Resolving Sets in Temporal Graphs

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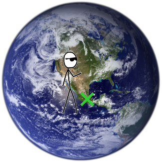

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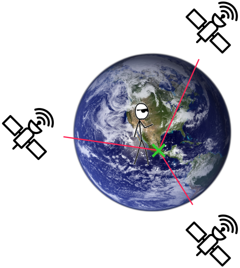

GPS, GLONASS, Galileo, Beidou, IRNSS, QZSS: use of at least four satellites

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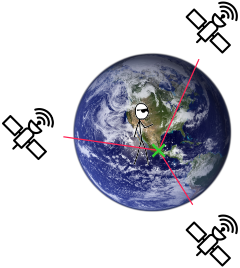

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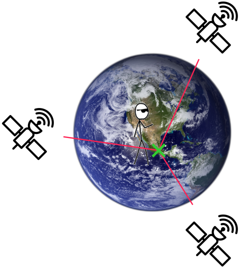

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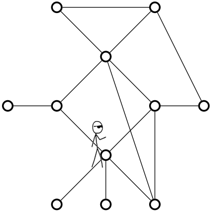

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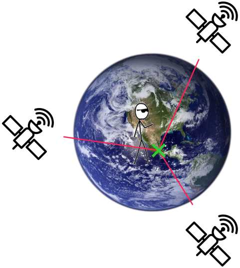

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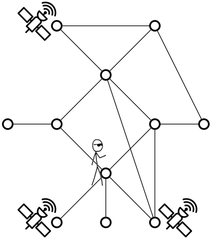

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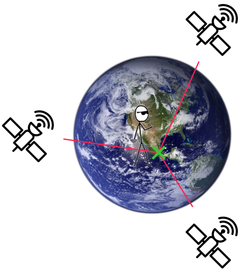

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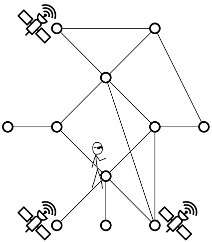

How many "satellites" would I need in a given graph?
⇒ Well-studied Metric Dimension

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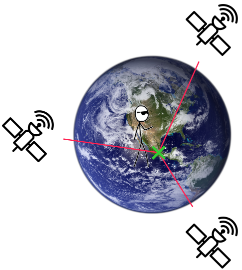

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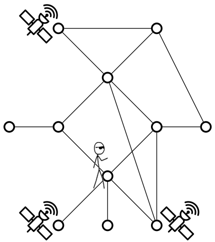

How many "satellites" would
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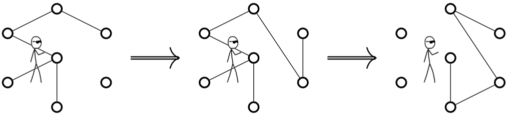
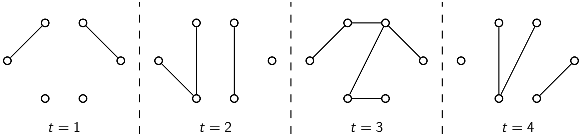
$\Rightarrow$ Well-studied Metric Dimension



What if the graph **changes over time**?

Temporal graphs

$$\mathcal{G} = (V, E_1, E_2, \ldots, E_{t_{\max}})$$ [Ferreira & Viennot, 2002]



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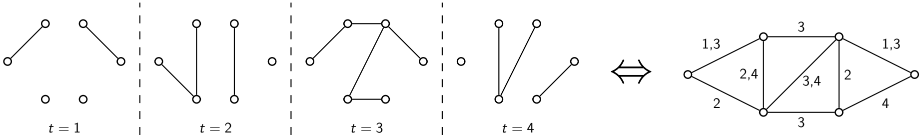

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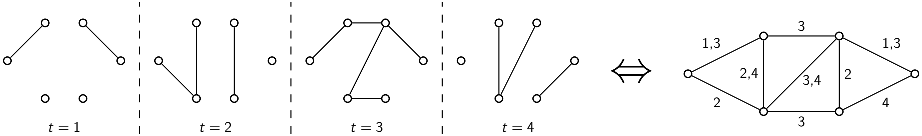

Well-studied in distributed algorithms and dynamic networks (biological, transportation...)

