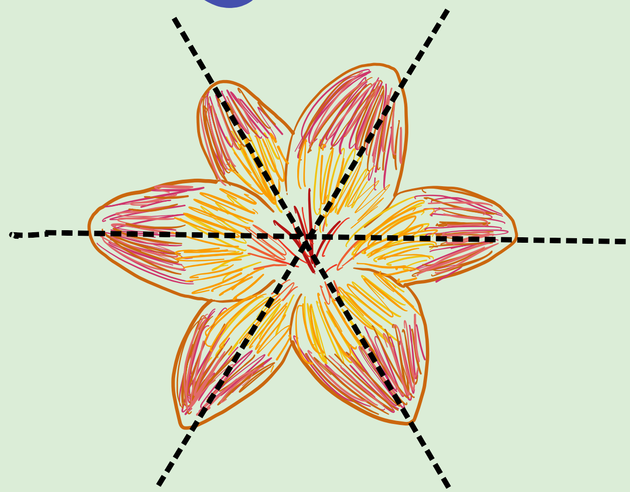
Symmetries describing equational non-triviality

Johanna Brunar
TU Wien



Algebra Week '24
Siena

European Research Council
Established by the European Commission

ÖAW
ÖSTERREICHISCHE AKADEMIE DER WISSENSCHAFTEN

PART I

STRUCTURE
OF AN ALGEBRA

$\updownarrow$

SATISFACTION OF
IDENTITIES

PART II

LOOP CONDITIONS

PART I

STRUCTURE OF AN ALGEBRA

$\updownarrow$

SATISFACTION OF IDENTITIES

- EQUATIONAL NON-TRIVIALITY
- CSP
- DICHOTOMY-THEOREM

PART II

LOOP CONDITIONS

$\underline{A} = (A; (f_i)_{i \in I})$ ALGEBRA

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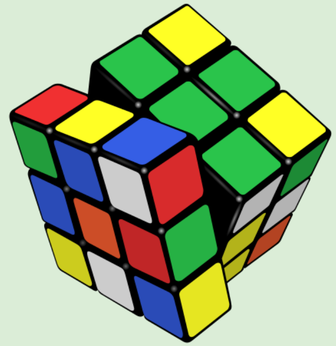
WE ARE ONLY INTERESTED IN $\mathrm{Clo}(\underline{A})$ = TERM OPERATIONS OF $\underline{A}$

$\underline{A} = (A; (f_i)_{i \in I})$ ALGEBRA

WE ARE ONLY INTERESTED IN $Clo(\underline{A})$ = TERM OPERATIONS OF $\underline{A}$

$\boxed{\text{EXAMPLES}}$



- GROUPS: $(A; +, -)$
- SEMIGROUPS: $(A; \circ)$
- VECTOR SPACES OVER A FIXED FIELD $\underline{F}$: $(A; +, (\lambda)_{\lambda \in F})$
- ROCK-PAPER-SCISSORS

$(\{\text{ROCK}, \text{PAPER}, \text{SCISSORS}\}; \text{WINNER}(x, y))$

$\underline{A} = (A; (f_i)_{i \in I})$ ALGEBRA

"STRUCTURE" OF $\underline{A}$ $\longleftrightarrow$ IDENTITIES THAT HOLD IN $\underline{A}$

$\underline{A} = (A; (f_i)_{i \in I})$ ALGEBRA

MEANING VARIES DEPENDING ON CONTEXT

"STRUCTURE" OF $\underline{A}$ $\leftrightsquigarrow$ IDENTITIES THAT HOLD IN $\underline{A}$

$\underline{A} = (A; (f_i)_{i \in I})$ ALGEBRA

MEANING VARIES DEPENDING ON CONTEXT

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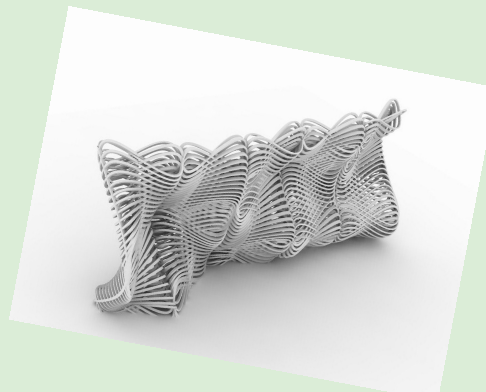
EXAMPLES

- VARIETY GENERATED BY $\underline{A}$
- CONGRUENCES OF $\underline{A}$
- SUBALGEBRAS OF $\underline{A}^n$



$\underline{A} = (A; (f_i)_{i \in I})$ ALGEBRA

MEANING VARIES DEPENDING ON CONTEXT

ABSTRACT EXPRESSION

$u(x_1 \ldots x_m) \approx v(y_1 \ldots y_n)$

"STRUCTURE" OF $\underline{A}$ $\quad \leftrightsquigarrow \quad$ IDENTITIES THAT HOLD IN $\underline{A}$

$\underline{A} = (A; (f_i)_{i \in I})$ ALGEBRA

MEANING VARIES DEPENDING ON CONTEXT

ABSTRACT EXPRESSION

$u(x_1 \dots x_m) \approx v(y_1 \dots y_n)$

"STRUCTURE" OF $\underline{A}$ $\leftrightsquigarrow$ IDENTITIES THAT HOLD IN $\underline{A}$

EXAMPLES

- $f(x,y) \approx f(y,x)$
- $u(u(x,y),z) \approx u(x,u(y,z))$
- $m(x,y,y) \approx m(y,y,x) \approx x$

$\underline{A} = (A; (f_i)_{i \in I})$ ALGEBRA

MEANING VARIES DEPENDING ON CONTEXT

ABSTRACT EXPRESSION

$u(x_1 \dots x_m) \approx v(y_1 \dots y_n)$

"STRUCTURE" OF $\underline{A}$ $\longleftrightarrow$ IDENTITIES THAT HOLD IN $\underline{A}$

$$\underline{A} \models u(x_1 \dots x_m) \approx v(y_1 \dots y_m)$$

$$:\Leftrightarrow \exists \bar{u}, \bar{v} \in \mathrm{Clo}(\underline{A})\ \forall x_1 \dots x_m, y_1 \dots y_n \in A: \\ (\bar{u}(x_1 \dots x_m) = \bar{v}(y_1 \dots y_m))$$

$\underline{A} = (A; (f_i)_{i \in I})$ ALGEBRA

MEANING VARIES DEPENDING ON CONTEXT

ABSTRACT EXPRESSION $u(x_1 \dots x_m) \approx v(y_1 \dots y_n)$

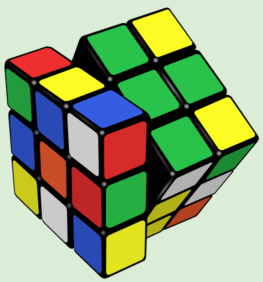
"STRUCTURE" OF $\underline{A}$ $\longleftrightarrow$ IDENTITIES THAT HOLD IN $\underline{A}$

$$\underline{A} \models u(x_1 \dots x_m) \approx v(y_1 \dots y_m)$$

$$:\Leftrightarrow \exists\, \bar{u}, \bar{v} \in Clo(\underline{A})\ \forall x_1 \dots x_m, y_1 \dots y_n \in A: \left(\bar{u}(x_1 \dots x_m) = \bar{v}(y_1 \dots y_m) \right)$$

EXAMPLE: GROUPS

$\underline{A} = (A, +, -)$

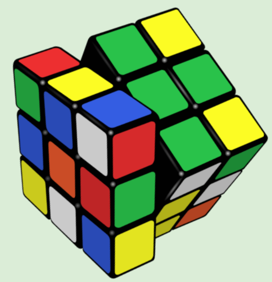


IDENTITIES

- $\underline{A} \models f(x,y) \approx f(y,x)$?
- $\underline{A} \models u(u(x,y),z) \approx u(x,u(y,z))$?
- $\underline{A} \models m(x,y,y) \approx m(y,y,x) \approx x$?

EXAMPLE: GROUPS

$\underline{A} = (A; +, -, 0)$



IDENTITIES

- $\underline{A} \models f(x,y) \approx f(y,x)$?
- $\underline{A} \models u(u(x,y),z) \approx u(x,u(y,z))$?
- $\underline{A} \models m(x,y,y) \approx m(y,y,x) \approx x$?

