycle



$$\mathcal{F} = \{\overrightarrow{C}_3\}$$

The $\mathcal{F}$-free orientation problem

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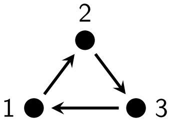


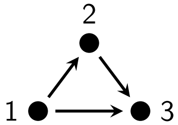
$$\mathcal{F} = \{\overrightarrow{C}_3\}$$

Remark: $\mathcal{F}$-free orientation problem is trivial

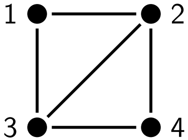
But: Orientation completion not necessarily trivial.

The $\mathcal{F}$-free orientation problem

Example 2: T_3-free orientation (completion) problem.



$\mathcal{F} = \{T_3\}$



G

Code orientations of G as solutions to the sys. lin. eq. over $\mathbb{Z}_2$

$$x_{ij} + x_{ji} = 0 \text{ for } ij \in E$$

The $\mathcal{F}$-free orientation problem

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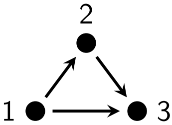

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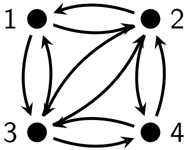

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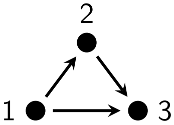

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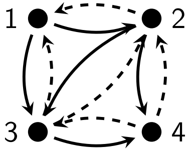
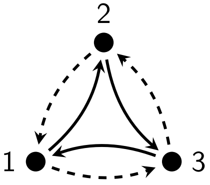


G'

$$x_{12} = 1,\ x_{13} = 1,\ x_{23} = 1,\ x_{24} = 1,\ x_{34} = 1$$
$$x_{21} = 0,\ x_{31} = 0,\ x_{32} = 0,\ x_{42} = 0,\ x_{43} = 0$$

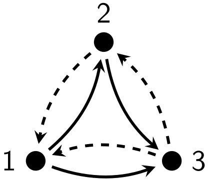
The $\mathcal{F}$-free orientation problem



$\overrightarrow{C}_3$

For each triangle i, j, k the following equality holds:

$$x_{ij} + x_{jk} = 0.$$

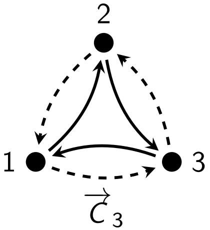


T_3

There exists a triangle i, j, k such that the following equality holds:

$$x_{ij} + x_{jk} = 1 \text{ for instance } x_{23} + x_{31} = 1.$$

The $\mathcal{F}$-free orientation problem



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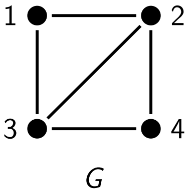
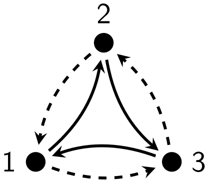


Code T_3-free orientation of G as solutions to

$$x_{ij} + x_{ji} = 0 \text{ for } ij \in E$$
$$x_{ij} + x_{jk} = 0 \text{ for } ijk \in T$$

The $\mathcal{F}$-free orientation problem



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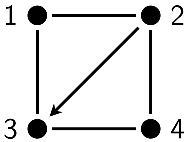
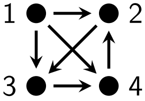


G

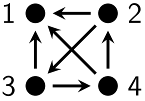
Code T_3-free orientation completions of G as solutions to

$$x_{ij} + x_{ji} = 0 \text{ for } ij \in E$$
$$x_{ij} + x_{jk} = 0 \text{ for } ijk \in T$$
$$x_{ij} = 1 \text{ for } ij \in A$$

The $\mathcal{F}$-free orientation problem



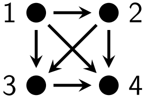
C_3^-



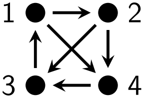
C_3^+

For each i, j, k, l in C_3^- and in C_3^+

$$x_{ij} + x_{jk} + x_{kl} + x_{li} = 1.$$



T_4



TC_4

$$x_{12} + x_{24} + x_{43} + x_{31} = 0 \text{ in } T_4$$
$$x_{12} + x_{24} + x_{43} + x_{31} = 0 \text{ in } TC_4$$

