GRAPH ORIENTATION PROBLEMS WITH FORBIDDEN TOURNAMENTS

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The graph orientation problem

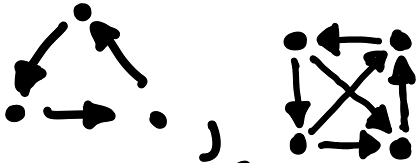
$\mathcal{F}$ ... finite set of finite directed graphs

OP($\mathcal{F}$): the "$\mathcal{F}$-free graph orientation problem"

INPUT: finite graph G

DECIDE: Does G admit orientation that does not embed any member of $\mathcal{F}$?

EXAMPLES ① $\mathcal{F} = \{$

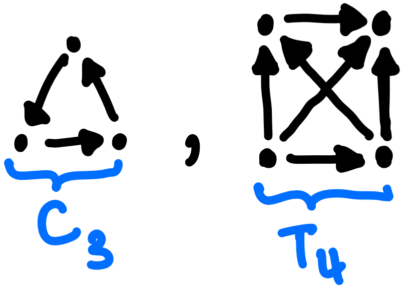


$\}$ Answer always YES:

Acyclic orientation always possible!

solvable in constant time

② $\mathcal{F} = \{$

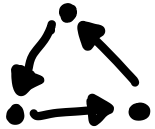


$\}$ Answer YES, iff input graph does not contain a 4-clique.

P

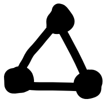
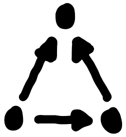
C_3 or T_4 embeds into every orient. of K_4. O/w acyclic orientation possible!

The graph orientation problem

EXAMPLES ③ $\mathcal{F} = \{ \bullet \to \bullet \to \bullet \to ,$  $\}$

follows from Gallai-Hasse-Roy-Vitaver Thm

OUTPUT : YES iff input graph is 3-colorable. NP-complete

$h: G \to$  $\leftrightsquigarrow$ orientation G' s.t. $h: G' \to$ 

$\updownarrow$

$\mathcal{F}$-free orientation G'.

THEOREM (Bodirsky, Guzmán-Pro 2023)

If $\mathcal{F}$ consists of tournaments only, then $OP(\mathcal{F})$ is either tractable or NP-complete.

The graph orientation completion problem

OCP($\mathcal{F}$) : the "$\mathcal{F}$-free graph orientation completion problem"

INPUT : finite graph G with partial orientation of edges

DECIDE : Can the partial orientation be extended to $\mathcal{F}$-free orientation of G.

Again:

THEOREM (Bodirsky, Guzmán-Pro 2023)

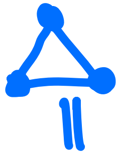
If $\mathcal{F}$ consists of tournaments only, then OCP($\mathcal{F}$) is either tractable or NP-complete.

Connection to CSPs (Constraint Satisfaction Problems)

$\mathbb{A}$ ... rel. structure with signature τ

- $\mathrm{CSP}(\mathbb{A}) = \{\mathbb{B}$ finite τ-structure : $\mathbb{B} \to \mathbb{A}\}$

- $\mathrm{CSP}(\mathbb{A})$ is also the decision problem: $\mathbb{B} \in \mathrm{CSP}(\mathbb{A})$?



EXAMPLES ① $\mathrm{CSP}(K_3) =$ "The 3-coloring problem"

② $\mathrm{CSP}(\mathbb{Q},<) =$ "decide if input is oriented acyclic graph"

③ $\mathrm{CSP}(\mathbb{Z},\{0\},\{1\},\underbrace{+,\cdot}_{\text{ternary relations}}) =$ "decide if Diophantine equation has integer solution."

Connection to CSPs (Constraint Satisfaction Problems)

$\mathbb{A}$ … rel. structure with signature τ

- $\mathrm{CSP}(\mathbb{A}) = \{\mathbb{B} \text{ finite } \tau\text{-structure} : \mathbb{B} \to \mathbb{A}\}$
- $\mathrm{CSP}(\mathbb{A})$ is also the decision problem: $\mathbb{B} \in \mathrm{CSP}(\mathbb{A})$?

Observation (Bodirsky, Guzmán-Pro)

If $\mathcal{F}$ consist of tournaments only ⇒ $\mathrm{OP}(\mathcal{F})$ and $\mathrm{OCP}(\mathcal{F})$ are CSPs of suitable structures.

complete oriented graphs

The templates are constructed using Fraissé's Thm.