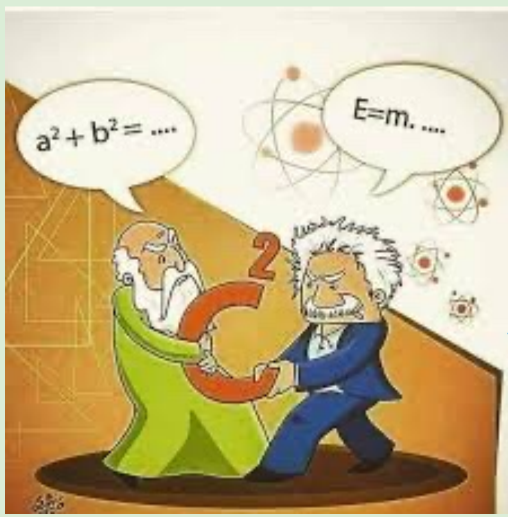
$\overline{m}(x,y,z) := x-y+z \in \mathrm{Clo}(\underline{A})$

then $\forall x,y,z \in A: \overline{m}(x,y,y) = \overline{m}(y,y,x) = x$

"STRUCTURE" OF $\underline{A}$ $\longleftrightarrow$ IDENTITIES THAT HOLD IN $\underline{A}$

BIRKHOFF '35

VARIETY GENERATED BY $\underline{A}$ $\longleftrightarrow$ IDENTITIES SATISFIED BY $\underline{A}$

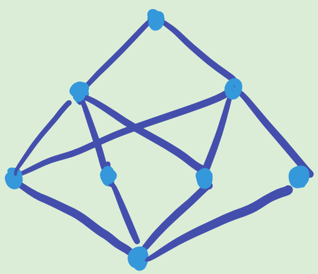


"STRUCTURE" OF $\underline{A}$ $\leftrightsquigarrow$ IDENTITIES THAT HOLD IN $\underline{A}$

SHAPE OF CON($\underline{A}$)

KEARNES + KISS '99 $\leftrightsquigarrow$ HOBBY + MCKENZIE '88

HOBBY-MCKENZIE TERM



"STRUCTURE" OF $\underline{A}$ $\leftrightsquigarrow$ IDENTITIES THAT HOLD IN $\underline{A}$

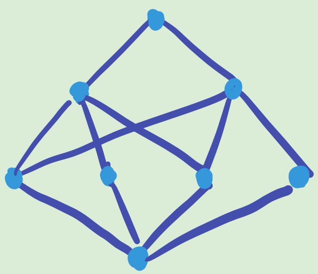
SHAPE OF CON($\underline{A}$)

eg:
MAL'CEV '54

$\leftrightsquigarrow$

KEARNES + KISS '99

CERTAIN SETS OF IDENTITIES



"STRUCTURE" OF $\underline{A}$

IDENTITIES THAT HOLD IN $\underline{A}$

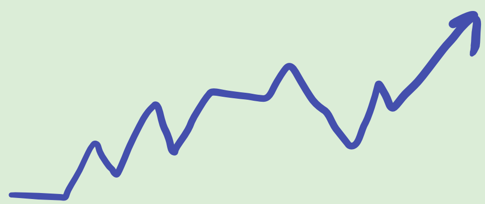
GROWTH RATE IN n

$\#\{$SUBALGS OF $\underline{A}^n\}$

BERMAN+
IDZIAK+
MARKOVIĆ+

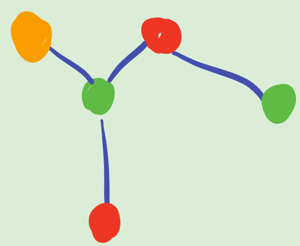
MCKENZIE+
VALERIOTE+
WILLARD
'08

K-EDGE TERM



"STRUCTURE" OF $\underline{A}$ $\longleftrightarrow$ IDENTITIES THAT HOLD IN $\underline{A}$

COMPLEXITY OF CSP($\underline{A}$) $\xleftrightarrow[\text{ZHUK '17}]{\text{BULATOV '17}}$ EQUATIONAL NON-TRIVIALITY



"STRUCTURE" OF $\underline{A}$

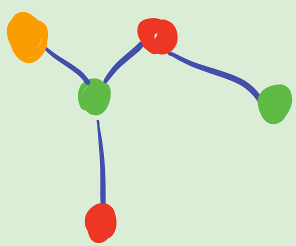
IDENTITIES THAT HOLD IN $\underline{A}$

COMPLEXITY OF CSP($\underline{A}$)

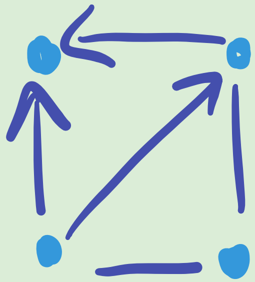
BULATOV '17

ZHUK '17

EQUATIONAL NON-TRIVIALITY



$\mathbb{A} = (A; (R_i)_{i \in I})$ RELATIONAL STRUCTURE



(POSSIBLY INFINITE)

CSP($\mathbb{A}$):

- INPUT: $\mathbb{X}$, finite
- QUESTION: DOES THERE EXIST A HOMOMORPHISM $\mathbb{X} \to \mathbb{A}$

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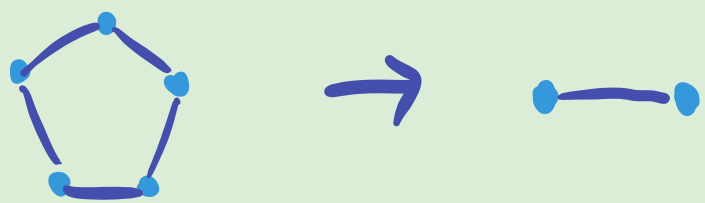
MAPS EDGES TO EDGES

$\mathbb{A}$ = 

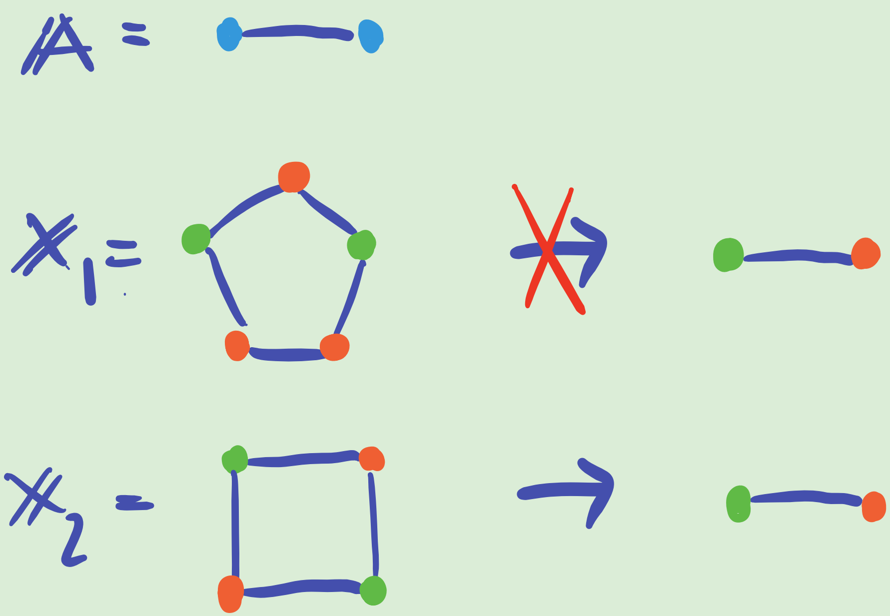
$\mathbb{X}_1$ = 

$\mathbb{X}_2$ = 

CSP($\mathbb{A}$):

- INPUT: $\mathbb{X}$, finite
- QUESTION: DOES THERE EXIST A HOMOMORPHISM $\mathbb{X} \to \mathbb{A}$

MAPS EDGES TO EDGES



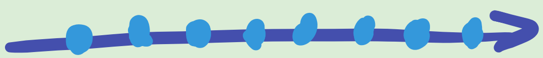
2-COLOURING!

$CSP(\mathbb{A})$:

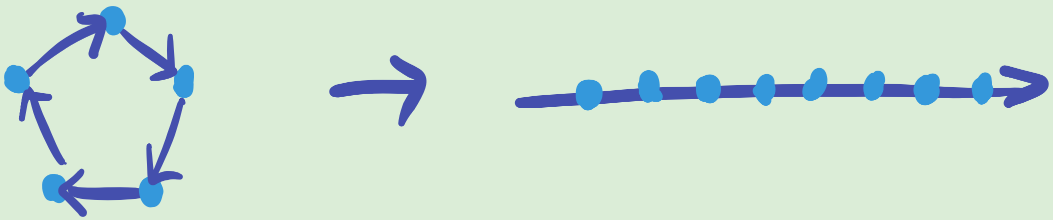
- INPUT: $\mathbb{X}$, finite
- QUESTION: DOES THERE EXIST A HOMOMORPHISM $\mathbb{X} \to \mathbb{A}$