[Casteigts, 2018] for a thorough introduction

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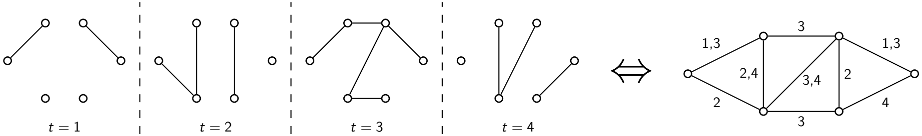

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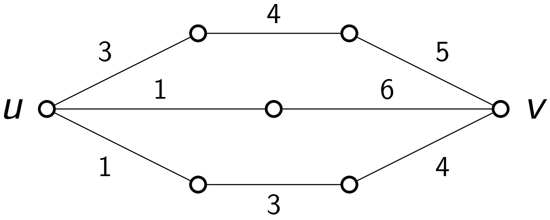
Useful terms

- The static graph $G = (V, E)$ is the **underlying graph**
- λ is the **time labeling**
- A **journey** is a path in the underlying graph with **strictly increasing** time-steps

Distance in temporal graphs

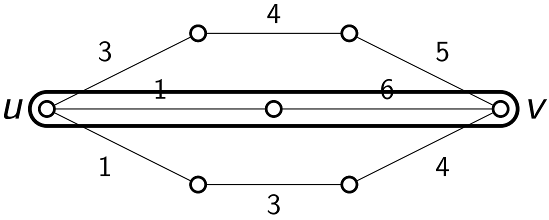
Three possible variables to minimize in a journey



Distance in temporal graphs

Three possible variables to minimize in a journey

- **Shortest:** Number of edges in the underlying graph



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- **Shortest:** Number of edges in the underlying graph
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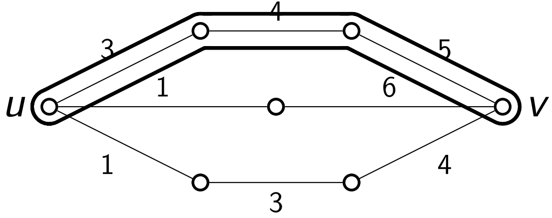

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- **Shortest:** Number of edges in the underlying graph
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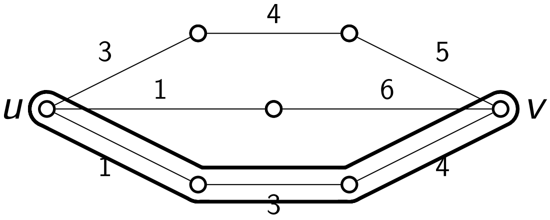
- **Foremost:** Arrival time



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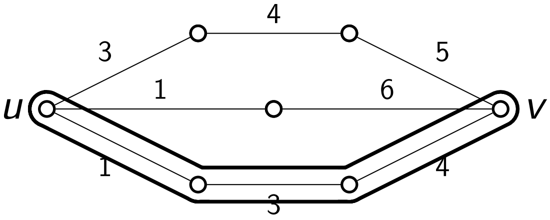


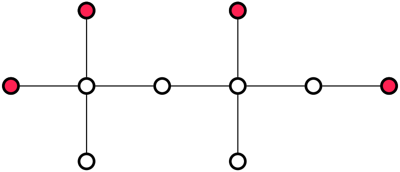
Temporal distance

The **temporal distance** from u to v is the last time-step of a **foremost** journey from u to v.

Resolving Set

Definition

$R \subseteq V$ s.t., $\forall u, v \in V$, $u \neq v$, $\exists r \in R$ with $\text{dist}(r, u) \neq \text{dist}(r, v)$.



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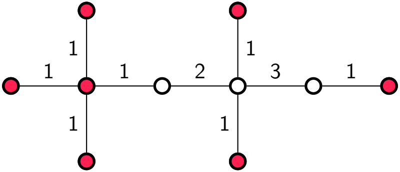

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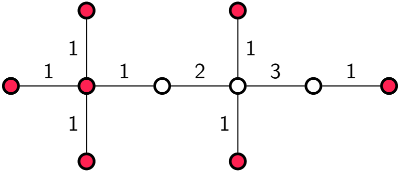

- All vertices must be reached from some vertex of R
- If $\lambda(e) = \{1, \ldots, \text{diam}(G)\}$ for every edge e, then we get the standard resolving set $\rightarrow$ Generalization
- Temporal Resolving Set: find a minimum-size set

Previous work on resolving sets

Resolving set

- Defined by [Harary & Melter, 1976] and [Slater, 1975], well-studied since
- NP-hard, even on restricted classes (bipartite...)
- W[2]-hard (wrt solution size) on subcubic graphs
- Polynomial algorithms for trees and complete graphs

