A complexity dichotomy for graph orientation problems

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The graph orientation problem

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Fix $\mathcal{F}$ finite set of finite directed graphs

GOP($\mathcal{F}$) the "$\mathcal{F}$-free graph orientation problem":

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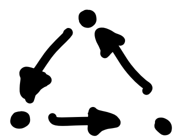
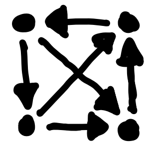
EXAMPLES ① $\mathcal{F} = \{$  ,  $\}$

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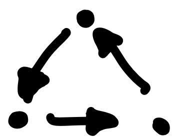
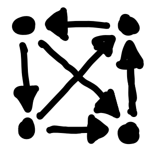
EXAMPLES ① $\mathcal{F} = \{$  ,  $\}$ OUTPUT: always YES :

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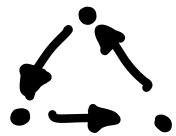
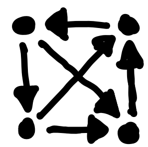
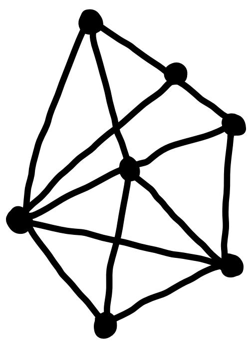
EXAMPLES ① $\mathcal{F} = \{$  ,  $\}$ OUTPUT: always YES :

G : 

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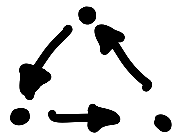
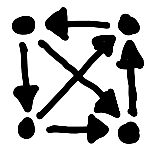
EXAMPLES ① $\mathcal{F} = \{$  ,  $\}$ OUTPUT: always YES:

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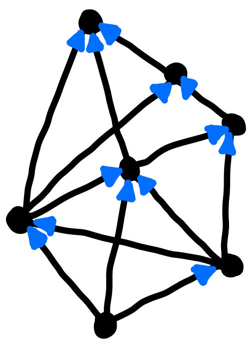

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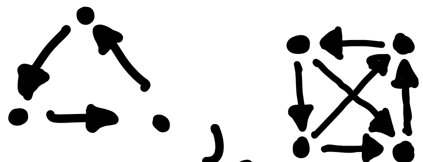
EXAMPLES ① $\mathcal{F} = \{$  $\}$ OUTPUT: always YES:

Acyclic orientation always possible!

solvable in constant time

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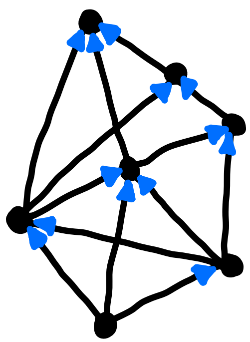

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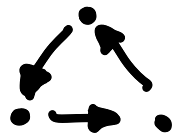
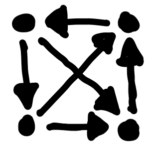
EXAMPLES ① $\mathcal{F} = \{$  ,  $\}$ OUTPUT: always YES:

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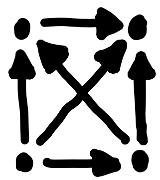
② $\mathcal{F} = \{$  C_3 ,  T_4 $\}$

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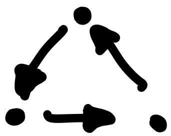
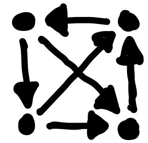
① $\mathcal{F} = \{$  ,  $\}$ OUTPUT: always YES:

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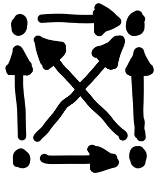
② $\mathcal{F} = \{$  ,  $\}$

C_3 , T_4

OUTPUT: YES iff input graph does not contain a 4-clique.

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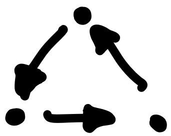
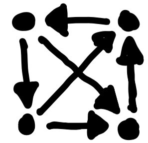
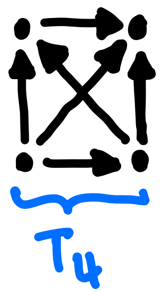
EXAMPLES ① $\mathcal{F} = \{$  ,  $\}$ OUTPUT: always YES: solvable in constant time

Acyclic orientation always possible!

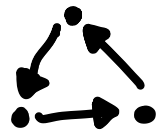
② $\mathcal{F} = \{$  ,  $\}$ OUTPUT: YES iff input graph does not contain a 4-clique. tractable

C_3 or T_4 embeds into every orient. of K_4. O/w acyclic orientation possible!

The graph orientation problem

EXAMPLES ③ $$\mathcal{F} = \left\{ \bullet \to \bullet \to \bullet \;,\; \begin{tikzcd}[column sep=small, row sep=small] & \bullet \arrow[dl] & \\ \bullet \arrow[rr] & & \bullet \arrow[ul] \end{tikzcd} \right\}$$

The graph orientation problem

EXAMPLES ③ $F = \{ \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow ,$  $\}$

follows from Gallai-Hasse-Roy-Vitaver Thm

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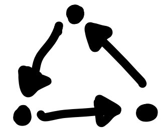
EXAMPLES ③ $\mathcal{F} = \{ \bullet \to \bullet \to \bullet ,$  $\}$

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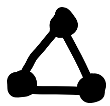
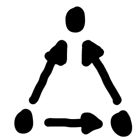
$h : G \longrightarrow$  $\rightsquigarrow$ orientation G' s.t. $h : G' \to$ 

$\updownarrow$

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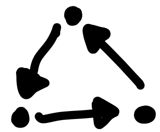
EXAMPLES ③ $\mathcal{F} = \{ \bullet \to \bullet \to \bullet \to ,$  $\}$

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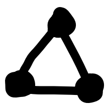
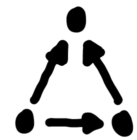
$h: G \to$  ⇝ orientation G' s.t. $h: G' \to$ 

$\mathcal{F}$-free orientation G'.