MAPS EDGES TO EDGES

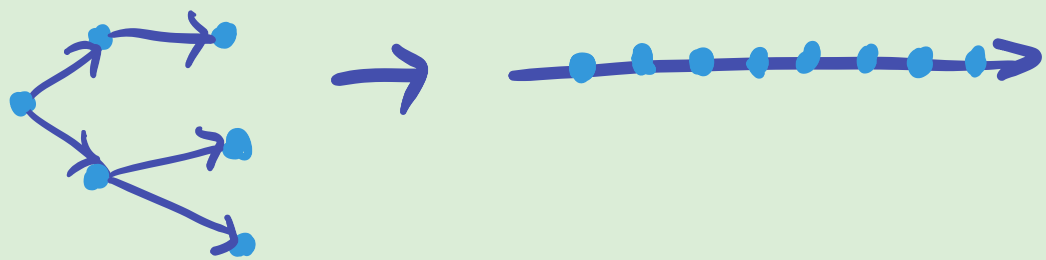
$\mathbb{A} = (\mathbb{Z}; <)$



$\mathbb{X}_1 =$



$\mathbb{X}_2 =$



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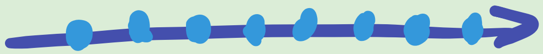
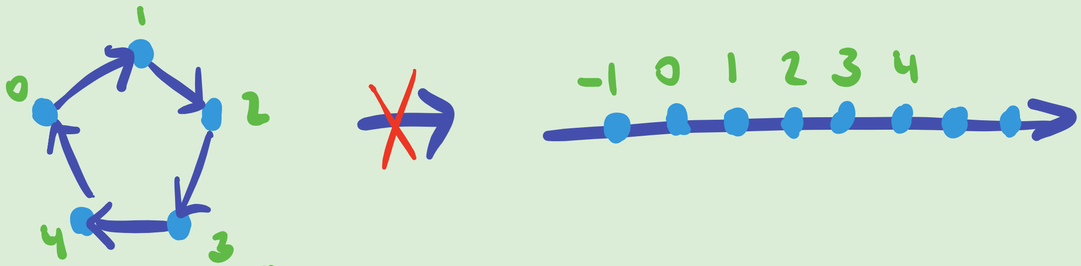
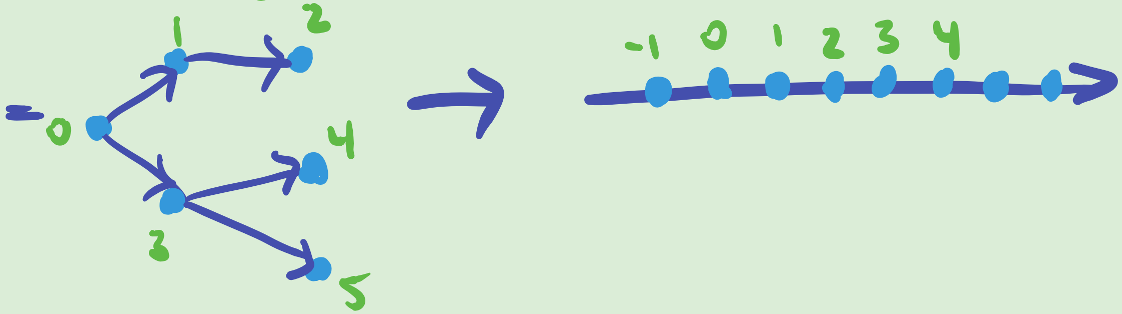


$\mathbb{X}_1 =$



$\mathbb{X}_2 =$



DIGRAPH-ACYCLICITY!

CSP($\mathbb{A}$):

- INPUT: $\mathbb{X}$, finite
- QUESTION: DOES THERE EXIST A HOMOMORPHISM $\mathbb{X} \to \mathbb{A}$

- $\mathbb{A}$ =  2-COLOURING P

CSP($\mathbb{A}$):

- INPUT: $\mathbb{X}$, finite
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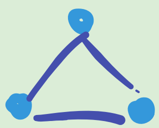
- $\mathbb{A}$ =  2-COLOURING P
- $\mathbb{A}$ =  3-COLOURING NP-C LOVÁSZ '73

CSP($\mathbb{A}$):

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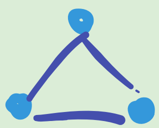
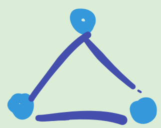
- $\mathbb{A}$ =  2-COLOURING P
- $\mathbb{A}$ =  3-COLOURING NP-C LOVÁSZ '73
- $\mathbb{A}$ FINITE, LOOPLESS UNDIRECTED: $\mathbb{A}$-COLOURING

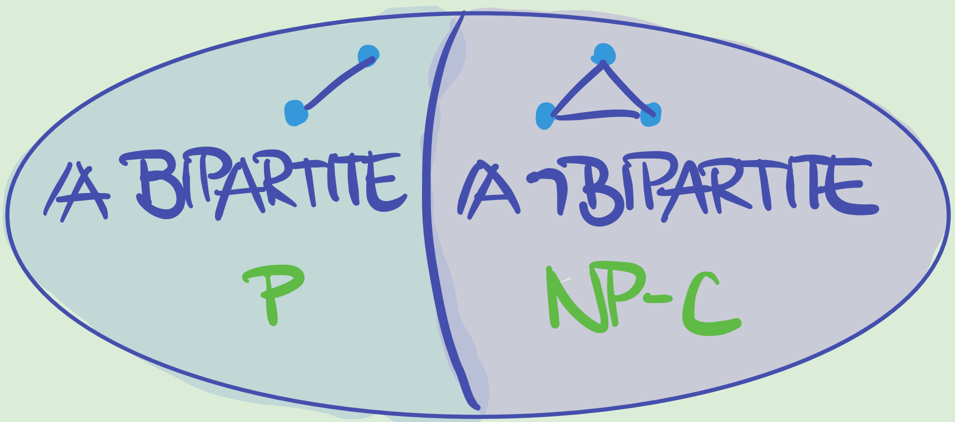
CSP(A):

- INPUT: X, finite
- QUESTION: DOES THERE EXIST A HOMOMORPHISM $X \to A$

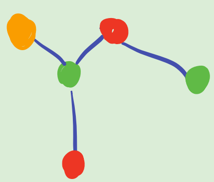
- A =  2-COLOURING P
- A =  3-COLOURING NP-C LOVÁSZ '73

THM: A FINITE LOOPLESS UNDIRECTED.

COMPLEXITY OF A-COLOURING:



HELL + NEŠETRIL '90



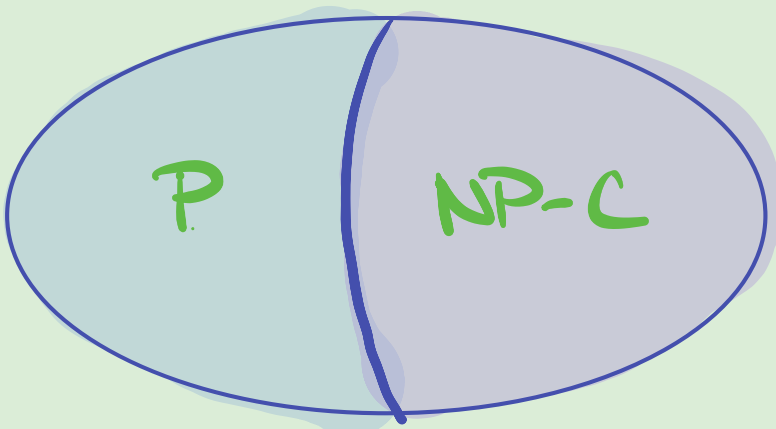
DICHOTOMY THEOREM

'93 CONJECTURE BY FEDER + VARDI

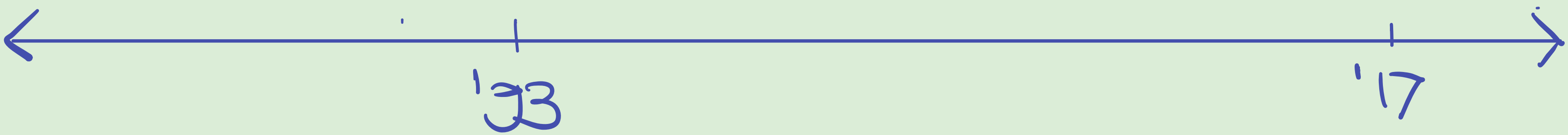
'17 PROOF BY BULATOV, ZHUK

$\mathbb{A}$ FINITE.

THEN CSP($\mathbb{A}$) IS TRACTABLE OR NP-COMPLETE.



HOW DID THE DICHOTOMY CONJECTURE BECOME A THEOREM?



ALGEBRAIC APPROACH TO CSP:

$\mathrm{Pol}(\mathbb{A})$ = CLONE OF OPERATIONS THAT LEAVE ALL RELATIONS FROM $\mathbb{A}$ INVARIANT

= HOMOMORPHISMS $\mathbb{A}^n \to \mathbb{A}$ $\quad n \geq 1$

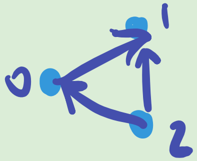
HOW DID THE DICHOTOMY CONJECTURE BECOME A THEOREM?

ALGEBRAIC APPROACH TO CSP:

$\mathrm{Pol}(\mathbb{A})$ = CLONE OF OPERATIONS THAT LEAVE ALL RELATIONS FROM $\mathbb{A}$ INVARIANT

= HOMOMORPHISMS $\mathbb{A}^n \to \mathbb{A}$ $\quad n \geq 1$

EXAMPLE $\quad \mathbb{A} =$



$\max, \min : \mathbb{A}^2 \to \mathbb{A}$

$$\max\begin{pmatrix} 0 & 2 \\ \downarrow & \downarrow \\ 1 & 0 \end{pmatrix} = \begin{matrix} 2 \\ \downarrow \\ 1 \end{matrix} \qquad \max\begin{pmatrix} 0 & 2 \\ \downarrow & \downarrow \\ 1 & 1 \end{pmatrix} = \begin{matrix} 2 \\ \downarrow \\ 1 \end{matrix} \qquad \max\begin{pmatrix} 2 & 2 \\ \downarrow & \downarrow \\ 0 & 1 \end{pmatrix} = \begin{matrix} 2 \\ \downarrow \\ 1 \end{matrix} \qquad \min\begin{pmatrix} 0 & 2 \\ \downarrow & \downarrow \\ 1 & 0 \end{pmatrix} = \begin{matrix} 0 \\ \cancel{\downarrow} \\ 0 \end{matrix}$$

HOW DID THE DICHOTOMY CONJECTURE BECOME A THEOREM?

ALGEBRAIC APPROACH TO CSP:

$\mathrm{Pol}(\mathbb{A})$ = CLONE OF OPERATIONS THAT LEAVE ALL RELATIONS FROM $\mathbb{A}$ INVARIANT

= HOMOMORPHISMS $\mathbb{A}^n \to \mathbb{A}$, $n \geq 1$

IF $f : \mathbb{A}^n \to \mathbb{A}$, $g_1 \cdots g_n : \mathbb{X} \to \mathbb{A}$ THEN $f \circ (g_1 \cdots g_n) : \mathbb{X} \to \mathbb{A}$

$\Rightarrow$ $\mathrm{POL}(\mathbb{A})$ CAN BE USED TO STUDY $\mathrm{CSP}(\mathbb{A})$

HOW DID THE DICHOTOMY CONJECTURE BECOME A THEOREM?