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k-truncated resolving set

$\mathrm{dist}_k(u, v) = \min(\mathrm{dist}(u, v), k + 1)$

- Defined by [Estrada-Moreno *et al.*, 2021], generalizing adjacency resolving sets for any k
- NP-hard on trees, but XP (wrt k) algorithm
- Polynomial algorithms for subdivided stars and complete graphs

Our results: complexity of Temporal Resolving Set

	Standard resolving set	k-truncated resolving set	**Temporal resolving set**
Trees	poly	NP-hard XP wrt k	NP-hard (2 consecutive time labels per edge)
Subdivided stars	poly	poly	NP-hard (2 time labels per edge) poly (time labels 1 and 2, one per edge)
Stars, Paths	poly	poly	poly (1 time label per edge)
Complete graphs	poly	poly	NP-hard (time labels 1 and 2, one per edge)

NP-hardness

Theorem [B., Dailly & Lehtilä, 2024]

TEMPORAL RESOLVING SET is **NP-hard**, even if the underlying graph is a **tree** and each edge appears **at most twice**.

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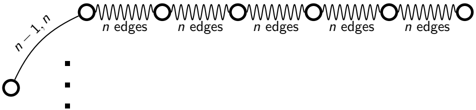

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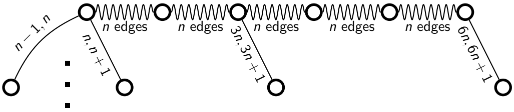

triple $(1, 3, 6)$

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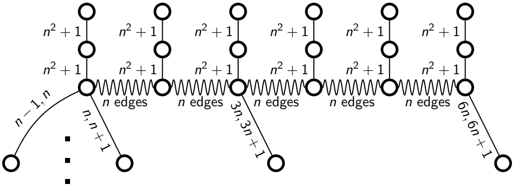

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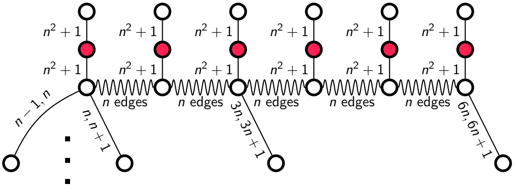

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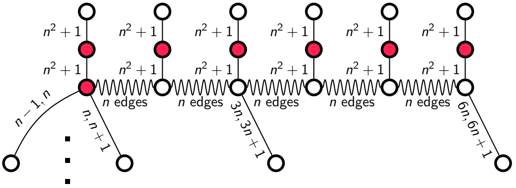

triple $(1, 3, 6)$
not in the matching

- Control vertices are always in the set
- First vertex in the set for every triple **not** in the matching

Paths

Theorem [B., Dailly & Lehtilä, 2024]

TEMPORAL RESOLVING SET can be solved in **linear** time if the underlying graph is a **path** and each edge appears **once**.

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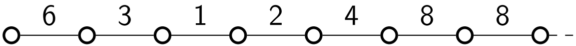

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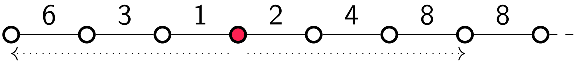

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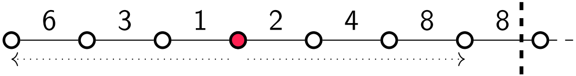

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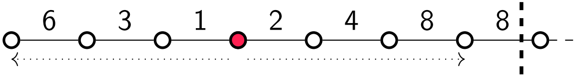

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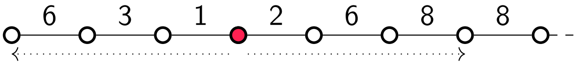

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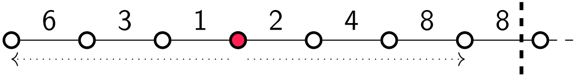

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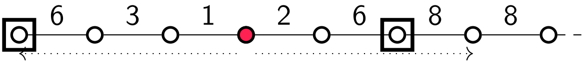

Conflict

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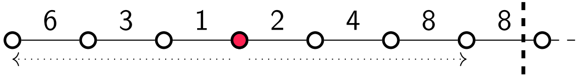

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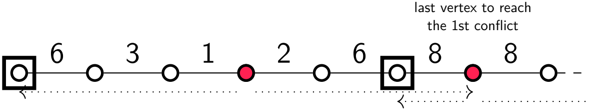