Interlude: Fraïssé Theory

$\mathcal{C}$ class of finite τ-structures (τ relational language) is a **Fraïssé-class** if:

- $\mathcal{C}$ is **hereditary**: $\mathbb{A} \in \mathcal{C}$, $\mathbb{B} \hookrightarrow \mathbb{A} \Rightarrow \mathbb{B} \overset{\text{up to isomorphism}}{\in} \mathcal{C}$
- $\mathcal{C}$ is **essentially countable**: Up to isomorphism only ctbly many structures in $\mathcal{C}$
- $\mathcal{C}$ has **amalgamation**:

$$\begin{tikzcd}
\mathbb{B}_2 & \\
\mathbb{A} \arrow[u, hook] \arrow[r, hook] & \mathbb{B}_1
\end{tikzcd}$$

in $\mathcal{C}$ $\Rightarrow \exists\, \mathbb{C}$ + embeddings

$$\begin{tikzcd}
\mathbb{B}_2 \arrow[r, hook] & \mathbb{C} \\
\mathbb{A} \arrow[u, hook] \arrow[r, hook] \arrow[ur, phantom, "?"] & \mathbb{B}_1 \arrow[u, hook]
\end{tikzcd}$$

"$\mathbb{B}_1, \mathbb{B}_2$ can be glued together along $\mathbb{A}$ within $\mathbb{C}$"

EXAMPLES

- finite sets
- finite graphs
- finite K_n-free graphs
- finite linear orders
- finite fields of char. p
  (need signature with function symbols; possible w/ additional assumptions)

Interlude: Fraïssé Theory

FRAÏSSÉ'S THEOREM

If $\mathcal{C}$ is a Fraïssé-class of τ-structures, then there is a countable, homogeneous τ-structure $\mathrm{Flim}(\mathcal{C})$ (unique up to isomorphism) whose finite substructures are precisely $\mathcal{C}$ (up to isomorphism).

EXAMPLES

- $\mathrm{Flim}(\text{finite sets}) = \omega$
- $\mathrm{Flim}(\text{fin. graphs}) =$ Rado/random graph
- $\mathrm{Flim}(\text{fin. } K_n\text{-free graphs}) =$ Henson graphs
- $\mathrm{Flim}(\text{fin. lin. orders}) = (\mathbb{Q}, <)$
- $\mathrm{Flim}(\text{fin. char } p \text{ fields}) = \overline{\mathbb{F}_p}$

Connection to CSPs (Constraint Satisfaction Problems)

$\mathcal{F}$ ... finite set of tournaments

$\mathcal{C}_{\mathcal{F}}$ ... class of finite $\mathcal{F}$-free directed graphs (a Fraïssé class)

$D_{\mathcal{F}} = (V, \to) =: \mathrm{Flim}(\mathcal{C}_{\mathcal{F}})$

$H_{\mathcal{F}} := (V, \overbrace{\to \cup \leftarrow}^{=: E})$ the graph reduct of $D_{\mathcal{F}}$

Observation

- $CSP(H_{\mathcal{F}}) = \{G \text{ finite graph} : G \to H_{\mathcal{F}}\}$

  $= \{G \text{ finite graph} : G \hookrightarrow H_{\mathcal{F}}\} = \left\{\begin{array}{c}\text{finite graphs that admit}\\ \mathcal{F}\text{-free orientation}\end{array}\right\}$;

  i.e. $CSP(H_{\mathcal{F}}) = OF(\mathcal{F})$

- $CSP(H_{\mathcal{F}}, \to) = OCP(\mathcal{F})$

Connection to CSPs

$\mathcal{C}_{\mathcal{F}}$ the class of finite directed $\mathcal{F}$-free graphs is finitely bounded :

$\exists k \in \mathbb{N} : \; \mathbb{B} \in \mathcal{C}_{\mathcal{F}} \Leftrightarrow$ all substructures of $\mathbb{B}$ of size $\leq k$ belong to $\mathcal{C}_{\mathcal{F}}$.

take $k = \max\{2\} \cup \{|T| : T \in \mathcal{F}\}$

Tractability Conjecture (Bodirsky, Pinsker 2011)

If $\mathbb{A}$ is f.o. reduct of a finitely bounded homogeneous structure, CSP($\mathbb{A}$) is either tractable or NP-complete.

- $H_{\mathcal{F}}$ and $(V, E, \rightarrow)$ are in the scope of this conjecture
- current proof doesn't use recent theory developed for that scope

Why reprove the Thm. by Bodirsky & Guzmán-Pro?