'93   '05   '17

ALGEBRAIC APPROACH TO CSP:

Thm: BULATOV + JEAVONS + KROKHIN '05

- COMPLEXITY OF CSP(A) IS COMPLETELY DETERMINED BY THE EQUATIONAL VARIETY GENERATED BY Pol(A)

⇝ STUDY IDENTITIES SATISFIED BY Pol(A)

- ENOUGH TO ONLY CONSIDER IDEMPOTENT POLYMORPHISMS i.e. $f(x \cdots x) \approx x$

EQUATIONAL (NON-) TRIVIALITY

- SET OF IDENTITIES IS NON-TRIVIAL IF IT CANNOT BE WITNESSED BY PROJECTIONS

$\boxed{\text{Ex:}}$ $m(x,y,y) \approx m(y,y,x) \approx x$ (MAL'CEV)

$f(x,y) \approx f(y,x)$ (COMMUTATIVITY)

$\boxed{\text{Non-Ex:}}$ $f(f(x,y),z) \approx f(x,f(x,y))$ (ASSOCIATIVITY)

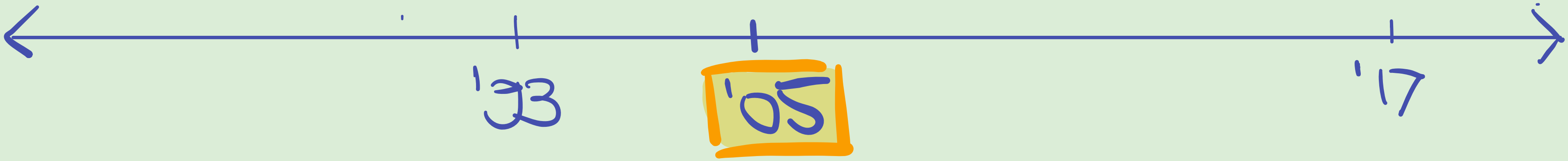
- $\mathcal{C}$ CLONE IS EQUATIONALLY NON-TRIVIAL IF $\exists \Sigma$ FINITE SET OF NON-TRIVIAL IDENTITIES S.T. $\mathcal{C} \models \Sigma$

$\boxed{\text{Ex:}}$ GROUPS: $\overline{m}(x,y,z) := x - y + z$

ROCK-PAPER-SCISSORS: $\text{winner}(x,y)$

$\boxed{\text{Non-Ex:}}$ SEMIGROUPS

HOW DID THE DICHOTOMY CONJECTURE BECOME A THEOREM?



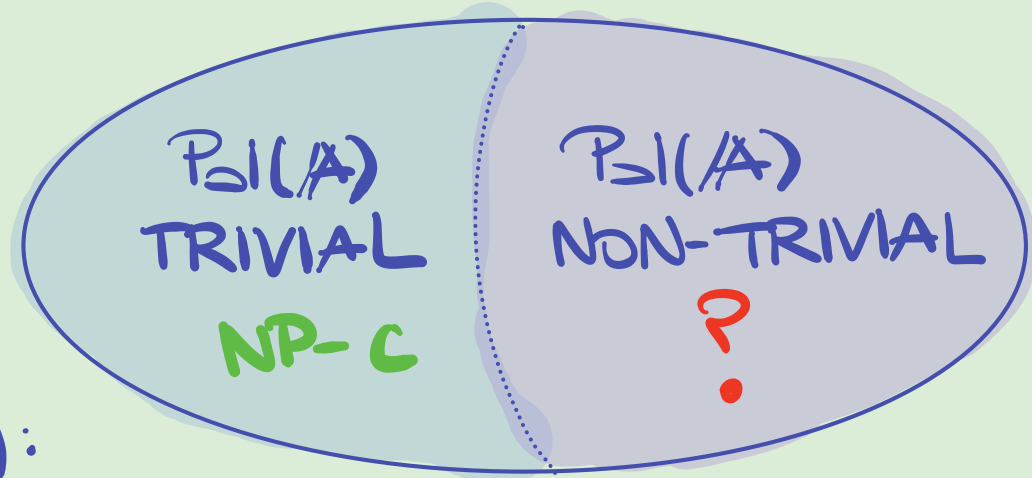
ALGEBRAIC APPROACH TO CSP:

<u>THM</u>: BULATOV + JEAVONS + KROKHIN '05

A FINITE

Pol(A) IDEMPOTENT.

COMPLEXITY OF CSP(A):



HOW DID THE DICHOTOMY CONJECTURE BECOME A THEOREM?

'77 '93 '05 '17

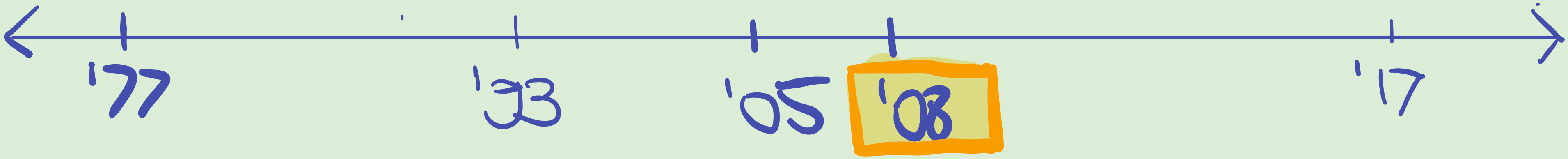
<u>THM</u>: TAYLOR '77

$\underline{A}$ FINITE, $Clo(\underline{A})$ IDEMPOTENT

THEN $Clo(\underline{A})$ IS EQU. NON-TRIVIAL IFF IT SATISFIES IDENTITY OF THE FORM

$$+\begin{pmatrix} x & * & \cdots & & * \\ * & x & * & \cdots & * \\ & & \ddots & & \vdots \\ & & & \ddots & * \\ * & * & \cdots & * & x \end{pmatrix} \approx +\begin{pmatrix} y & * & \cdots & & * \\ * & y & * & \cdots & * \\ & & \ddots & & \vdots \\ & & & \ddots & * \\ * & * & \cdots & * & y \end{pmatrix} \leftarrow \text{TAYLOR TERM}$$

HOW DID THE DICHOTOMY CONJECTURE BECOME A THEOREM?



THM: MARÓTI + MCKENZIE '08

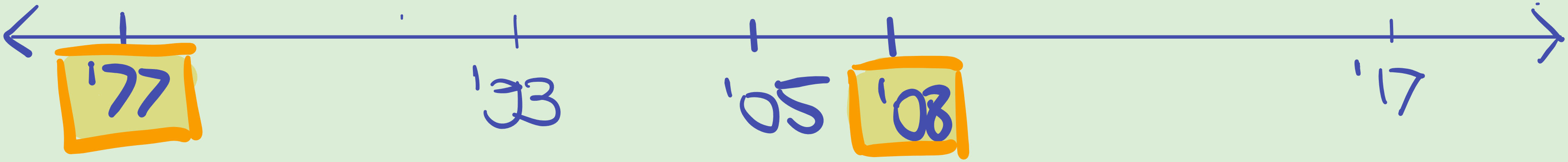
$\underline{A}$ FINITE, $Clo(\underline{A})$ IDEMPOTENT

THEN $Clo(\underline{A})$ IS EQU. NON-TRIVIAL IFF IT SATISFIES IDENTITY OF THE FORM

$$w(xx\cdots xy) \approx w(xx\cdots xyx) \approx \ldots \approx w(yx\cdots x)$$

WNU-TERM

HOW DID THE DICHOTOMY CONJECTURE BECOME A THEOREM?



<u>THM</u>: $\underline{A}$ FINITE, $Clo(\underline{A})$ IDEMPOTENT. TFAE:

- $Clo(\underline{A})$ IS $\boxed{\text{EQUATIONALLY NON-TRIVIAL}}$
- ∃ TAYLOR TERM $t(x,*\dots*) \approx t(y,*\dots*), \dots$ TAYLOR '77
- ∃ WNU TERM $w(xx\cdots xy) \approx \dots \approx w(yx\cdots x)$ MARÓTI + McKENZIE '08

HOW DID THE DICHOTOMY CONJECTURE BECOME A THEOREM?

'77 '93 '05 '08 '10 '17

THM: $\underline{A}$ finite, $\mathrm{Clo}(\underline{A})$ idempotent. TFAE:

- $\mathrm{Clo}(\underline{A})$ is EQUATIONALLY NON-TRIVIAL
- $\exists$ TAYLOR TERM $t(x,*\dots*) \approx t(y,*\cdots*), \dots$ — TAYLOR '77
- $\exists$ WNU TERM $w(xx\cdots xy) \approx \dots \approx w(yx\cdots x)$ — MARÓTI + McKENZIE '08
- $\exists$ SIGGERS TERM $s(x,y,z,x,y,z) \approx s(y,x,x,z,z,y)$ — SIGGERS '10

HOW DID THE DICHOTOMY CONJECTURE BECOME A THEOREM?

'77   '93   '05   '08   '10   '12   '17

<u>THM</u>: $\underline{A}$ FINITE, $\mathrm{Clo}(\underline{A})$ IDEMPOTENT. TFAE:

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- $\exists$ CYCLIC TERM $c(x_1\cdots x_p) \approx c(x_2\cdots x_p,x_1)$ — BARTO + KOZIK '12

HOW DID THE DICHOTOMY CONJECTURE BECOME A THEOREM?