The $\mathcal{F}$-free orientation problem

Example 3: The $\{T_4, TC_4\}$-free orientation (completion) problem is in P

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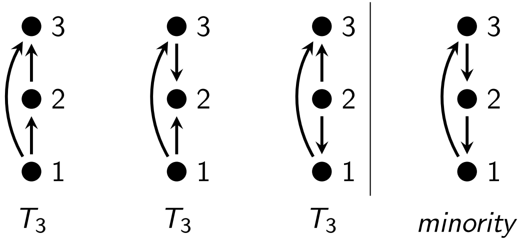
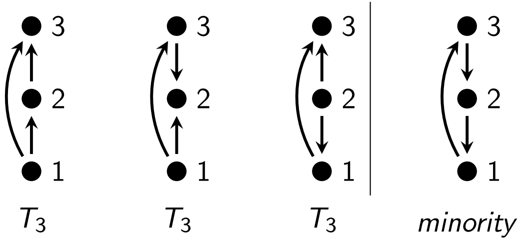
Question: For which finite sets of tournaments $\mathcal{F}$ the $\mathcal{F}$-free does this method work?

The $\mathcal{F}$-free orientation problem



- $\vec{C}_3$-free tournaments **are not** *preserved* by the *minority* operation.
- T_3-free tournaments **are** *preserved* by the *minority* operation.

The $\mathcal{F}$-free orientation problem



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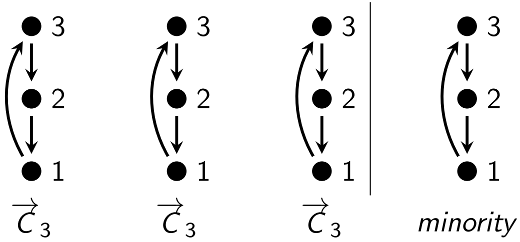

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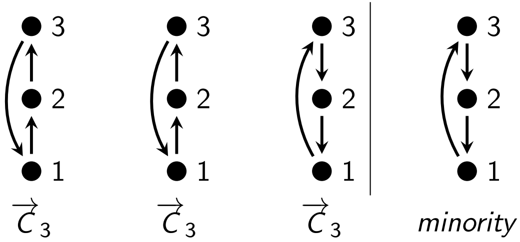

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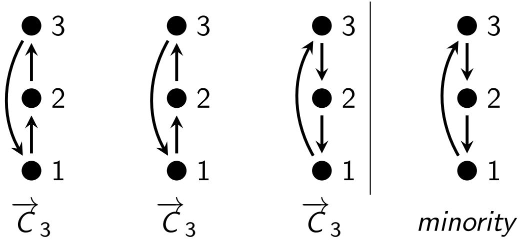
The $\mathcal{F}$-free orientation problem



Lemma

Let $\mathcal{F}$ be a finite set of tournaments. The $\mathcal{F}$-free orientations of a graph G correspond to the solution space of some system of linear equations if and only if the $\mathcal{F}$-free tournaments are preserved by the minority operation.

The $\mathcal{F}$-free orientation problem

Example 4: The $\overrightarrow{C}_3$-free orientation completion problem is NP-complete

The $\mathcal{F}$-free orientation problem

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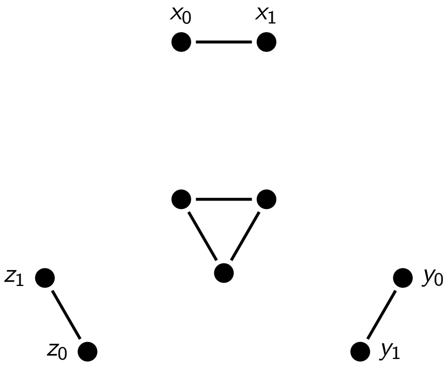
Reduction from NAE 3-SAT with Input: $(x \vee y \vee z) \wedge \dots$



The $\mathcal{F}$-free orientation problem

Example 4: The $\overrightarrow{C}_3$-free orientation completion problem is NP-complete

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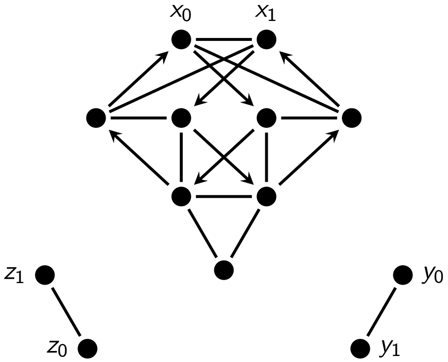