Smooth Approximations (Mottet, Pinsker 2020)

Use recently developed theory to redo proof for the complexity dichotomy of $OP(\mathcal{F})$ and $OCP(\mathcal{F})$ to:

- aligning proof with previous applications of the theory to provide structured overview of current situation
- make proof amenable to further generalizations:
  - → $\mathcal{F}$ not just tournaments, but other directed graphs
  - → edge coloring problems instead of edge orientation problems
- test limitations of the theory

Polymorphisms the symmetries of a CSP

- comp. complexity of CSP($\mathbb{A}$) captured by polymorphism clone $\mathrm{Pol}(\mathbb{A}) := \bigcup_{n \in \mathbb{N}} \{ h : \mathbb{A}^n \to \mathbb{A} \mid h \text{ homom.} \}$:

$\mathbb{B}$ τ-structure, $h_1, \ldots, h_n : \mathbb{B} \to \mathbb{A}$ homom. , $f \in \mathrm{Pol}(\mathbb{A})$ n-ary, then $f \circ (h_1, \ldots, h_n)$ is homom. $\mathbb{B} \to \mathbb{A}$

"Polymorphisms are symmetries of solution space $\{ h : \mathbb{B} \to \mathbb{A} \mid h \text{ homom.} \}$ of an instance $\mathbb{B}$ of CSP($\mathbb{A}$)"

few symmetries (only trivial ones) ⇒ hard CSP

non-trivial symmetries ⇒ CSP is tractable

Polymorphisms the symmetries of a CSP

THEOREM (Barto, Opršal, Pinsker 2017)

$\mathbb{A}$ ω-cat. with only trivial symmetries (exists $\phi : \underbrace{\mathrm{Pol}(\mathbb{A}) \xrightarrow{u.c.} \mathrm{Proj}}$) then CSP($\mathbb{A}$) is NP-complete.

uniformly continuous: $\exists F \subseteq_{\text{fin.}} A$ s.t. $\phi(f)$ uniquely determined by restriction to (appropriate power of) F, $\forall f \in \mathrm{Pol}(\mathbb{A})$

minion homomorphism: $\phi(f \circ (\pi^k_{i_1}, \dots, \pi^k_{i_n})) = \phi(f) \circ (\pi^k_{i_1}, \dots, \pi^k_{i_n})$, $\forall f \in \mathrm{Pol}(\mathbb{A})$

THEOREM (Bulatov, Zhuk 2017)

If $\mathbb{A}$ finite, then CSP($\mathbb{A}$) is tractable iff exists $f \in \mathrm{Pol}(\mathbb{A})$ that is cyclic, i.e. $\forall x_1, \dots, x_n$ $f(x_1, \dots, x_n) = f(x_n, x_1, \dots, x_{n-1})$.

THEOREM (F., Pinsker)

$\mathcal{F}$ finite set of tournaments, $R_1, \ldots, R_n$ relations f.o. definable in $D_{\mathcal{F}}$ consisting of tuples inducing tournaments in $H_{\mathcal{F}}$.

Exactly one of the two statements holds:

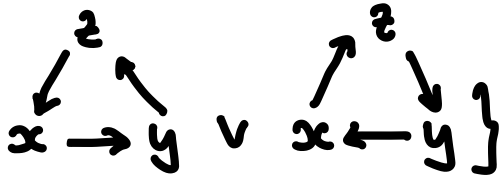
(1) $\mathbb{A} := (H_{\mathcal{F}}, R_1, \ldots, R_n)$ has only trivial symmetries, i.e.
$\exists \, \mathrm{Pol}(\mathbb{A}) \xrightarrow{u.c.} \mathrm{Proj}$, so $\mathrm{CSP}(\mathbb{A})$ is NP-complete.

(2) $\mathrm{Pol}(H_{\mathcal{F}}, R_1, \ldots, R_n)$ contains canonical ternary function that is cyclic. In this case $\mathrm{CSP}(H_{\mathcal{F}}, R_1, \ldots, R_n)$ is in P.