'77 '93 '05 '08 '10 '12 '15 '17

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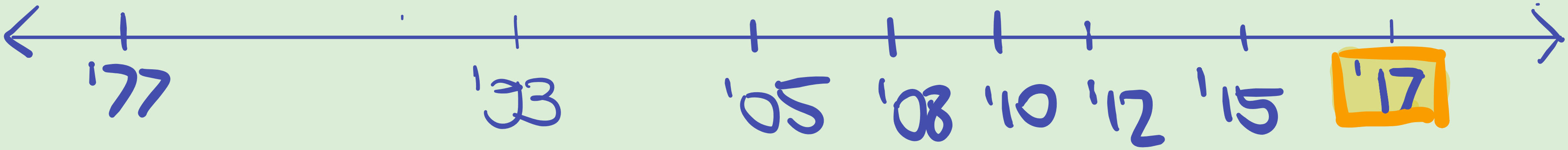
- $\exists$ CYCLIC TERM $c(x_1\cdots x_p) \approx c(x_2\cdots x_p,x_1)$ — BARTO + KOZIK '12
- $\exists$ 4-ARY SIGGERS TERM $s(a,r,e,a) \approx s(r,a,r,e)$ — KEARNES + MARKOVIĆ + MCKENZIE '15

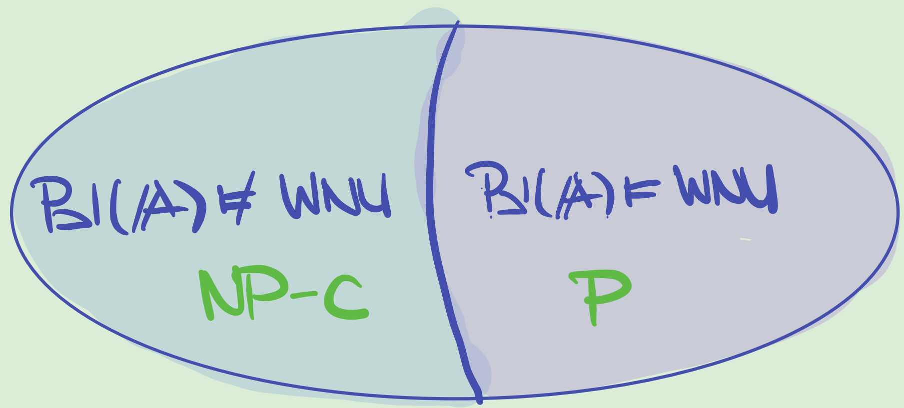
HOW DID THE DICHOTOMY CONJECTURE BECOME A THEOREM?



THM: ZHUK '17

A FINITE, Pol(A) IDEMPOTENT, $\boxed{\text{Pol(A)} \models \text{WNU.}}$
THEN CSP(A) IS TRACTABLE.

COMPLEXITY OF CSP(A):



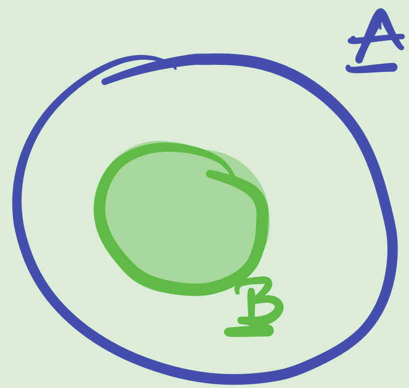
ZHUK'S PROOF OF THE DICHOTOMY THEOREM

THM: ZHUK'S CASES — ZHUK '17

$\underline{A}$ FINITE, $Clo(\underline{A})$ IDEMPOTENT, EQU. NON-TRIVIAL

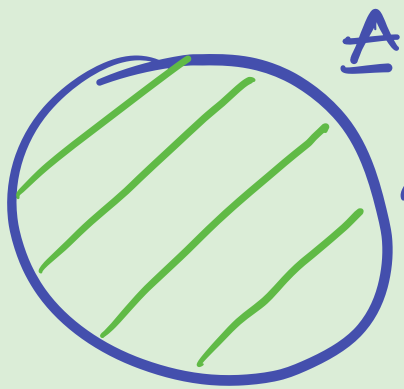
THEN EITHER

(i) $\underline{A}$ HAS NON-TRIVIAL STRONG SUBUNIVERSE



OR

(ii) $\underline{A}/\theta$ IS p-AFFINE FOR SOME $\theta \in Con(\underline{A})$, $p \in \mathbb{P}$.



$\simeq \mathbb{Z}_p \times \dots \times \mathbb{Z}_p$

ZHUK'S PROOF OF THE DICHOTOMY THEOREM

OUTLINE OF THE ALGORITHM:

- FORCE CONSISTENCY
  ⇒ NO SOLUTIONS WILL BE LOST
  WHEN REDUCING TO STRONG SUBSETS
- AS LONG AS EXISTS:
  REDUCE TO STRONG SUBSET +
  FORCE CONSISTENCY
- ELSE: SOLVE LINEAR EQUATIONS

CAVEAT OF THE ALGORITHM

$\underline{A} = (A; w)$; w WNU

USE BEHAVIOUR OF w IN

- STRONG SUBALG. CASE
- p-AFFINE CASE

NEED KNOWLEDGE OF:

- STRONG SUBALGEBRAS
- CONGRUENCES

CAVEAT OF THE ALGORITHM

NOT "UNIVERSAL":

- ALGEBRA + WNU NEED TO BE FIXED
- DEPENDS EXPONENTIONALLY ON SIZE OF DOMAIN

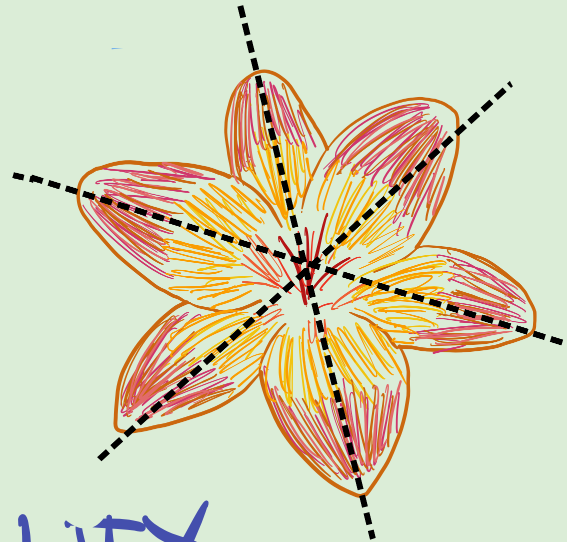
IS THERE A "TRULY POLYNOMIAL ALGORITHM"?

LIMITS OF UNIVERSAL ALGORITHMS

CSP(A) IS SOLVED BY ...

- LOCAL CONSISTENCY IFF ∃ WNUs OF ALL ARITIES — BULATOV, BARTO+KOZIK '09, ZHUK '20
- BLP IFF ∃ SYMMETRIC OPERATIONS OF ALL ARITIES — KOLMOGOROV+THAPPER+ŽIVNÝ '15
- BLP+AIP IFF ∃ SYMMETRIC OPERATIONS OF INFINITELY MANY ARITIES — BRAKENSIEK+GURUSWAMI '20
- ⋮

WHY STUDY SYMMETRIES?



- CHARACTERISE NON-TRIVIALITY
- NON-TRIVIALITY IS BORDER-LINE OF CSP-DICHOTOMY
- ALGORITHMS FOR CSPS BASED ON SYMMETRIC OPERATIONS
- BEAUTY REASONS!

PART I

STRUCTURE
OF AN ALGEBRA
$\updownarrow$
SATISFACTION OF
IDENTITIES

PART II

LOOP CONDITIONS
☺

- LOOPS &
  IDENTITIES
- WNUS
  REPROVED
- OTHER
  IDENTITIES

LOOP LEMMATA

= ANY STATEMENT OF THE FORM

"IF $(V;E)$ IS STRUCTURE SATISFYING CONDITION (C), THEN $(V;E)$ HAS A LOOP" i.e. $\exists a \in V: E(a \dots a)$"

USUALLY: (C) = (A) $\wedge$ (G) WHERE

(A) .. ALGEBRAIC CONDITION

(G) .. GRAPH THEORETIC CONDITION

LOOP LEMMATA: EXAMPLES

$\mathbb{G} := (V, E)$. IF

- $\mathbb{G}$ UNDIRECTED (G)
- $\exists f \in \mathrm{Bl}(\mathbb{G})\ \forall x,y: f(x,y) = f(y,x)$ (A)

THEN $\mathbb{G}$ HAS LOOP.

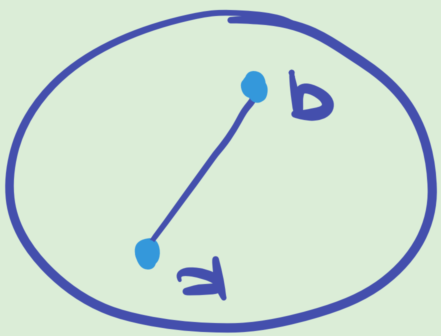
LOOP LEMMATA: EXAMPLES

$\mathbb{G} := (V; E)$. IF

- $\mathbb{G}$ UNDIRECTED (G)
- $\exists f \in \mathrm{Pol}(\mathbb{G})\ \forall x,y: f(x,y) = f(y,x)$ (A)

THEN $\mathbb{G}$ HAS LOOP.