Conflict solved

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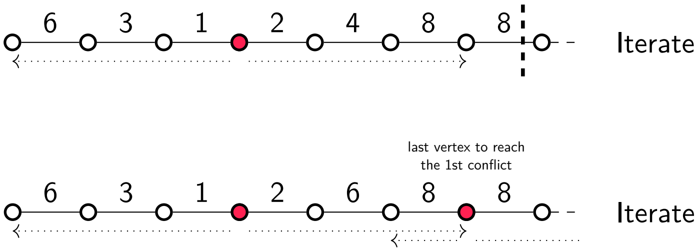

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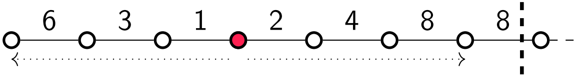

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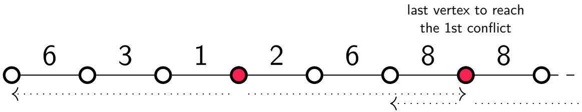


Iterate

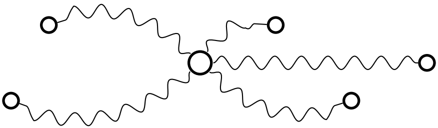
- A few more details for edge cases
- Proof of optimality: technical analysis

Subdivided stars

Theorem [B., Dailly & Lehtilä, 2024]

TEMPORAL RESOLVING SET can be solved in **polynomial** time if the underlying graph is a **subdivided star**, each edge appears **once** and the time-steps are **1 or 2**.

Algorithm idea



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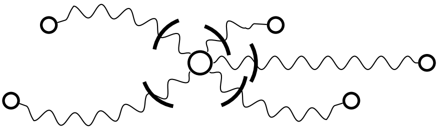

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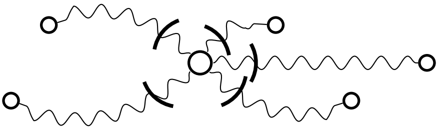
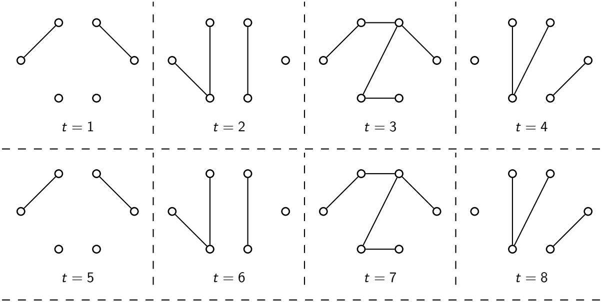


- Apply the path algorithm on every branch
- Manage the center and the link with the branches (lots of cases to consider carefully!)

Our results: periodic time labelings



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$\rightarrow$ 4-periodic time labeling

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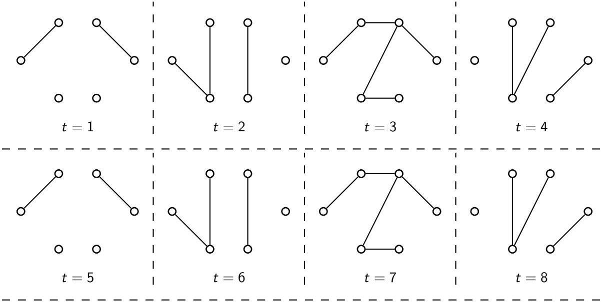

• • •

$\rightarrow$ 4-periodic time labeling

Results for 1 time label per edge

- Combinatorial results (bounds) for paths, cycles, complete graphs, subdivided stars, and complete binary trees
- XP algorithm (wrt **period**) for subdivided stars

Periodic time labelings on subdivided stars

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Proposition [B. Dailly, Lehtilä, 2024]

Any leaf is a temporal resolving set of a path with a periodic time labeling.

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Periodic time labelings on subdivided stars

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Proof idea

- Lemma: for a tree in this setting, there is a minimum-size temporal resolving set containing only leaves
- At least $\ell - p$ leaves are necessary ($\ell =$ number of leaves)
- Test every set of $\ell - p$, $\ell - p + 1$, ..., ℓ leaves

Final words

Our contributions

- Introducing temporal resolving sets
- NP-hard even on subdivided stars and complete graphs
- Poly-time algorithms for paths, stars (under constraints)
- XP on subdivided stars for periodic labelings (wrt period)

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Future work

- Other parameters: number of time-steps, ...
- Large number of labels per edge might help!
- Other notions of distance (shortest and fastest journey)

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