EXAMPLE

- no R_i : $\mathcal{F}$-free orientation problem

- $R = \{(x,y) \in V^2 : x \to y\} = \to$: orientation completion problem

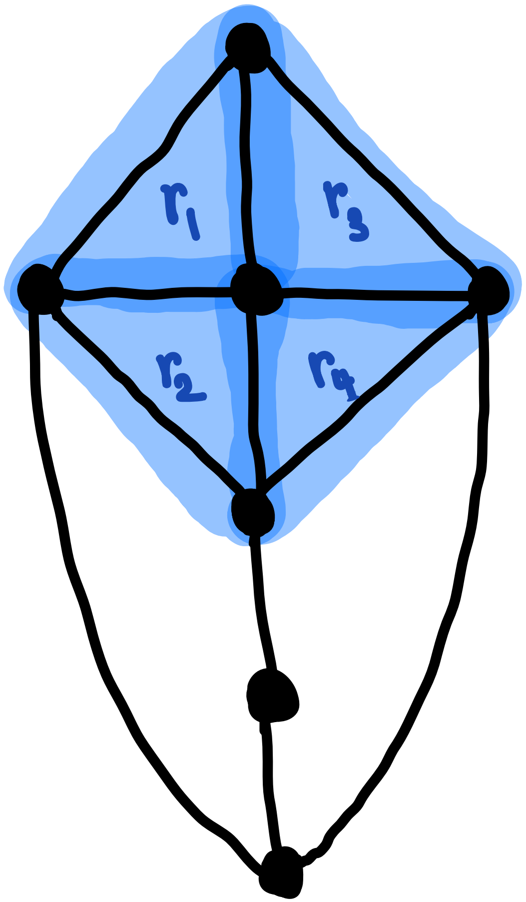
- $R = \{(x,y,z) \in V^3 :$ 

$\}$: orientation s.t. some marked 3-cliques have cyclic orientation

- $R = \left\{ (x,y,z) \in V^3 : \begin{tikzcd}[sep=small] & z \arrow[dl] & \\ x \arrow[rr] & & y \arrow[ul] \end{tikzcd} \vee \begin{tikzcd}[sep=small] & z \arrow[dr] & \\ x \arrow[ur] & & y \arrow[ll] \end{tikzcd} \right\}$: orientation s.t. some marked 3-cliques have cyclic orientation

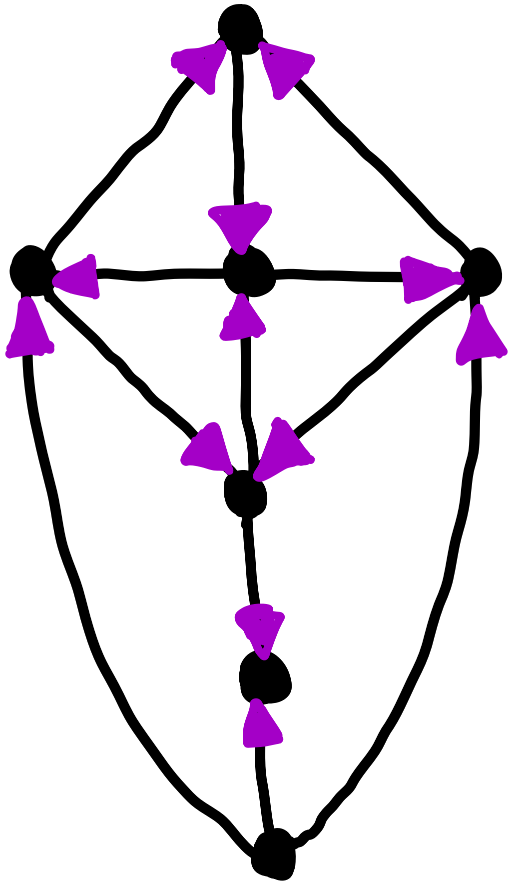
Instance to CSP(H_F, R):

$(V, E, R = \{r_1, r_2, r_3, r_4\})$



Solution to instance

(V, E, R)



HOW IS THE THEOREM PROVED?

- Follow blueprint that was used to establish complexity dichotomies for:
  - all reducts of the Rado/Henson graphs
  - all reducts of the universal homogeneous tournament (Flim(fin. tournaments))

  Mottet, Pinsker '21

  - all reducts of certain homogeneous k-Hypergraphs

  Mottet, Nagy, Pinsker '23

Idea: • Use that $CSP(H_{\mathcal{F}}, R_1, \ldots, R_n)$ can be reduced to finite "orbit CSP".

- **Problem:** orbit CSP might be harder than initial CSP
- Quite some work: This is not the case!