THEOREM (Bodirsky, Guzmán-Pro 2023)

If $\mathcal{F}$ consists of tournaments only, then $\mathrm{GOP}(\mathcal{F})$ is either tractable or NP-complete.

Connection to CSPs

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Recall : $\mathbb{A}$ rel. structure with signature τ

- $\mathrm{CSP}(\mathbb{A}) = \{\mathbb{B}$ finite τ-structure : $\mathbb{B} \to \mathbb{A}\}$
- $\mathrm{CSP}(\mathbb{A})$ is also the decision problem: $\mathbb{B} \in \mathrm{CSP}(\mathbb{A})$?

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③ $CSP(\mathbb{Z}, \{0\}, \{1\}, \underbrace{+, \cdot}_{\text{ternary relations}}) =$ "decide if Diophantine equation has integer solution."

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$h: G \to H_{\mathcal{F}}$

pull back orientation $D_{\mathcal{F}}$ of $H_{\mathcal{F}}$ along h

$G' \hookrightarrow D_{\mathcal{F}}$,

so $G \hookrightarrow H_{\mathcal{F}}$

$\mathcal{F}$-free orientation G' of G

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Connection to CSPs

$\mathcal{C}_{\mathcal{F}}$ the class of finite directed $\mathcal{F}$-free graphs is **finitely bounded**:

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Conjecture (Bodirsky, Pinsker 2011)

If A is f.o. reduct of a **finitely bounded** homogeneous structure, CSP(A) is either tractable or NP-complete.

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If $\mathbb{A}$ is f.o. reduct of a **finitely bounded** homogeneous structure, CSP($\mathbb{A}$) is either tractable or NP-complete.

- $\mathbb{H}_\mathcal{F}$ is in scope of conjecture
- current proof doesn't use recent theory developed for that scope

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- provide good challenge for beginning PhD student

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- comp. complexity of CSP(A) captured by polymorphism clone $\mathrm{Pol}(A) := \bigcup_{n\in\mathbb{N}} \{h : A^n \to A \mid h \text{ homom.}\}$

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few symmetries (only trivial ones) $\Rightarrow$ hard CSP

non-trivial symmetries $\Rightarrow$ CSP is tractable

Polymorphisms the symmetries of a CSP

THEOREM (Barto, Opršal, Pinsker 2017)

If $\mathbb{A}$ ω-cat. with only trivial symmetries (exists $\mathrm{Pol}(\mathbb{A}) \xrightarrow{u.c.} \mathrm{Proj}$) then $\mathrm{CSP}(\mathbb{A})$ is NP-complete.

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THEOREM (Bulatov, Zhuk 2017)

If $\mathbb{A}$ finite, then $\mathrm{CSP}(\mathbb{A})$ is tractable iff exists $f \in \mathrm{Pol}(\mathbb{A})$ that is cyclic, i.e. $\forall x_1, \dots, x_n \; f(x_1, \dots, x_n) = f(x_n, x_1, \dots, x_{n-1})$.

THEOREM (F., Pinsker)

$\mathcal{F}$ finite set of tournaments. Exactly one of the two holds:

(1) $H_{\mathcal{F}}$ has only trivial symmetries (exists $\mathrm{Pol}(H_{\mathcal{F}}) \xrightarrow{u.c.} \mathrm{Proj}$), so $\mathrm{CSP}(H_{\mathcal{F}}) = \mathrm{GOP}(\mathcal{F})$ is NP-complete.

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<u>pf. sketch</u>. If in case (2): Reduce $CSP(H_{\mathcal{F}})$ to $CSP(\mathbb{A})$, where $\mathbb{A}$ is finite orbit structure.

Problem: $CSP(\mathbb{A})$ in general harder than $CSP(H_{\mathcal{F}})$.

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<u>pf. sketch</u>. If in case (2): Reduce $\mathrm{CSP}(H_{\mathcal{F}})$ to $\mathrm{CSP}(\mathbb{A})$, where $\mathbb{A}$ is finite orbit structure.

Problem: $\mathrm{CSP}(\mathbb{A})$ in general harder than $\mathrm{CSP}(H_{\mathcal{F}})$.

But: $\mathrm{Pol}(H_{\mathcal{F}})$ contains canonical ternary function, inducing cyclic polymorphism of $\mathbb{A}$.

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$\mathcal{F}$ finite set of tournaments. Exactly one of the two holds:

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Bulatov, Zhuk $\Rightarrow$ $\mathrm{CSP}(\mathbb{A})$ tractable, in particular $\mathrm{CSP}(H_{\mathcal{F}})$.

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Smooth Approx.: $\exists$ mapping $\psi : \mathrm{Pol}(H_{\mathcal{F}}) \to \mathrm{Pol}(H_{\mathcal{F}})^{\mathrm{can}}$ s.t.

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$\phi \circ \psi : \mathrm{Pol}(H_{\mathcal{F}}) \xrightarrow{\text{u.c.}} \mathrm{Proj}$. Barto, Opršal, Pinsker $\Rightarrow$ $\mathrm{CSP}(H_{\mathcal{F}})$ NP-c.

Thank you!

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