PROOF:



$$f\begin{pmatrix} a & b \\ | & | \\ b & a \end{pmatrix} = \begin{matrix} c \\ | \\ c \end{matrix} \quad \blacksquare$$

LOOP LEMMATA: EXAMPLES

$\mathcal{G} := (V, E)$. IF

- ~~$\mathcal{G}$ UNDIRECTED~~ (G)
- $\exists f \in \mathrm{Bl}(\mathcal{G})\ \forall x, y: f(x,y) = f(y,x)$ (A)

THEN $\mathcal{G}$ HAS LOOP.

LOOP LEMMATA: EXAMPLES

$G := (V, E)$. IF

- ~~G UNDIRECTED~~ (G)
- $\exists f \in \mathrm{Bi}(G)\ \forall x, y: f(x,y) = f(y,x)$ (A)

THEN G HAS LOOP.

<u>COUNTEREX:</u>

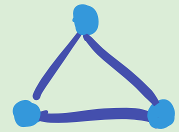
$$\begin{array}{c} b \\ \bullet \\ \uparrow \\ \bullet \\ a \end{array}$$

$$\max(x, y)$$

$$\max\left(\begin{array}{c} b \\ \uparrow \\ a \end{array} \ \begin{array}{c} b \\ \uparrow \\ a \end{array} \right) = \begin{array}{c} b \\ \uparrow \\ a \end{array}$$

LOOP LEMMATA: EXAMPLES

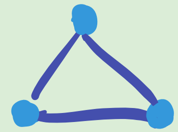
$\mathbb{G} := (V, E)$. IF

- $\mathbb{G}$ UNDIRECTED, CONTAINS  (G)
- $\exists s \in \mathrm{Pol}(\mathbb{G})\ \forall x, y, z:$
  $s(x, y, x, z, y, z) = s(y, x, z, x, z, y)$ (A)

THEN $\mathbb{G}$ HAS LOOP.

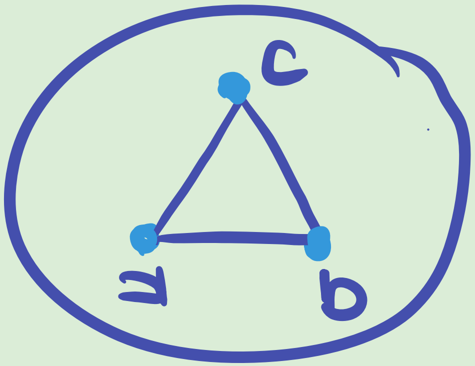
LOOP LEMMATA: EXAMPLES

$\mathbb{G} := (V, E)$. IF

- $\mathbb{G}$ UNDIRECTED, CONTAINS  (G)
- $\exists s \in \mathrm{Pol}(\mathbb{G})\ \forall x, y, z$:
  $s(x, y, x, z, y, z) = s(y, x, z, x, z, y)$ (A)

THEN $\mathbb{G}$ HAS LOOP.

<u>PROOF</u>:



$$s\begin{pmatrix} a & b & a & c & b & c \\ | & | & | & | & | & | \\ b & a & c & a & c & b \end{pmatrix} = \begin{matrix} a \\ | \\ a \end{matrix}$$

LOOP LEMMATA: EXAMPLES

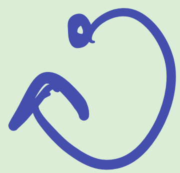
<u>THM</u>: HELL + NEŠETŘIL '91
BULATOV '05

$\mathbb{A}$-COLOURING REVISITED:

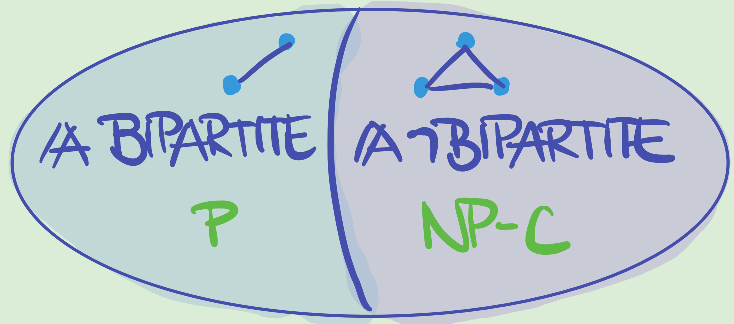
$\mathbb{A} := (V; E)$ FINITE. IF

- $\mathbb{A}$ UNDIRECTED, NON-BIPARTITE (G)
- $Pol(\mathbb{A})$ EQUATIONALLY NON-TRIVIAL (A)

THEN $\mathbb{A}$ HAS LOOP.



$\mathbb{A}$ FINITE, LOOPLESS, UNDIRECTED.
$\mathbb{A}$-COLOURING:



SATISFACTION OF IDENTITIES

THM: HELL+NEŠETŘIL '91, BULATOV '05

$\mathbb{A}$ FINITE, UNDIRECTED, NON-BIPARTITE, $\mathrm{Pol}(\mathbb{A})$ EQU. NON-TRIVIAL $\Rightarrow$ LOOP

COR: SIGGERS '10

$\underline{A}$ FINITE, $\mathrm{Clo}(\underline{A})$ EQU. NON-TRIVIAL

$\Rightarrow \underline{A} \models s(x,y,x,z,y,z) \approx s(y,x,z,x,z,y)$

SIGGERS TERM

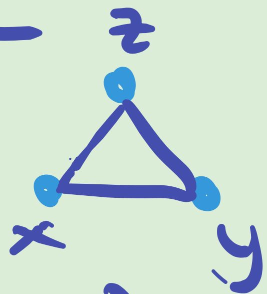
SATISFACTION OF IDENTITIES

<u>PROOF</u>: $\underline{F} := \mathcal{F}_{\mathcal{V}(\underline{A})}(\{x,y,z\})$ FREE ALG.

$\underline{F}$ FINITE, $\mathrm{Clo}(\underline{F})$ EQU. NON-TRIVIAL

$$R := \left\langle \left\{ \binom{x}{y}, \binom{y}{x}, \binom{x}{z}, \binom{z}{x}, \binom{y}{z}, \binom{z}{y} \right\} \right\rangle_{\underline{F}}$$

- R SYMMETRIC
- R CONTAINS



- $\mathrm{Pol}(R) \supseteq \mathrm{Clo}(\underline{F})$

Thm. $\Rightarrow \exists (a,a) \in R$

$$\Rightarrow \exists s \in \mathrm{Clo}(\underline{A}):\quad s\begin{pmatrix} x & y & x & z & y & z \\ y & x & z & x & z & y \end{pmatrix} = \begin{pmatrix} a \\ a \end{pmatrix}$$

WNUS REPROVED LOOP LEMMA - STYLE

$$
\begin{aligned}
& w(yxx \cdots x) \\
\approx\; & w(xyx \cdots x) \\
\approx\; & w(xxyx \cdots x) \\
& \vdots \\
\approx\; & w(x \cdots xy)
\end{aligned}
$$

⚠ THIS PART CONTAINS SOME LIES

WNUS REPROVED LOOP LEMMA - STYLE

$$
\begin{aligned}
& w(y\,x\,x\,\cdots\,x) \\
\approx\ & w(x\,y\,x\,\cdots\,x) \\
\approx\ & w(x\,x\,y\,x\,\cdots\,x) \\
& \vdots \\
\approx\ & w(x\,\cdots\,x\,y)
\end{aligned}
$$

$$R := \langle \quad \cdots \quad \rangle_I$$

FIND LOOP LEMMA THAT GIVES LOOP IN R!

WNUS REPROVED LOOP LEMMA - STYLE

$$
\begin{aligned}
& w(y\,x\,x \;\cdots\; x) \\
\approx\; & w(x\,y\,x \;\cdots\; x) \\
\approx\; & w(x\,x\,y\,x \;\cdots\; x) \\
& \quad \vdots \\
\approx\; & w(x \;\cdots\; x\,y)
\end{aligned}
$$

$$R := \langle (\;)(\;) \cdots (\;) \rangle_I$$

FIND LOOP LEMMA THAT GIVES LOOP IN R!

WHAT PROPERTIES DOES R HAVE?

WNUS REPROVED LOOP LEMMA - STYLE

$$
\begin{aligned}
& w(y\, x\, x \cdots x) \\
\approx\ & w(x\, y\, x \cdots x) \\
\approx\ & w(x\, x\, y\, x \cdots x) \\
& \vdots \\
\approx\ & w(x \cdots x\, y)
\end{aligned}
$$

$R := \langle (\,)(\,) \cdots (\,) \rangle_I$

FIND LOOP LEMMA THAT GIVES LOOP IN R!

WHAT PROPERTIES DOES R HAVE?

- SYMMETRIC
- INVARIANT

WNUs REPROVED LOOP LEMMA - STYLE

QUESTION:

$\underline{A}$ FINITE, $Clo(\underline{A})$ IDEMPOTENT, EQU. NON-TRIVIAL. $R \leq \underline{A}^n$ SYMMETRIC. DOES R HAVE CONSTANT TUPLE?