THE ORBIT REDUCTION FOR $\mathrm{CSP}(H_{\mathcal{F}})$

due to Bodirsky, Mottet '18

$D_{\mathcal{F}} = \mathrm{Flim}(\text{finite } \mathcal{F}\text{-free directed graphs})$

$H_{\mathcal{F}} = \text{graph reduct of } D_{\mathcal{F}}$

- G finite graph: $G \to H_{\mathcal{F}} \Leftrightarrow \exists\, \mathcal{F}\text{-free orientation } G' \text{ of } G.$
- $D_{\mathcal{F}}$ fin. bounded: $\exists k \in \mathbb{N}$ s.t. dir. graph D $\mathcal{F}$-free $\Leftrightarrow$ all subgraphs of D of size k are $\mathcal{F}$-free

$$\Rightarrow \quad G \to H_{\mathcal{F}} \leftrightsquigarrow \text{map } \phi : \overbrace{\{A \subseteq G : |A| = k\}}^{\text{domain of } I_G} \to \overbrace{\left\{\begin{array}{l}\text{oriented } \mathcal{F}\text{-free graphs}\\ \text{of size } k \text{ up to } \cong\end{array}\right\}}^{\text{domain of } \mathrm{Orb}_{D_{\mathcal{F}}}(H_{\mathcal{F}})}$$

s.t. (1) $G|_A \cong \text{graph reduct of } \phi(A)$

(2) $\phi(A)|_{A \cap A'} = \phi(A')_{A \cap A'} \quad \forall A, A' \subseteq G \text{ of size } k$

(1) + (2) are captured by (unary & binary) relations on I_G, $\mathrm{Orb}_{D_{\mathcal{F}}}(H_{\mathcal{F}})$:

maps $I_G \to \mathrm{Orb}_{D_{\mathcal{F}}}(H_{\mathcal{F}})$ satisfying (1)+(2) $\leftrightsquigarrow$ homomorphisms of relational structures $I_G \to \mathrm{Orb}_{D_{\mathcal{F}}}(H_{\mathcal{F}})$

IN TOTAL: $G \in \mathrm{CSP}(H_{\mathcal{F}}) \Leftrightarrow I_G \in \mathrm{CSP}(\mathrm{Orb}_{D_{\mathcal{F}}}(H_{\mathcal{F}}))$

THEOREM (F., Pinsker)

Exactly one of the two statements holds:

(1) $\mathbb{A} := (H_{\mathcal{F}}, R_1, \dots, R_n)$ has only trivial symmetries, i.e. $\exists\ \mathrm{Pol}(\mathbb{A}) \xrightarrow{\text{u.c.}} \mathrm{Proj}$, so $\mathrm{CSP}(\mathbb{A})$ is NP-complete.

(2) $\mathrm{Pol}(H_{\mathcal{F}}, R_1, \dots, R_n)$ contains canonical ternary function that is cyclic. In this case $\mathrm{CSP}(H_{\mathcal{F}}, R_1, \dots, R_n)$ is in P.

pf. sketch. **If in case (2)**: The canonical ternary cyclic $f \in \mathrm{Pol}(H_{\mathcal{F}}, R_1, \dots, R_n)$ induces a ternary cyclic $\tilde{f} \in \mathrm{Pol}(\mathrm{Orb}_{D_{\mathcal{F}}}(H_{\mathcal{F}}, R_1, \dots, R_n))$. Bulatov, Zhuk $\Rightarrow$ Orbit CSP in P,
$\Rightarrow$ $\mathrm{CSP}(H_{\mathcal{F}}, R_1, \dots, R_n)$ in P.

If not in case (2): elementary (tedious) combinatorial arguments + compactness of f.o. logic $\Rightarrow$ $\underbrace{\mathrm{Pol}(H_{\mathcal{F}}, R_1, \dots, R_n)^{\text{can}}}_{\text{subclone}} \leq \mathrm{Pol}(H_{\mathcal{F}}, R_1, \dots, R_n)$ has u.c. clone homomorphism into Proj

$$\phi : \mathrm{Pol}(H_{\mathcal{F}}, R_1, \dots, R_n)^{\text{can}} \xrightarrow{\text{u.c.}} \mathrm{Proj}$$

Smooth Approximations: $\exists$ mapping $\psi : \mathrm{Pol}(H_{\mathcal{F}}, R_1, \dots, R_n) \xrightarrow{\text{u.c.}} \mathrm{Proj}$ s.t.
$\phi \circ \psi : \mathrm{Pol}(H_{\mathcal{F}}, R_1, \dots, R_n) \xrightarrow{\text{u.c.}} \mathrm{Proj}$ clone homomorphism, especially minion homomorphism.

Thank you!

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