The $\mathcal{F}$-free orientation problem

Example 4: The $\overrightarrow{C}_3$-free orientation completion problem is NP-complete

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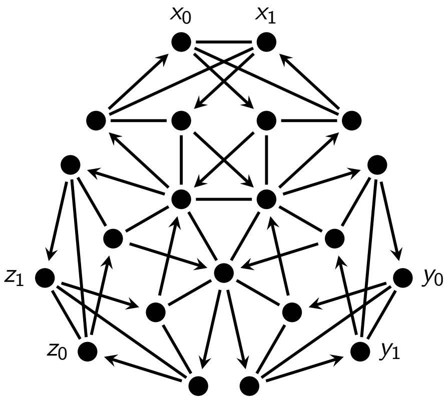


The $\mathcal{F}$-free orientation problem

Is that all?

The $\mathcal{F}$-free orientation problem

Theorem (Bodirsky, G.P., 24+)

For every finite set of finite tournaments $\mathcal{F}$ one of the following cases holds.

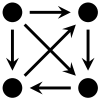
1. $\mathcal{F}_f$ is preserved by the minority operation. In this case, the $\mathcal{F}$-free orientation completions of a partially oriented graph G correspond to the solution space of a system of linear equations over $\mathbb{Z}_2$.
2. Otherwise, $\mathcal{F}$-free orientation completion problem is NP-complete.

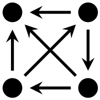
In the first case, the $\mathcal{F}$-free orientation completion problem is in P.

The $\mathcal{F}$-free orientation problem

Corollary

If every tournament in $\mathcal{F}$ contains a directed cycle, then the $\mathcal{F}$-free orientation completion problem is NP-complete.





The $\mathcal{F}$-free orientation problem

Theorem (Bodirsky, G.P., 24+)

For every finite set of finite tournaments $\mathcal{F}$ one of the following cases holds.

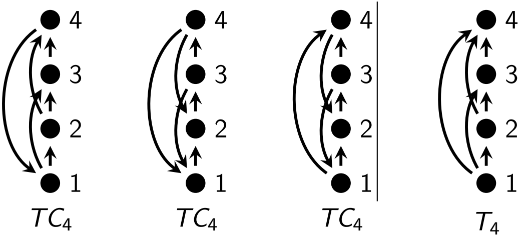
1. $\mathcal{F}$ contains no transitive tournament. In this case, every graph admits an $\mathcal{F}$-free orientation.
2. $\mathcal{F}_f$ is preserved by the minority operation. In this case, the $\mathcal{F}$-free orientations of a graph G correspond to the solution space of a system of linear equations over $\mathbb{Z}_2$.
3. Otherwise, $\mathcal{F}$-free orientation problem is NP-complete.

In cases 1 and 2, the $\mathcal{F}$-free orientation problem is in P.

The $\mathcal{F}$-free orientation problem

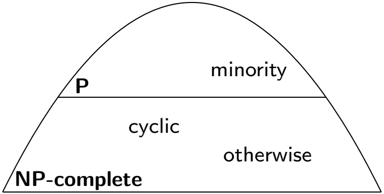
Corollary

The T_k-free orientation problem is NP-complete for each $k \geq 4$.

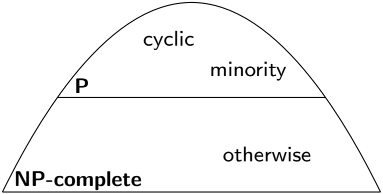


If the $\mathcal{F}$-free orientation problem is NP-hard, then it is still NP-hard for K_f-free graphs.

Thank you!