WNUs REPROVED LOOP LEMMA - STYLE

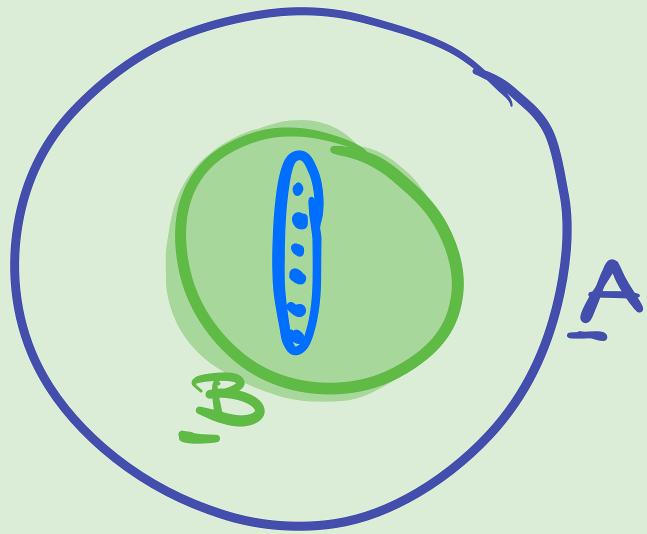
<u>QUESTION</u>:

$\underline{A}$ FINITE, $Clo(\underline{A})$ IDEMPOTENT, EQU. NON-TRIVIAL. $R \leq \underline{A}^n$ SYMMETRIC. DOES R HAVE CONSTANT TUPLE?

⇝ INDUCTION ON $|A|$

WANT $\underline{B} <_{\times} \underline{A}$ S.T.

$$\underbrace{B^n \cap R}_{\text{INVARIANT SYMMETRIC}} \neq \emptyset$$



WNUS REPROVED LOOP LEMMA - STYLE

"LEMMA"

$R \leq \underline{A}^n$ SYMMETRIC $\Rightarrow$ R HAS LOOP

COUNTEREXAMPLE:

$$\underline{A} = (\mathbb{Z}_2 ; \ x+y+z)$$

$$R = \left\{ \binom{x}{y} : x+y=1 \right\}$$

WNUs REPROVED LOOP LEMMA - STYLE

"LEMMA"

$R \leq \underline{A}^n$ SYMMETRIC $\Rightarrow$ R HAS LOOP

COUNTEREXAMPLE:

$\underline{A} = (\mathbb{Z}_2 ; \ x+y+z\ )$

$R = \left\{ \binom{x}{y} : x+y = 1 \right\} \subseteq A^2$

⚠ $\mathfrak{z}r(R) \nmid 2$!

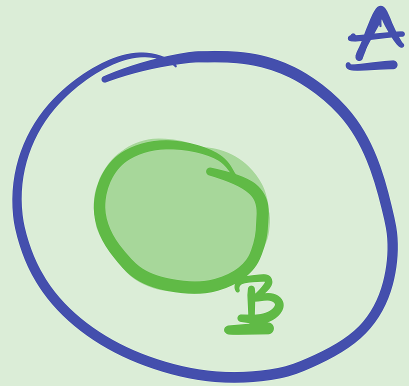
WNUs REPROVED LOOP LEMMA - STYLE

Thm: ZHUK'S CASES — ZHUK '17

$\underline{A}$ FINITE, $Clo(\underline{A})$ IDEMPOTENT, EQU. NON-TRIVIAL

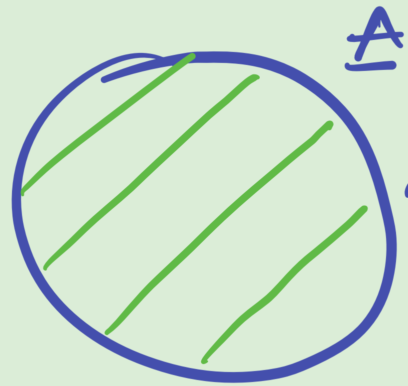
THEN EITHER

(i) $\underline{A}$ HAS NON-TRIVIAL STRONG SUBUNIVERSE



OR

(ii) $\underline{A}/\theta$ IS p-AFFINE FOR SOME $\theta \in Con(\underline{A})$, $p \in \mathbb{P}$.



$\simeq \mathbb{Z}_p \times \dots \times \mathbb{Z}_p$

WNUs REPROVED LOOP LEMMA - STYLE

LEMMA: STRONG SUBALG.- CASE ZHUK '20

$\underline{A}$ FINITE, $Clo(\underline{A})$ IDEMPOTENT.
IF $\underline{A}$ HAS STRONG SUBALG., THEN R CAN BE RESTRICTED TO SMALLER DOMAIN.

WNUs REPROVED LOOP LEMMA - STYLE

LEMMA: STRONG SUBALG.- CASE ZHUK '20

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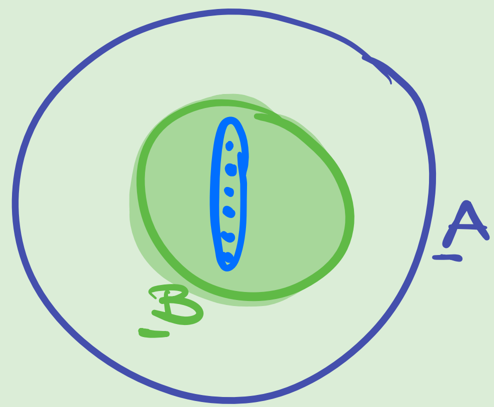
IF $\underline{A}$ HAS STRONG SUBALG., THEN R CAN BE RESTRICTED TO SMALLER DOMAIN.

PROOF

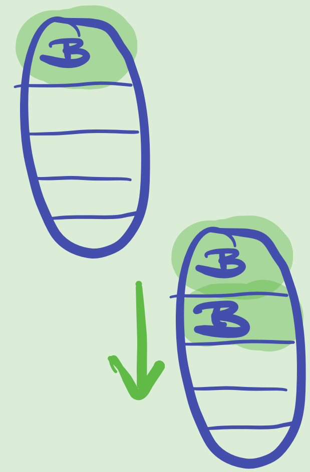


- $pr_1(R) = pr_2(R) = \dots = A$
- $R^B = R \cap (B \times A \times \dots \times A)$
- $\underline{B}$ STRONG $\Rightarrow$ $pr_2(R^B) = pr_3(R^B)$.. STRONG

WNUS REPROVED LOOP LEMMA - STYLE

- B s.t. R^B MAXIMAL SIZE

R^B:

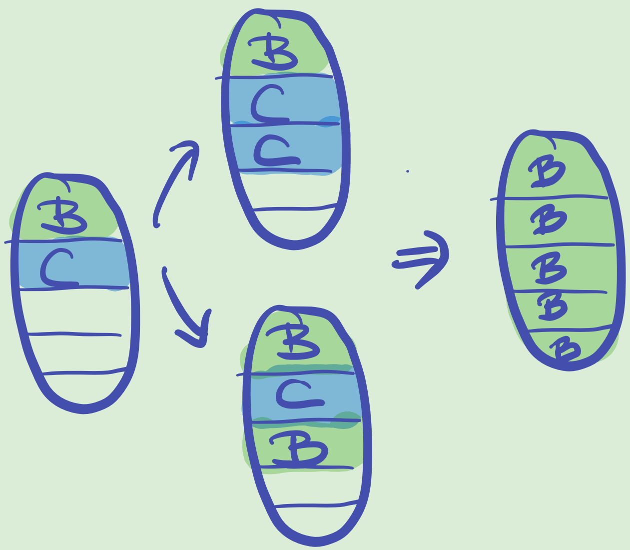


- ENOUGH: $R \cap (B \times B \times A \dots \times A) \neq \emptyset$

CLAIM: $C := pr_2(R^B)$ FULL OR $C = B$

$pr_1(R^C) \supseteq B \overset{R^B \text{ MAX}}{\Longrightarrow} pr_1(R^C) = B$

THUS,



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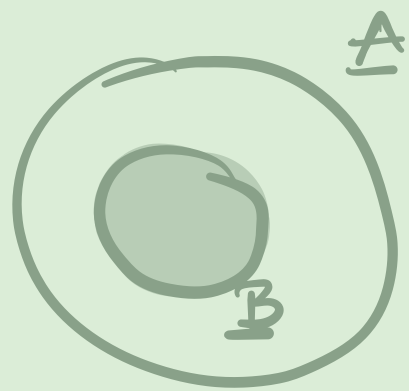
WNUs REPROVED LOOP LEMMA - STYLE

Thm: ZHUK'S CASES ZHUK '17

$\underline{A}$ FINITE, $\mathrm{Clo}(\underline{A})$ IDEMPOTENT, EQU. NON-TRIVIAL

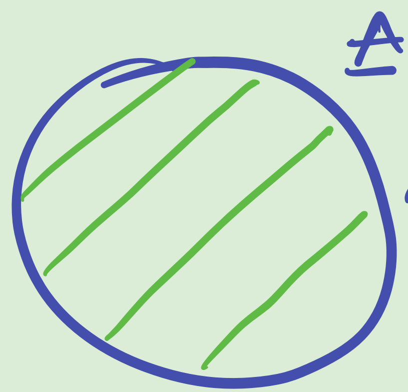
THEN EITHER

✓ (i) $\underline{A}$ HAS NON-TRIVIAL STRONG SUBUNIVERSE



OR

(ii) $\underline{A}/\vartheta$ IS p-AFFINE FOR SOME $\vartheta \in \mathrm{Con}(\underline{A})$, $p \in \mathbb{P}$.



$\simeq \mathbb{Z}_p \times \dots \times \mathbb{Z}_p$

WNUS REPROVED LOOP LEMMA - STYLE

P-AFFINE CASE

- $Cb(\underline{A})$ CONTAINS ALL P-AFFINE OPERATIONS $\lambda_1 x_1 + \dots + \lambda_m x_m$, $\sum \lambda_i \equiv 1 \ (p)$

- GIVEN $R \leq \underline{A}^n$ SYMMETRIC, WANT TO FIND $\sigma_1 \dots \sigma_{m-1} \in S_n$, $f(x_1 \dots x_m)$ P-AFFINE S.T. $f(\underbrace{r}_{\in R}, \underbrace{\sigma_1(r)}_{\in R} \dots \underbrace{\sigma_{m-1}(r)}_{\in R}) = \underbrace{(c \cdots c)}_{\in R}$ FOR SOME $r \in R$

- BY COUNTEREX. MUST HAVE $n \nmid p$.

WNUs REPROVED LOOP LEMMA - STYLE

LEMMA: P-AFFINE CASE ZHUK '20

$\underline{A}$ p-AFFINE, $p \nmid n$, $R \leq \underline{A}^n$ SYMMETRIC.
THEN R HAS CONSTANT TUPLE.

WNUs REPROVED LOOP LEMMA - STYLE

<u>LEMMA</u>: p-AFFINE CASE ZHUK '20

$\mathbb{A}$ p-AFFINE, $p \nmid n$, $R \leq \mathbb{A}^n$ SYMMETRIC.
THEN R HAS CONSTANT TUPLE.

<u>PROOF</u>: $\sigma := (1 \dots n) \in S_n$ CYCLIC PERM.

$$f(x_1 \dots x_n) := \frac{1}{n} \sum_{i=1}^{n} x_i \quad , \quad r \in R$$

NOW, $f(r, \sigma(r), \sigma^2(r) \dots \sigma^{n-1}(r)) =$

$$= \frac{1}{n} \left[\begin{pmatrix} r_1 \\ \vdots \\ r_n \end{pmatrix} + \begin{pmatrix} r_2 \\ \vdots \\ r_n \\ r_1 \end{pmatrix} + \dots \begin{pmatrix} r_n \\ r_1 \\ \vdots \\ r_{n-1} \end{pmatrix} \right] = \frac{1}{n} \begin{pmatrix} \sum r_i \\ \vdots \\ \sum r_i \end{pmatrix} \in R \qquad \blacksquare$$

WNUs REPROVED LOOP LEMMA - STYLE

COR: MARÓTI + MCKENZIE '08

$\underline{A}$ FINITE, $Clo(\underline{A})$ IDEMP., EQU. NON-TRIVIAL

$\forall p \leq |A|, p \in \mathbb{P}$: $p \nmid n$

THEN $Clo(\underline{A})$ HAS WNU OF ARITY n.

WNUS REPROVED LOOP LEMMA - STYLE

COR: MARÓTI + MCKENZIE '08

$\underline{A}$ FINITE, $Clo(\underline{A})$ IDEMP., EQU. NON-TRIVIAL

$\forall p \leq |A|, p \in \mathbb{P}: \quad p \nmid n$

THEN $Clo(\underline{A})$ HAS WNU OF ARITY n.

PROOF: $R := \langle () () \cdots () \rangle_{\underline{F}}$ SYMMETRIC

$\underline{F}$ FINITE, IDEMP., EQU. NON-TRIVIAL

W.T.S.: R HAS CONSTANT TUPLE

$\exists w \in Clo(\underline{A}): \quad w\left(\begin{pmatrix} x \\ x \\ \vdots \\ y \end{pmatrix} \cdots \begin{pmatrix} y \\ x \\ \vdots \\ x \end{pmatrix}\right) = \begin{pmatrix} c \\ \vdots \\ c \end{pmatrix}$

WNUS REPROVED LOOP LEMMA - STYLE

INDUCTION ON $|\underline{F}|$:

- IF $\underline{F}$ HAS STRONG SUBALGEBRA:
  CAN FIND $\underline{B} \lneq \underline{F}$ S.T. $B^n \cap R \neq \emptyset$
  + INDUCTIVE HYPOTHESIS ✓
- ELSE P-~~AFFINE~~, IN PARTICULAR: $p \leq |A|$,
  $\Rightarrow p \nmid n$ BY ASSUMPTION
  $\Rightarrow$ CONST. TUPLE BY LEMMA ✓

OTHER IDENTITIES

<u>THM</u>: $\underline{A}$ FINITE, $Clo(\underline{A})$ IDEMPOTENT. TFAE:

- $Clo(\underline{A})$ IS $\boxed{\text{EQUATIONALLY NON-TRIVIAL}}$
- $\exists$ TAYLOR TERM $t(x, *\ldots *) \approx t(y, *\cdots *), \ \cdots$ — TAYLOR '77
- $\exists$ WNU TERM $w(xx\cdots xy) \approx \ldots \approx w(yx\cdots x)$ — MARÓTI + MCKENZIE '08
- $\exists$ SIGGERS TERM $s(x,y,z,x,y,z) \approx s(y,x,x,z,z,y)$ — SIGGERS '10
- $\exists$ CYCLIC TERM $c(x_1\cdots x_p) \approx c(x_2\cdots x_p, x_1)$ — BARTO + KOZIK '12
- $\exists$ 4-ARY SIGGERS TERM $s(a,r,e,a) \approx s(r,a,r,e)$ — KEARNES + MARKOVIĆ + MCKENZIE '15

MY EXTEND LIST!

OTHER IDENTITIES

k-WNU:

$$\left.\begin{aligned} & w(\overbrace{y\cdots y}^{k}\overbrace{x\cdots x}^{n-k}) \\ \approx\; & w(y\cdots yxyx\cdots x) \\ & \vdots \\ \approx\; & w(xx\cdots xy\cdots y) \end{aligned}\right\} \binom{n}{k}$$

OTHER IDENTITIES

K-WNU:

$$\left.\begin{aligned} & w(\overbrace{y\cdots y}^{k}\overbrace{x\cdots x}^{n-k}) \\ \approx\ & w(y\cdot y x y x\cdots x) \\ & \vdots \\ \approx\ & w(x x\cdots x y\cdots y) \end{aligned}\right\} \binom{n}{k}$$

SYMMETRY?

OTHER IDENTITIES

k-WNU:

$$\begin{aligned} & w(\underbrace{y \cdots y}_{k}\,\underbrace{x \cdots x}_{n-k}) \\ \approx\; & w(y \cdot y x y x \cdots x) \\ & \vdots \\ \approx\; & w(x x \cdots x y \cdot y) \end{aligned} \quad \Bigg\}\ \binom{n}{k} = \{k\text{-ELEMENT SUBSETS OF } \{1..n\}\}$$

$$S_n \curvearrowright \binom{n}{k}$$

SYMMETRY?

$\sigma \in S_n$, $\quad r = (r_E)_{E \in \binom{n}{k}} \in R \quad$ "(n,k)-SYMMETRIC"

$\Rightarrow \sigma(r) = (r_{\sigma(E)})_{E \in \binom{n}{k}} \in R$

OTHER IDENTITIES

LET $R \leq \underline{A}^{\binom{n}{k}}$ (n,k)-SYMMETRIC
$\underline{A}$ FINITE, IDEMP., EQU. NON-TRIVIAL

LEMMA: ZHUK + B.

IF $\underline{A}$ HAS STRONG SUBALG., THEN R HAS NON-EMPTY RESTRICTION TO SOME SMALLER DOMAIN.

LEMMA: BRADY + PINSKER + ZHUK + B. $k := 2$

IF $\underline{A}$ P-AFFINE, $p \nmid n \cdot \binom{n}{2}$, THEN R HAS CONSTANT TUPLE

COR: BRADY + PINSKER + ZHUK + B. $k := 2$

IF $\forall p \leq |A|, p \in \mathbb{P}: p \nmid n \cdot \binom{n}{2}$, THEN $Cb(\underline{A})$ HAS 2-WNU OF ARITY n.

OTHER IDENTITIES

SIMULTANEOUS K-WNU: XY-SYMMETRIC

$f(x_1 \dots x_n)$ IS XY-SYMMETRIC $:\Leftrightarrow$ f IS k-WNU $\forall k \leq n$

i.e. f SYMMETRIC ON ALL TUPLES $J_k = (\underbrace{y \dots y}_{k} \underbrace{x \dots x}_{n-k})$

<u>THM</u>: ZHUK '24

$\underline{A}$ FINITE, IDEMPOTENT, n ODD.
IF $\underline{A}$ HAS WNU OF ARITY n, THEN
IT HAS AN XY-SYMMETRIC OPERATION
OF ARITY n.

OTHER IDENTITIES

XYZ - SYMMETRIC

$$A = \{0,1,2\} \qquad h(x,y,z) := \begin{cases} x+y+z & \text{if } x,y,z \in \{0,1\} \\ 2 & \text{if } x=y=z=2 \\ \text{first non-2} & \text{o/w} \end{cases}$$

$\underline{A} = (A; h)$ IS FINITE, IDEMP, EQU. NON-TRIVIAL

BUT $\forall k, l, m$ $Cl_3(\underline{A})$ DOES NOT HAVE

$$(\underbrace{x \cdots x}_{k}\underbrace{y \cdots y}_{l}\underbrace{z \cdots z}_{m})\text{-SYMMETRIC OPERATION}$$

OTHER IDENTITIES

<u>QUESTIONS</u>:

- $\underline{A}$ FINITE. FOR WHICH TUPLES $\mathcal{J}$ OF VARIABLES DOES $\underline{A}$ HAVE $\mathcal{J}$-SYMMETRIC OPERATION?

- $\mathcal{J}$ TUPLE OF VARIABLES. CAN WE DESCRIBE ALL ALGEBRAS THAT DO (NOT) HAVE $\mathcal{J}$-SYMMETRIC OPERATIONS?

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- ⋮

THANK YOU!

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ÖAW
ÖSTERREICHISCHE AKADEMIE DER WISSENSCHAFTEN