$\mathcal{F}$-free orientation completion



$\mathcal{F}$-free orientation

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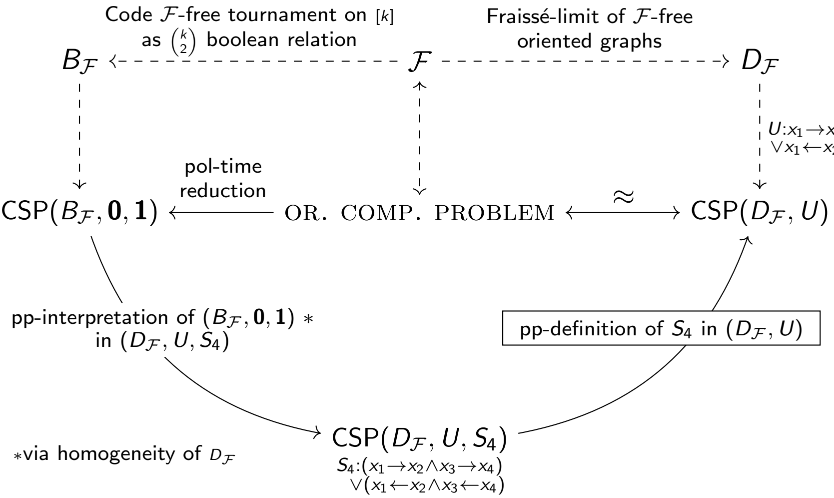
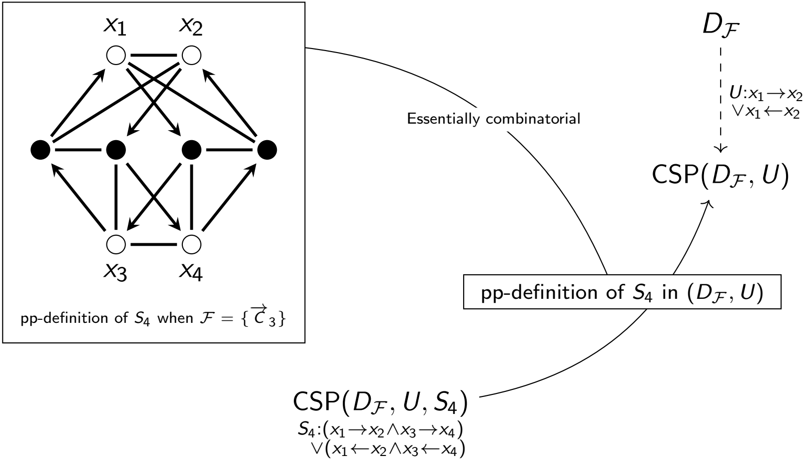
Proof overview



Proof overview



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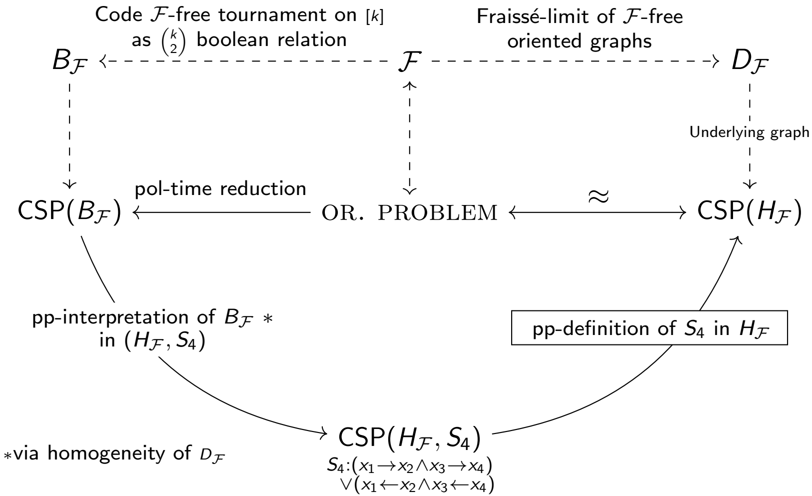
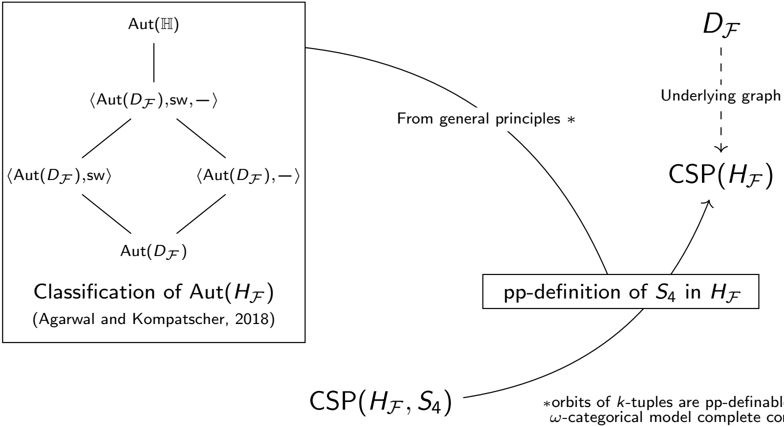


Proof overview



$*$orbits of k-tuples are pp-definable in ω-categorical model complete cores